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# Connecting far-from-equilibrium hydrodynamics to resummed transport coefficients and attractors


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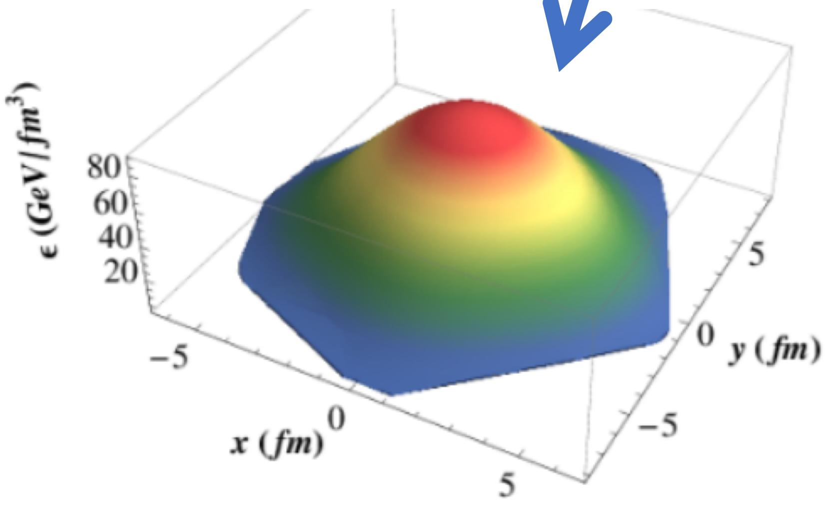
# Surprising effectiveness of hydro in heavy ions

At first, it seemed that hydrodynamics was "easily" justifiable

**Very smooth fluid over the size of a large nucleus**

**quark-gluon plasma**




**near equilibrium dynamics**

macro: $\partial\varepsilon/\varepsilon_0 \sim 1/L$

micro: $\ell \sim 1/T \sim 1/\Lambda_{QCD}$

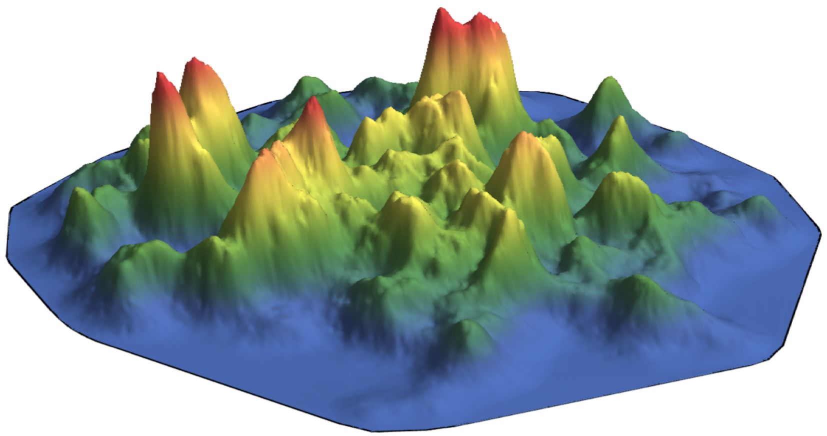
**Knudsen number**

$$\boxed{K_N \sim \ell\,\partial\varepsilon < 0.1}$$

Fluid dynamics over scales of the size of a large nucleus

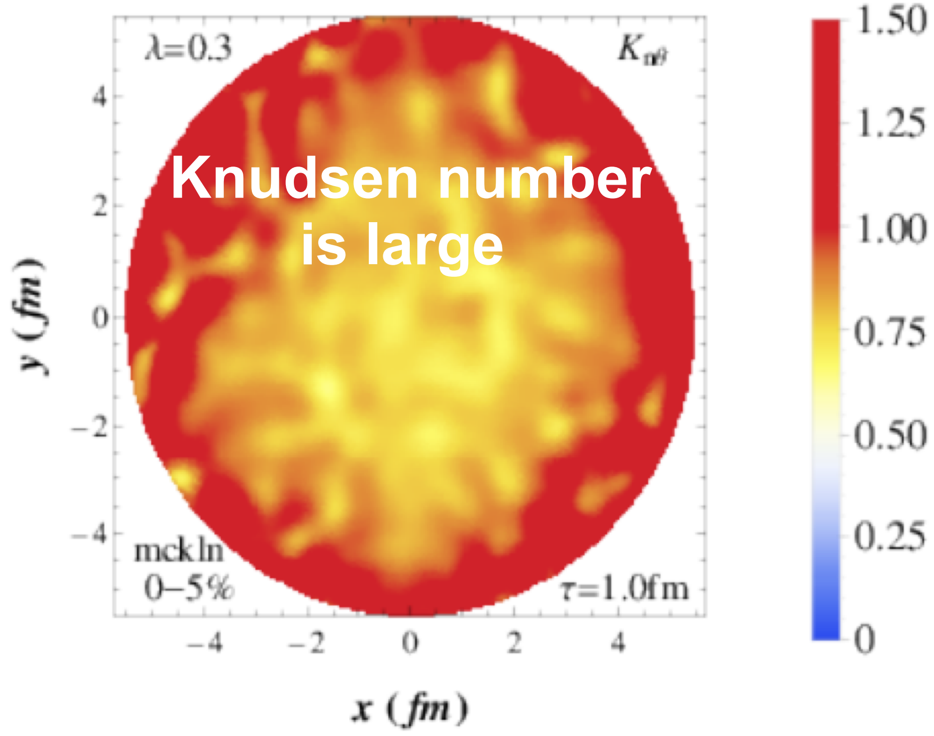
# However, even in realistic AA collisions ...

QGP energy density



**Large spatial gradients at early times**

J. Noronha-Hostler, JN, Gyulassy, PRC (2016)




- The Knudsen number is not small (in pA it should be worse).
- One cannot avoid the far-from-equilibrium regime.
- **What is the regime of applicability of hydrodynamics?**

Israel, Stewart, Ann. Phys. 118, 341 (1979)

# "Hydro" in our field is not simple textbook hydro → Israel-Stewart theory

See G. Denicol's talk

Energy-momentum tensor

$$T_{\mu\nu} \longrightarrow \varepsilon, u_\mu, \pi_{\mu\nu}, \Pi \quad \text{as dynamical variables}$$

An effective theory for hydrodynamic fields and non-hydrodynamic fields

**Dynamics:** $\nabla_\mu T^{\mu\nu} = 0$ (energy-momentum conservation)

$$u^\lambda \nabla_\lambda \pi^{\mu\nu} + F^{\mu\nu}(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0 \quad \text{(shear)}$$

