Including momentum and stress in a systematic framework for understanding the evolution of a heavy-ion collision

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work in progress with Jefferson de Souza and Jorge Noronha

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Quark Matter 2019
6 November

Introduction

- Heavy-ion collision: complicated, non-linear evolution
- Simple observed relations, e.g.,

$$v_n = \kappa_n \, \varepsilon_n$$

- Until now: only considered energy / entropy in initial state
- Goal: include effects from other components of $T^{\mu\nu}$.
- Motivation: determine relevant aspects, quantify contribution & interplay with eccentricity (important in small systems?)

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1. Reminder: $v_n = \kappa_n \varepsilon_n$ as leading term in systematic expansion
2. Ansatz for adding $T^{\mu\nu}$ contributions
3. Numeric validation

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Cumulant expansion: energy density

- $E\frac{dN}{d^3p}\Big|_{\text{final}} = \frac{N}{2\pi}\sum_{n=-\infty}^{\infty} V_n e^{-in\phi}$
- Assumption 1: $V_n = \mathcal{F}\left(T^{\mu\nu}\big|_{\tau=\tau_0}, j^{\mu}\big|_{\tau=\tau_0}\right)$
  - $T^{\mu\nu}, j^{\mu}$ at some time τ_0 determines final observable
- Assumption 2: Hierarchy of scales
  - Structure at large length scales more important

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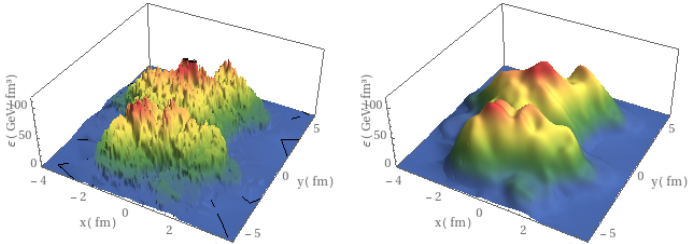

(arXiv:1807.05213)

Cumulant expansion: energy density

Cumulants

- Separate scales with Fourier transform

$$e^{W(\vec{k})} \equiv \int d^2x \; e^{i\vec{k}\cdot\vec{x}} T^{\tau\tau}(x,y)$$

- Isolate large scales with Taylor series around $k = |\vec{k}| = 0$

$$W(\vec{k}) = \sum_{m=0}^{\infty} W_m(\phi_k) k^m = \sum_{n=-\infty}^{\infty} \sum_{m=|n|}^{\infty} W_{n,m} k^m e^{in\phi_k}$$

- System fully characterized by *ordered* set $\{W_{n,m}\}$
- Smaller $m \implies$ larger length scales $\implies$ can truncate at some $m = m_{\max}$.
- $V_n = f(\{W_{n,m}\})$

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- $V_n \simeq f(\{W_{n,m}\}\big|_{m \leq m_{\max}})$

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Cumulant series

- $V_n = f(\{W_{n,m}\})$
- Power series in anisotropic ($n \neq 0$) cumulants:

$$V_n \simeq V_n^{(\text{est})} = \text{linear} + \text{quadratic} + \text{cubic} + \ldots$$

- Double expansion — small m; lower powers of anisotropic cumulants W
- (No guidance for which type of correction more important)

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$$V_n \simeq V_n^{(\text{est})} = \kappa_{n,n} W_{n,n} + \kappa_{n,n+2} W_{n,n+2} + \kappa_{n,n+4} W_{n,n+4} + \text{quadratic} + \text{cubic} + \ldots$$

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Cumulant expansion: energy density

Normalization

- V_n dimensionless, $W_{n,m}$ dimensionful
- $\implies$ compare $W_{n,m}$ to relevant scale(s) to make dimensionless ratios
- E.g., lowest isotropic cumulant $W_{0,2} = \langle r^2 \rangle_\epsilon - \left| \langle r e^{i\phi} \rangle_\epsilon \right|^2$
- $\mathcal{E}_{n,m} = -\dfrac{W_{n,m}}{(W_{0,2})^{m/2}}$
- E.g., $\mathcal{E}_{2,2} \equiv \mathcal{E}_2 = -\dfrac{\langle r^2 e^{i2\phi} \rangle_\epsilon - \langle r e^{i\phi} \rangle_\epsilon^2}{\langle r^2 \rangle_\epsilon - \left| \langle r e^{i\phi} \rangle_\epsilon \right|^2} \overset{\text{centered}}{=} -\dfrac{\langle r^2 e^{i2\phi} \rangle_\epsilon}{\langle r^2 \rangle_\epsilon}$
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Cumulant expansion: energy density