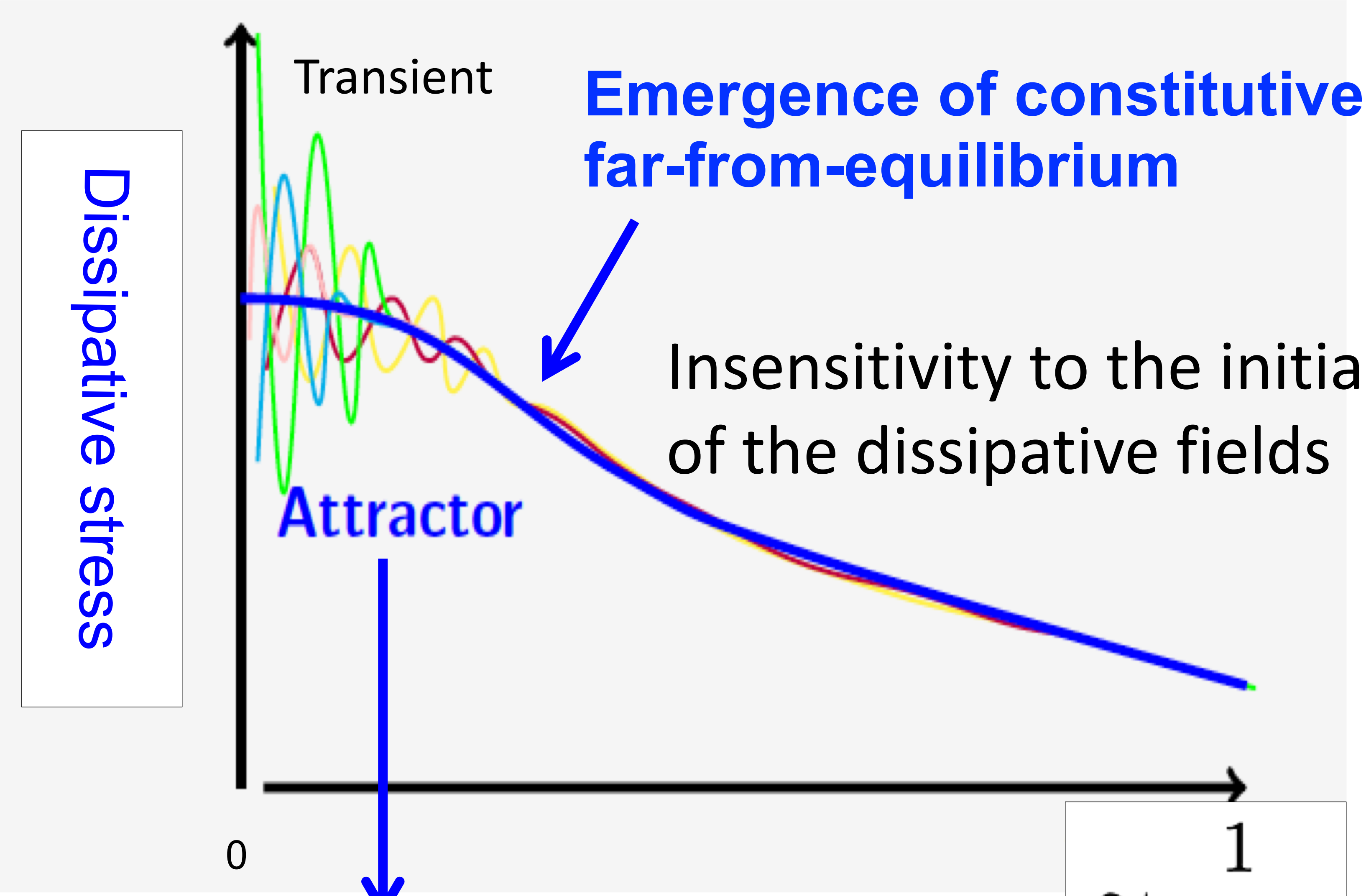
$$u^\lambda \nabla_\lambda \Pi + F(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0 \quad \text{(bulk)}$$

# Far-from-equilibrium hydrodynamics → Attractors

Heller and Spalinski, PRL (2015)

$$\frac{\Pi}{P}$$

$$\frac{\pi^{\mu\nu}}{P}$$




$$\sim \frac{1}{K_N}$$

**Emergence of constitutive relations far-from-equilibrium**

Insensitivity to the initial conditions of the dissipative fields

Resummation in Knudsen number

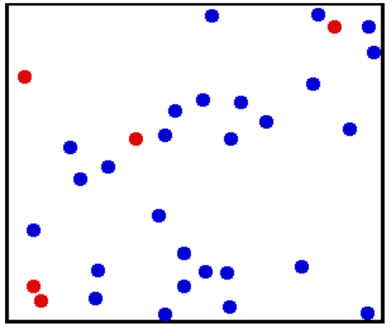
# New development: Hydrodynamic attractor for the full Boltzmann equation

G. Denicol and JN, arXiv:1908.09957 [nucl-th]

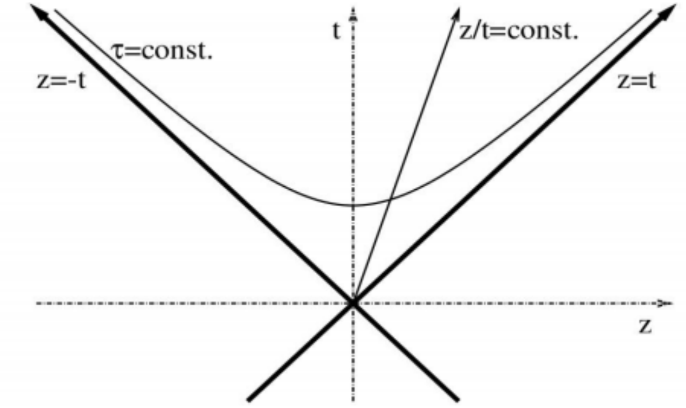
Gas of ultrarelativistic hard spheres undergoing Bjorken flow

## Boltzmann dynamics

"2 to 2" (conserved particle number)



## Bjorken flow




$$\tau = \sqrt{t^2 - z^2}$$

## *Constant* Knudsen number

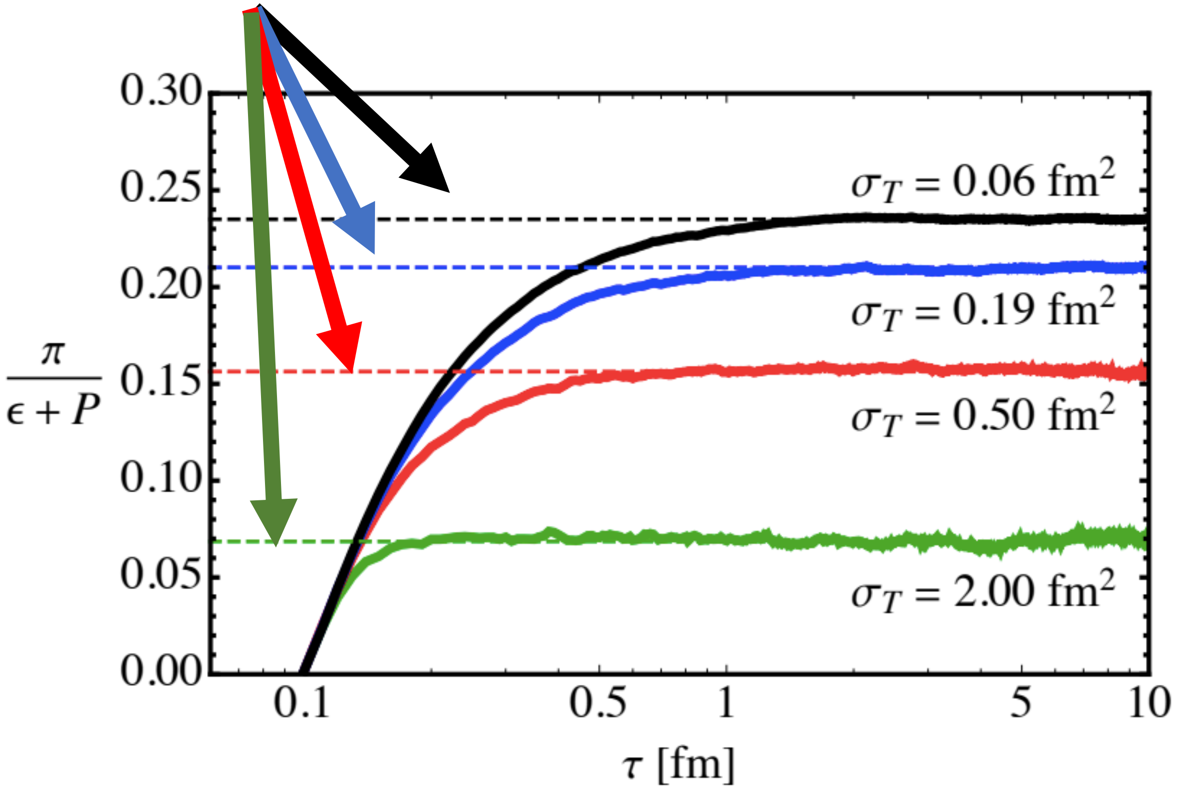
$$\mathrm{Kn} = \frac{\ell_{\mathrm{mfp}}}{\tau} = \frac{1}{n_0 \tau_0 \sigma_T}$$

- Tunable parameter
- Always far from equilibrium
- Attractor is a constant!!!

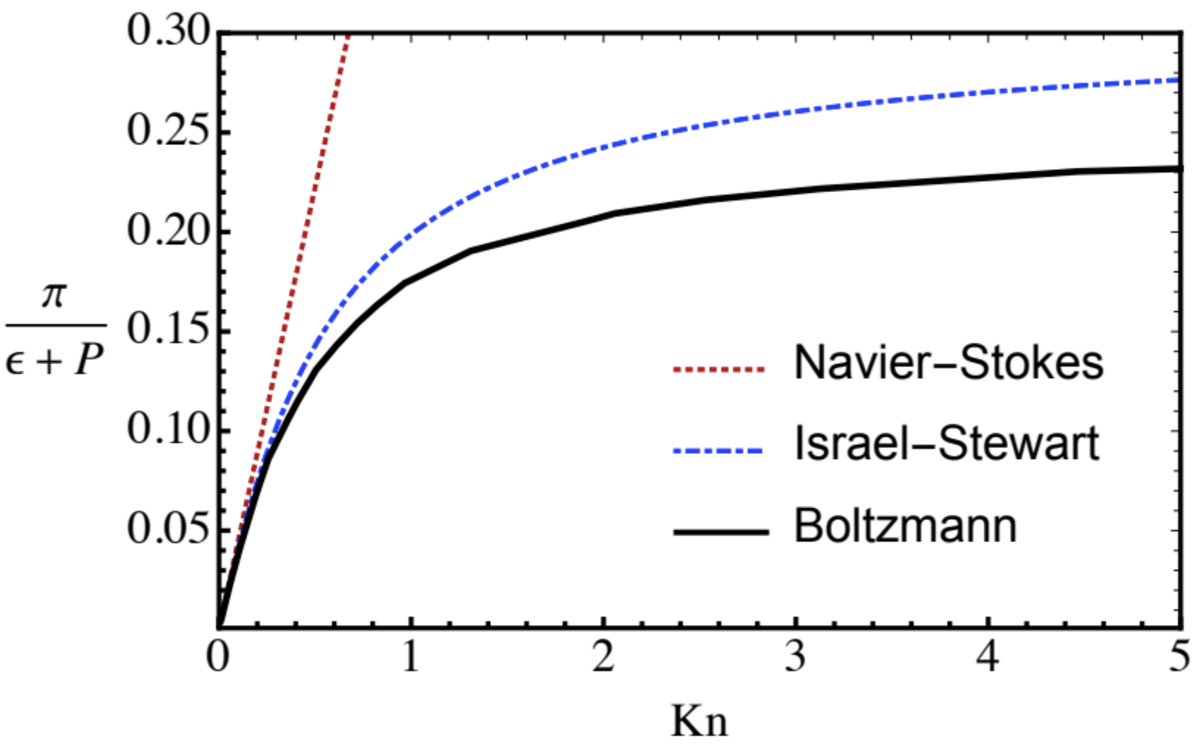
# The attractor from the full Boltzmann equation

G. Denicol and JN, arXiv:1908.09957 [nucl-th]

Boltzmann attractors




Boltzmann simulations: BAMPS algorithm



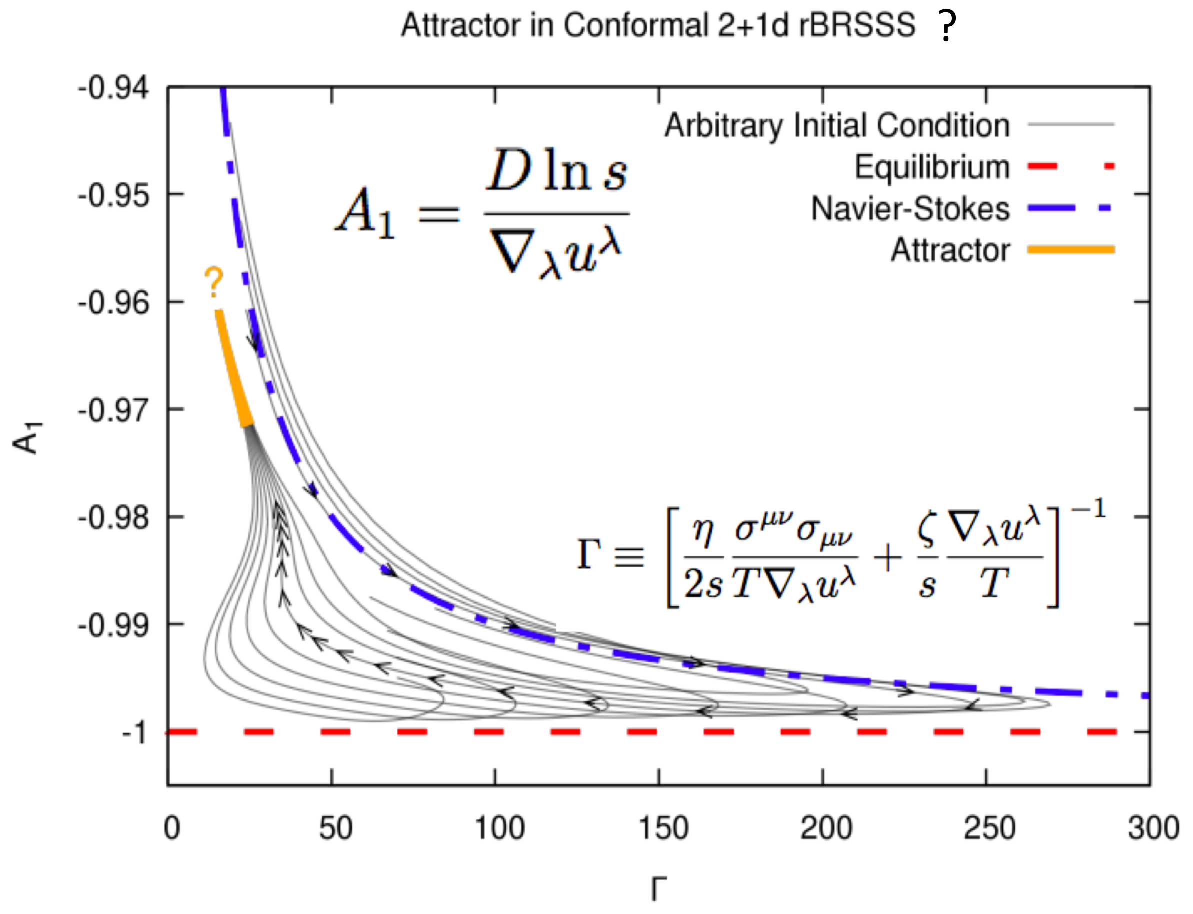

- First hydrodynamic attractor for a gas described by the full Boltzmann equation.
- Israel-Stewart hydro gives a good qualitative description of the large Knudsen regime.
- Gradient series converges, though the system is still rapidly expanding.