Eccentricity scaling

- Lowest order estimators:

$$V_2^{(\mathrm{est})} = \kappa_2 \mathcal{E}_{2,2}$$
$$V_3^{(\mathrm{est})} = \kappa_3 \mathcal{E}_{3,3}$$

- Can be systematically improved (see talk by Mauricio Hippert)

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Cumulant expansion: energy density

Cumulant expansion properties

1. Symmetries: well-defined rotation modes n
2. Symmetries: Translation-invariant
3. Ordered in length scales

- Note: for $T^{\tau\tau} \to 0 \implies \mathcal{E}_{n,m} \not\to 0$

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Anzatz: Including $T^{\mu\nu}$ components

$$\rho(\vec{x}) = T^{\tau\tau}(\vec{x})$$

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$$V_n \simeq \kappa_n \mathcal{E}_{n,n} + \ldots$$

- $(\alpha, \beta) =$ transport coefficients

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Additional justification

Thought experiment

- Consider a system with only $T^{\tau\tau}$ at some time τ_0
- Cumulant expansion of $\rho(\vec{x}) = T^{\tau\tau}(\tau_0)$ predicts final V_n
- At a slightly different time $\tau = \tau_0 + \delta_\tau$:

$$T^{\tau\tau}(\tau) = T^{\tau\tau}(\tau_0) + \delta\tau\partial_\tau T^{\tau\tau}|_{\tau_0} + \frac{\delta\tau^2}{2}\partial_\tau^2 T^{\tau\tau}|_{\tau_0} + O(\delta\tau^3)$$

- Other components are generated via conservation of energy/momentum

$$\partial_\tau T^{\tau\tau} = -\partial_i T^{\tau i}$$
$$\partial_\tau^2 T^{\tau\tau} = -\partial_i\partial_\tau T^{\tau i} = \partial_i\partial_j T^{ij}$$

- Final V_n still determined by $\rho(\vec{x}) = T^{\tau\tau}(\tau_0) \simeq T^{\tau\tau}(\tau) + \delta\tau\partial_i T^{\tau i}(\tau_0) - \frac{\delta\tau^2}{2}\partial_i\partial_j T^{ij}(\tau_0)$

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- Final V_n still determined by $\rho(\vec{x}) = T^{\tau\tau}(\tau_0) \simeq T^{\tau\tau}(\tau) + \delta\tau\partial_i T^{\tau i}(\tau_0) - \frac{\delta\tau^2}{2}\partial_i\partial_j T^{ij}(\tau_0)$

Additional justification

Thought experiment

- Consider a system with only $T^{\tau\tau}$ at some time τ_0
- Cumulant expansion of $\rho(\vec{x}) = T^{\tau\tau}(\tau_0)$ predicts final V_n
- At a slightly different time $\tau = \tau_0 + \delta_\tau$:

$$T^{\tau\tau}(\tau) = T^{\tau\tau}(\tau_0) + \delta\tau\partial_\tau T^{\tau\tau}|_{\tau_0} + \frac{\delta\tau^2}{2}\partial_\tau^2 T^{\tau\tau}|_{\tau_0} + O(\delta\tau^3)$$

- Other components are generated via conservation of energy/momentum

$$\partial_\tau T^{\tau\tau} = -\partial_i T^{\tau i}$$
$$\partial_\tau^2 T^{\tau\tau} = -\partial_i\partial_\tau T^{\tau i} = \partial_i\partial_j T^{ij}$$

- Final V_n still determined by $\rho(\vec{x}) = T^{\tau\tau}(\vec{x}) + \alpha\partial_i T^{\tau i}(\vec{x}) - \beta\partial_i\partial_j T^{ij}(\vec{x})$