# Are there attractors under general flow conditions?

Previous work by P. Romatschke, JHEP (2017)

Navier-Stokes regime

$$A_1 = -1 + \frac{1}{\Gamma}$$

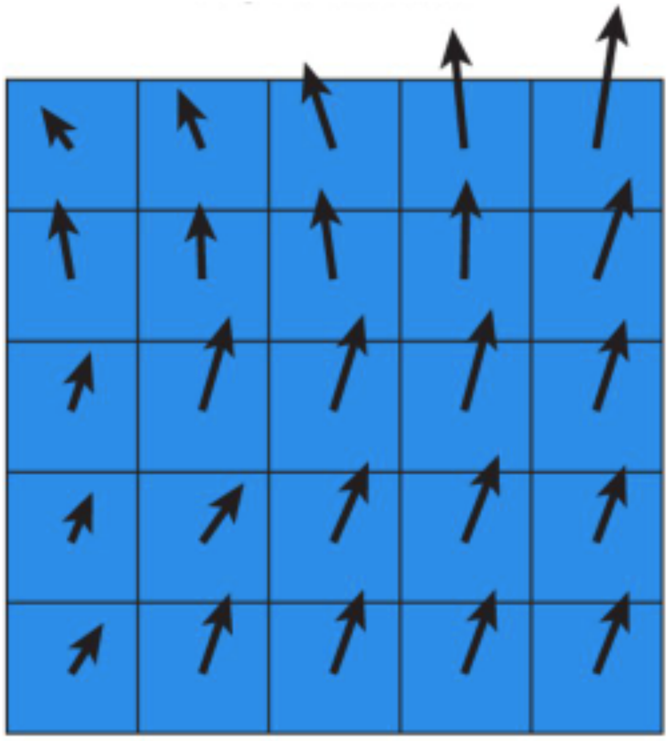



- Noncentral AuAu collision
- Indication of attractor behavior?
- Systematic study of the attractor?

Let us think about this differently.

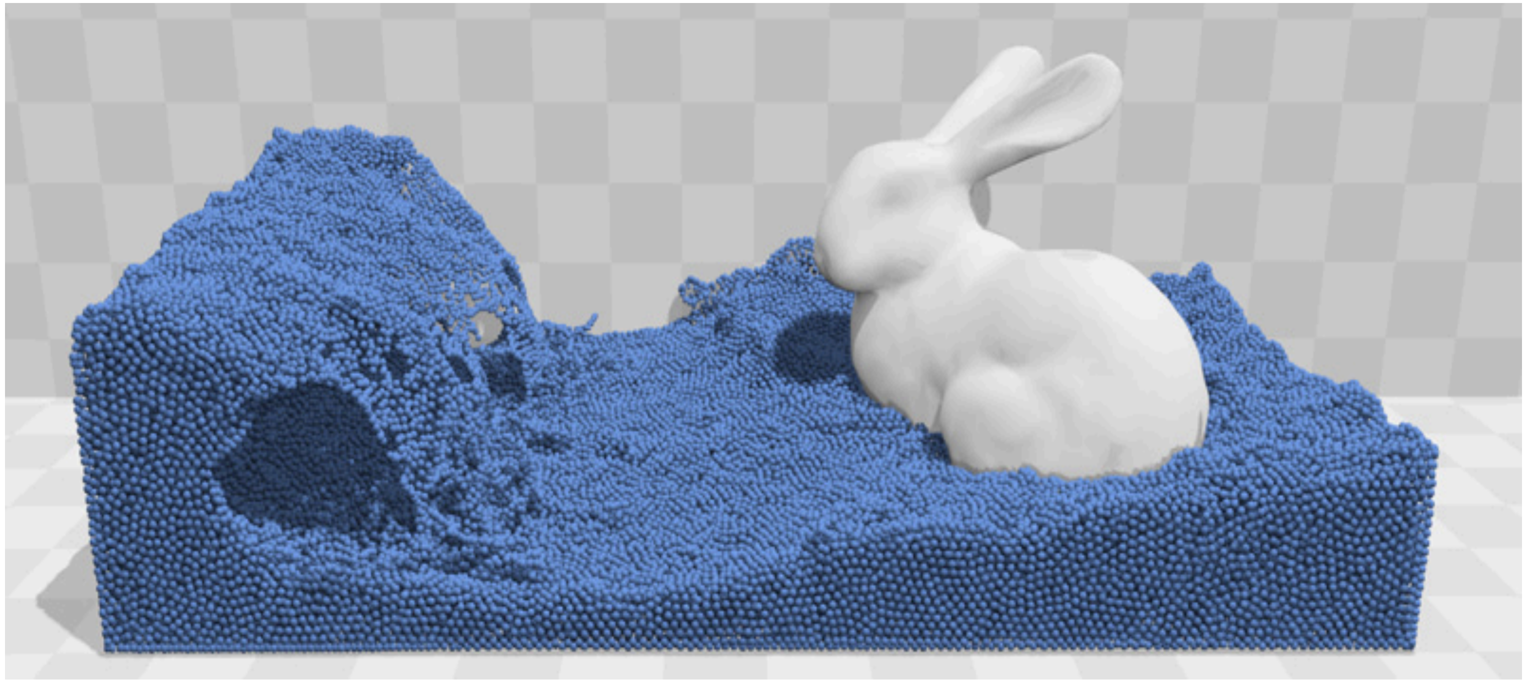
# Different ways to solve the fluid equations

## Eulerian description



- Fixed grid (e.g. MUSIC code)
- Track field configurations

## Lagrangian description



- No fixed grid (e.g. v-USPhydro code)
- Track the trajectories of fluid elements over time

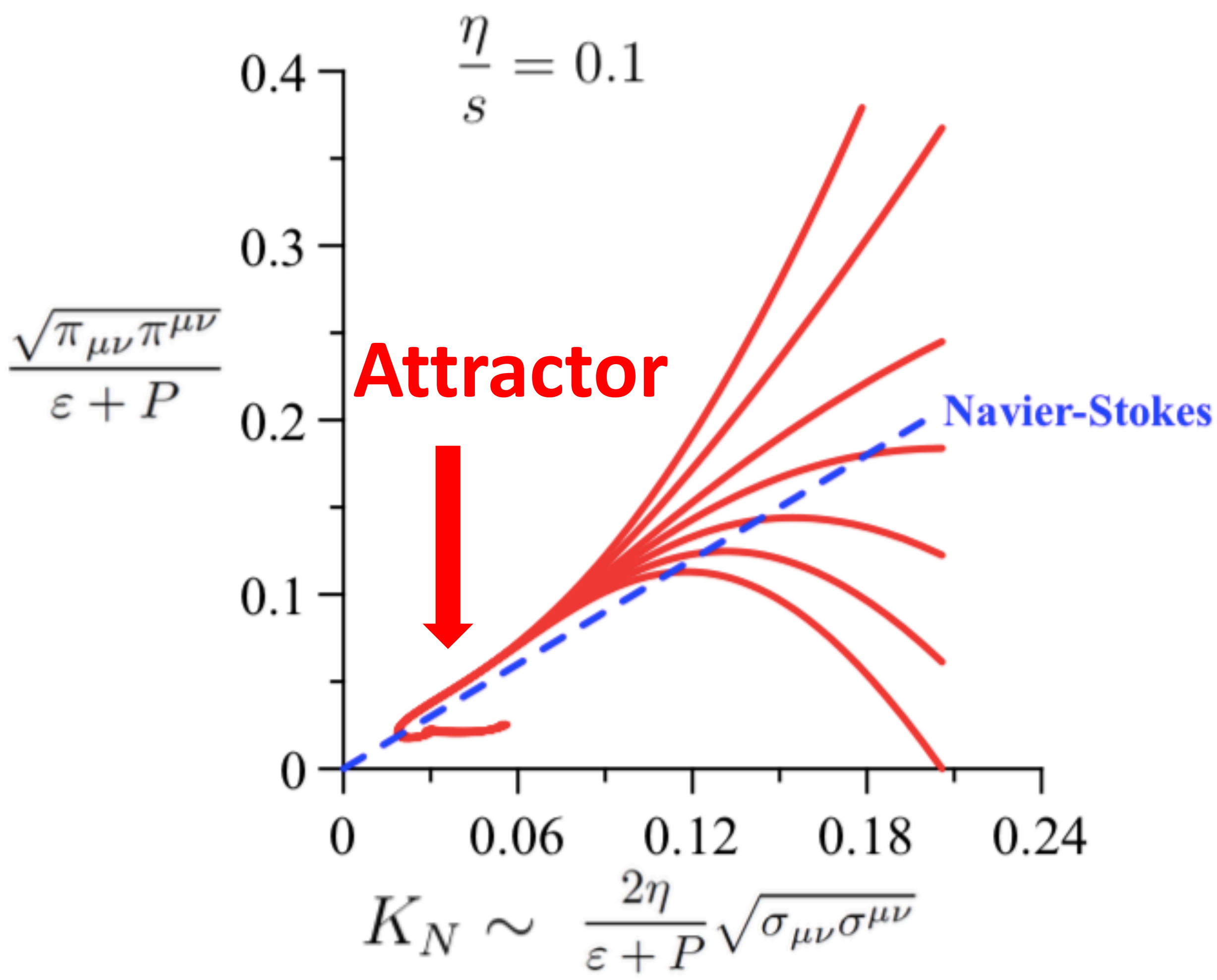
# How does one find attractors in a Lagrangian method?

- Track the values of dissipative tensors, e.g. $\pi^{\mu\nu}$, for each fluid element as a function of time.
- If there's an attractor, each fluid element should reach a universal regime at late times independent of initial conditions.
- However, due to transverse expansion, each fluid element should have its own attractor (in contrast, in Bjorken flow all fluid elements are the same).

# Presence of attractors in general flows in heavy ions

Ex: fluid cell at the center

G. Denicol, JN, to appear

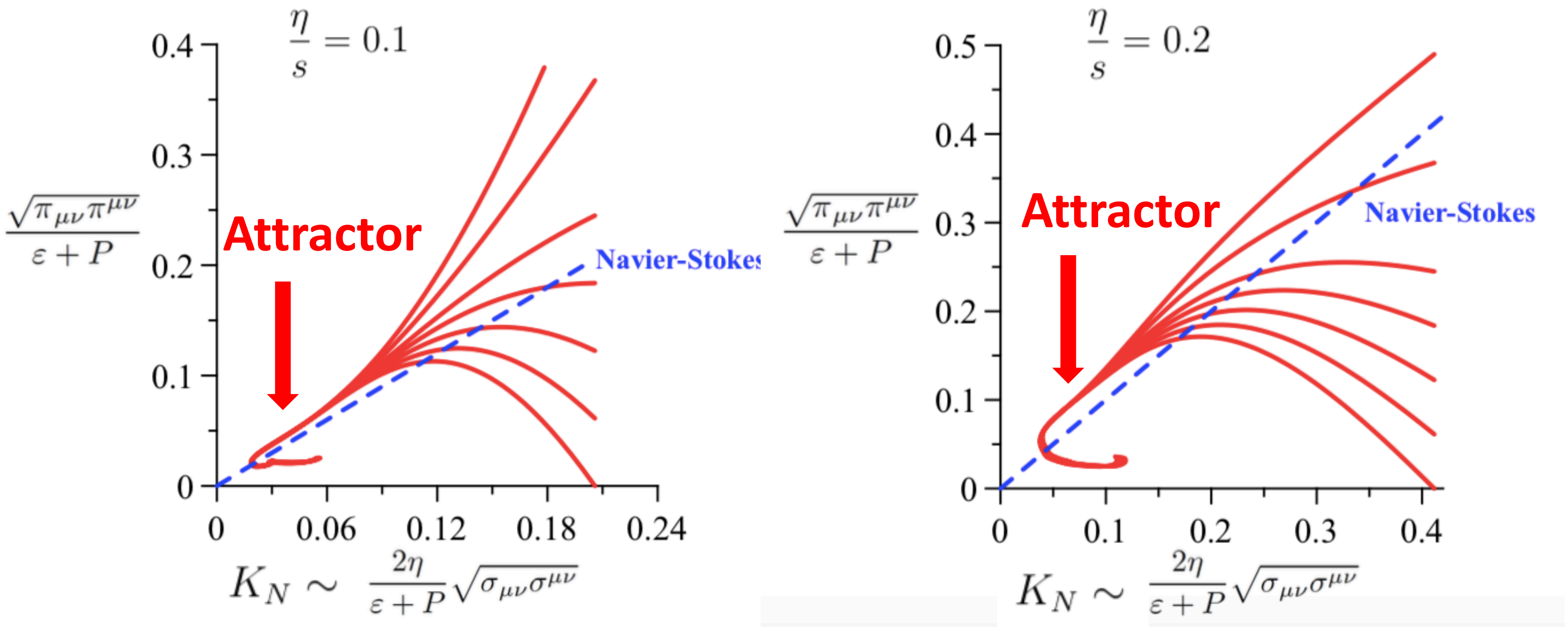



- Smoothed Particle Hydrodynamics (SPH) formalism (e.g. SPheRIO and v-USPhydro)
- Initial results for conformal Israel-Stewart in radially symmetric AA systems.
- Same behavior is found for other initial conditions and dissipative stresses.
- Attractors seem to persist even in inhomogeneous heavy ion collisions.

Fast convergence (roughly 1 fm/c) to the attractor

# Attractors go beyond Navier-Stokes theory

G. Denicol, JN, to appear

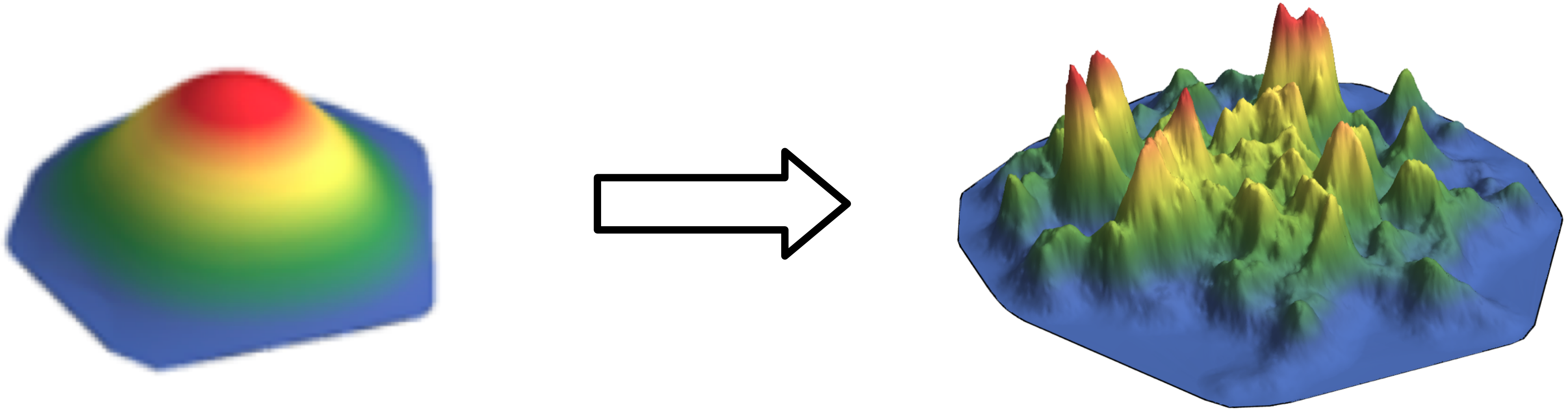


# Far-from-equilibrium hydrodynamics for general flows

Consider the usual gradient expansion ($K_N$ << 1)

$$\boxed{\begin{aligned}\pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + \eta_1\sigma_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \eta_2\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_3\omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_4\theta\sigma^{\mu\nu} + \eta_5\nabla_\perp^{\langle\mu}P\nabla_\perp^{\nu\rangle}P \\ &+ \eta_6\nabla_\perp^{\langle\mu}\nabla_\perp^{\nu\rangle}P + \mathcal{O}\left[K_N^3\right],\end{aligned}}$$

Now increase $K_N$ "adiabatically" towards $K_N$ ~ 1:



What can happen to $\pi^{\mu\nu}$ ?

- Large rearrangement of the series.
- 3rd order terms ~ $\sigma^{\mu\nu}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ may be grouped with $2\eta\sigma^{\mu\nu}$ and etc.
- **Resummation of transport coefficients:**

$$\eta \rightarrow \eta^R = \eta^R(K_N)$$

**Grouping all terms, the symmetries impose the following generalized tensorial expansion**

$$\boxed{\begin{aligned}\pi^{\mu\nu} &= 2\eta^R\sigma^{\mu\nu} + \eta_1^R\sigma_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \eta_2^R\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_3^R\omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5^R\nabla_\perp^{\langle\mu}P\nabla_\perp^{\nu\rangle}P \\ &+ \eta_6^R\nabla_\perp^{\langle\mu}\nabla_\perp^{\nu\rangle}P + \mathcal{O}\left[(K_N^R)^3\right],\end{aligned}}$$

This is far-from-equilibrium hydrodynamics

# Example: Israel-Stewart's equations undergoing general flow

Inverse Reynolds number: $\mathcal{X}^{\mu\nu} = \dfrac{\pi^{\mu\nu}}{\varepsilon + P}$ + using the conservation laws

DNMR, PRD 2012

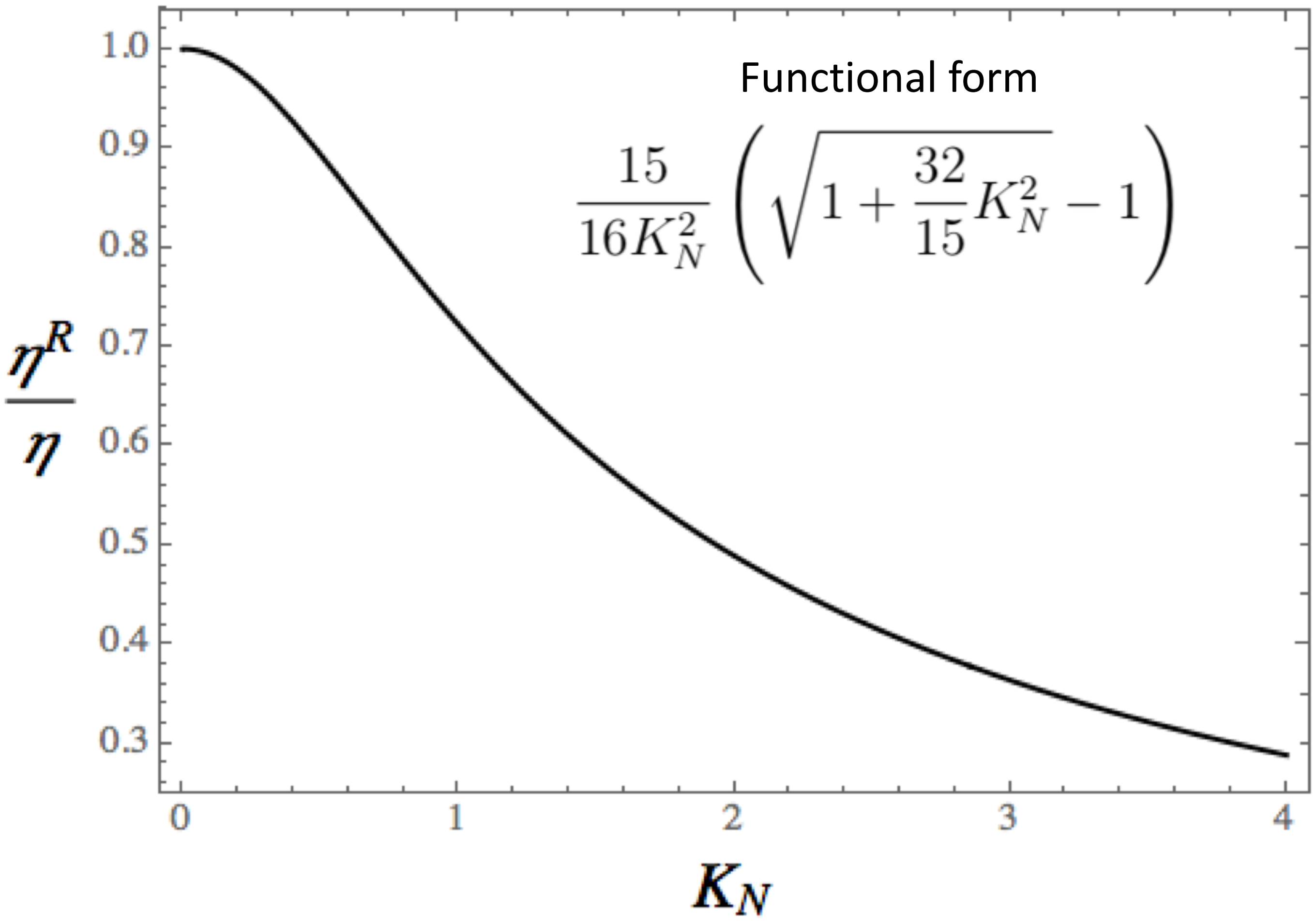
$$\Longrightarrow \quad \tau_R D\mathcal{X}^{\langle\mu\nu\rangle} = -\mathcal{X}^{\mu\nu} + \frac{2}{5}\tau_R\sigma^{\mu\nu} - \frac{4}{3}\tau_R\mathcal{X}^{\mu\nu}\mathcal{X}^{\alpha\beta}\sigma_{\alpha\beta}$$

**Slow-roll series** $\longrightarrow \quad D\chi^{\langle\mu\nu\rangle} \sim 0 \quad \longrightarrow \quad \pi^{\mu\nu} = 2\eta^R(K_N)\sigma^{\mu\nu}$

$$\boxed{\begin{array}{c}\textbf{Resummed shear viscosity} \\ \eta^R(K_N)\end{array}}$$

Resummed higher order coefficients still need to be determined.

# Resummed shear viscosity




First calculation for a general flow !!

- Effective shear viscosity becomes smaller far from equilibrium.
- Entropy production is bounded even if Kn is very large.

# Conclusions & Outlook

- First calculation of the hydro attractor for a relativistic gas described by the full Boltzmann equation (in Bjorken flow).

- Direct evidence of attractor behavior under heavy ion collision conditions (investigate differences between large and small systems!).

- When most fluid cells do not reach their attractor then the results become highly sensitive to initial conditions for full $T_{\mu\nu}$ (pA?).

- New way to understand far-from-equilibrium attractors using a generalized tensorial expansion with resummed transport coefficients.

# ADDITIONAL SLIDES

# Far-from-equilibrium hydrodynamics for general flows

G. Denicol, JN, to appear

Assumption → Dynamics described solely using $\{\varepsilon, u_\mu\}$

**Attractor → Constitutive form for viscous stresses beyond gradient expansion**

$$\boxed{\pi^{\mu\nu} \sim \sum_{n=0}^{\infty} c_n^{(0),\mu\nu} (K_N)^n + c^{(1),\mu\nu} (K_N)^\beta e^{-S/K_N} + \ldots}$$

Trans-series (schematic)

How does this come about?

# How can a pp system behave like a fluid?

Proton at high energies: average shape but with strong color fluctuations

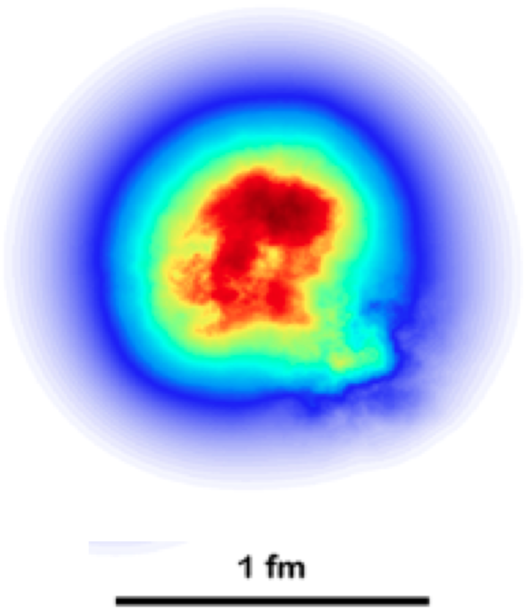



Fig. from Mantysaari, Schenke, PRL (2016)

**A simple uncertainty principle estimate**

$$\Delta x \sim 0.1\,\mathrm{fm}$$

$$\Delta p \sim \frac{1}{\Delta x} \sim 2\,\mathrm{GeV} > \langle p_T \rangle$$

**Quantum correlations should be important !!**

$$\langle \hat{T}^{\mu\nu} \rangle, \langle \hat{T}^{\mu\nu} \hat{T}^{\alpha\beta} \rangle, \ldots$$

- **Opportunity:** Investigate quantum entanglement in a non-Abelian theory.
- Here we should really go beyond the "everything is hydro approach" …

# The analytical hydrodynamic attractor

G. Denicol, JN, PRD (2018)

The equations of motion of Israel-Stewart theory with constant $\tau_R$

$$D\varepsilon + (\varepsilon + P)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0$$

$$(\varepsilon + P)Du^{\mu} - \Delta^{\mu}_{\lambda}\nabla^{\lambda}P + \Delta^{\mu}_{\lambda}\nabla_{\mu}\pi^{\mu\lambda} = 0$$

$$\tau_R \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \delta_{\pi\pi}\,\theta\pi^{\mu\nu} + \tau_{\pi\pi}\,\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\lambda}\sigma^{\beta}_{\lambda} - 2\,\tau_R \Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha}_{\lambda}\omega^{\beta\lambda} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$

+ Bjorken flow

$$\varepsilon = 3P$$

can be FULLY solved analytically

Energy density

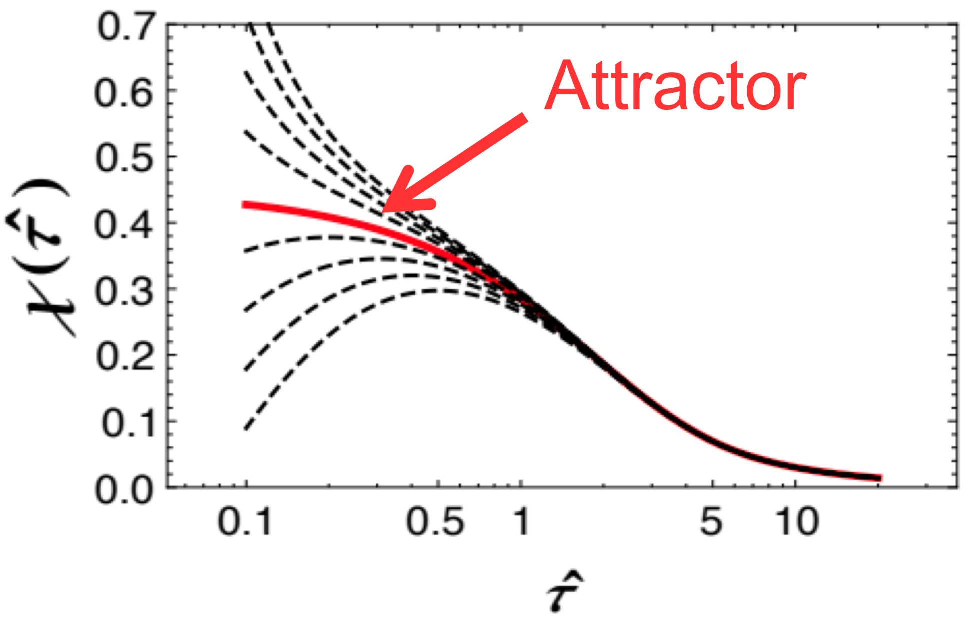
$$\varepsilon(\hat{\tau}) = \varepsilon_0\, e^{-\frac{1}{2}(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{\frac{5}{6}} \left[\frac{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - K_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}}{2}\right)}{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}_0}{2}\right) - K_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}_0}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}_0}{2}\right) + I_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}_0}{2}\right)}\right]$$

# Full solution for shear stress tensor

$$\chi(\hat{\tau}) = \frac{\pi}{\varepsilon + P} = \frac{3\sqrt{a}}{4}\left[\frac{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}\right]$$

**First analytical expression for a hydrodynamic attractor**

$$\chi(\hat{\tau}) \rightarrow \chi_{att}(\hat{\tau}) = \frac{3\sqrt{a}}{4}\left[\frac{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}\right]$$




**Non-perturbative behavior**

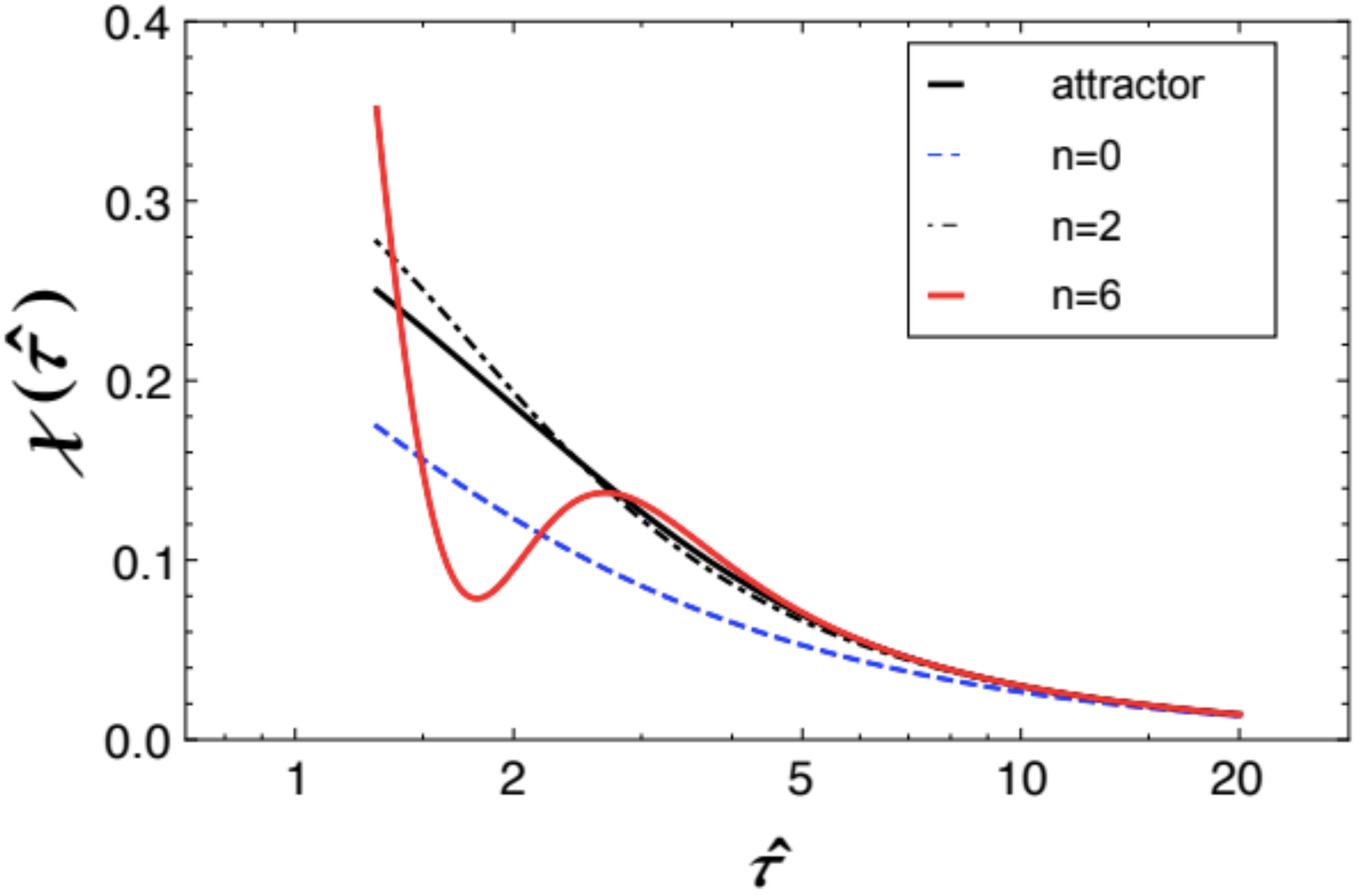
$$\exp\{-1/K_N\}$$

Resummation of gradient expansion

Trans-series

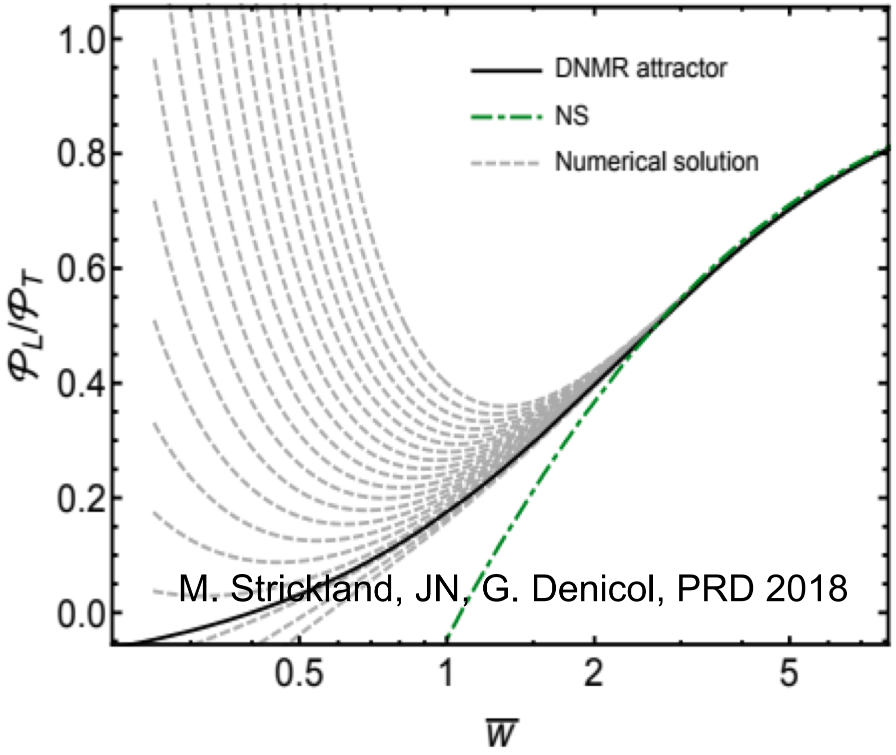
# Divergence of the slow-roll series

G. Denicol, JN, PRD (2018)

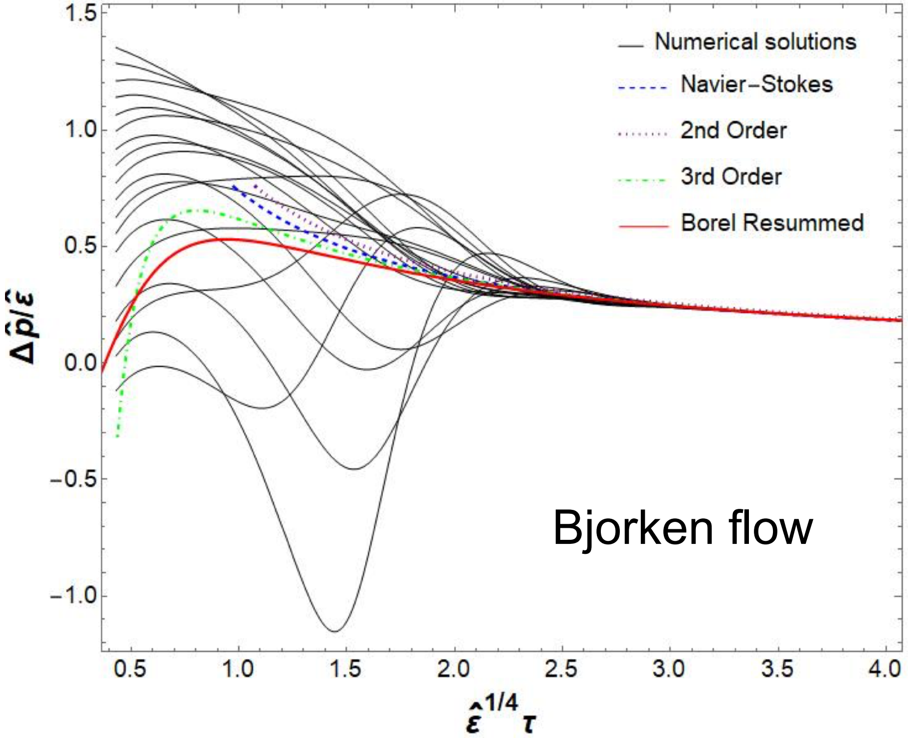


# Far-from-equilibrium hydro – Attractor solutions

Boltzmann → Israel-Stewart equations




N = 4 SYM at strong coupling




- Very different transient behavior at weak vs. strong coupling.

- Presence of far-from-equilibrium attractor in both cases.