Generalized estimators

- With this ansatz, we find,

$$V_2^{(\mathrm{est})} = \kappa_2 \mathcal{E}_{2,2}(\alpha,\beta) = -\kappa_2 \frac{\langle r^2 e^{i2\phi}\rangle_\epsilon - 2\alpha\langle re^{i\phi}\rangle_u - 4\beta\langle 1\rangle_c - \left(\langle re^{i\phi}\rangle_\epsilon - \alpha\langle 1\rangle_u\right)^2}{\langle r^2\rangle_\epsilon - |\langle re^{i\phi}\rangle_\epsilon|^2}$$

$$V_3^{(\mathrm{est})} = \kappa_3 \mathcal{E}_{3,3}(\alpha,\beta) = -\kappa_3 \frac{\langle r^3 e^{i3\phi}\rangle_\epsilon - 3\alpha\langle r^2 e^{i2\phi}\rangle_u - 12\beta\langle re^{i\phi}\rangle_c - 2\left(\langle re^{i\phi}\rangle_\epsilon - \alpha\langle 1\rangle_u\right)^3}{\left(\langle r^2\rangle_\epsilon - |\langle re^{i\phi}\rangle_\epsilon|^2\right)^{\frac{3}{2}}} + \kappa_3 \frac{\left(\langle re^{i\phi}\rangle_\epsilon - \alpha\langle 1\rangle_u\right)\left(3\langle r^2 e^{i2\phi}\rangle_\epsilon - 6\alpha\langle re^{i\phi}\rangle_u - 12\beta\langle 1\rangle_c\right)}{\left(\langle r^2\rangle_\epsilon - |\langle re^{i\phi}\rangle_\epsilon|^2\right)^{\frac{3}{2}}}$$

Generalized estimators

- With this ansatz, we find, (in coordinate system with $W_{1,1}(\alpha,\beta)=0$)

$$V_2^{(\mathrm{est})} = \kappa_2 \mathcal{E}_{2,2}(\alpha,\beta) = -\kappa_2 \frac{\langle r^2 e^{i2\phi}\rangle_\epsilon - 2\alpha\langle re^{i\phi}\rangle_u - 4\beta\langle 1\rangle_c - \left(\langle re^{i\phi}\rangle_\epsilon - \alpha\langle 1\rangle_u\right)^2}{\langle r^2\rangle_\epsilon - |\langle re^{i\phi}\rangle_\epsilon|^2}$$

$$V_3^{(\mathrm{est})} = \kappa_3 \mathcal{E}_{3,3}(\alpha,\beta) = -\kappa_3 \frac{\langle r^3 e^{i3\phi}\rangle_\epsilon - 3\alpha\langle r^2 e^{i2\phi}\rangle_u - 12\beta\langle re^{i\phi}\rangle_c - 2\left(\langle re^{i\phi}\rangle_\epsilon - \alpha\langle 1\rangle_u\right)^3}{\left(\langle r^2\rangle_\epsilon - |\langle re^{i\phi}\rangle_\epsilon|^2\right)^{\frac{3}{2}}}$$

$$+ \kappa_3 \frac{\left(\langle re^{i\phi}\rangle_\epsilon - \alpha\langle 1\rangle_u\right)\left(3\langle r^2 e^{i2\phi}\rangle_\epsilon - 6\alpha\langle re^{i\phi}\rangle_u - 12\beta\langle 1\rangle_c\right)}{\left(\langle r^2\rangle_\epsilon - |\langle re^{i\phi}\rangle_\epsilon|^2\right)^{\frac{3}{2}}}$$

Generalized estimators

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$$V_2^{(\mathrm{est})} = \kappa_2 \mathcal{E}_{2,2}(\alpha,\beta) = -\kappa_2 \frac{\langle r^2 e^{i2\phi}\rangle_\epsilon - 2\alpha\langle re^{i\phi}\rangle_u - 4\beta\langle 1\rangle_c}{\langle r^2\rangle_\epsilon - |\langle re^{i\phi}\rangle_\epsilon|^2}$$

$$V_3^{(\mathrm{est})} = \kappa_3 \mathcal{E}_{3,3}(\alpha,\beta) = -\kappa_3 \frac{\left\langle r^3 e^{i3\phi}\right\rangle_\epsilon - 3\alpha\left\langle r^2 e^{i2\phi}\right\rangle_u - 12\beta\left\langle r^{i\phi}\right\rangle_c}{\left(\left\langle r^2\right\rangle_\epsilon - \left|\left\langle re^{i\phi}\right\rangle_\epsilon\right|^2\right)^{\frac{3}{2}}}$$

$$U(\vec{x}) \equiv T^{\tau x} + iT^{\tau y}$$

$$C(\vec{x}) \equiv \frac{1}{2}\left(T^{xx} - T^{yy}\right) + iT^{xy}$$

$$\langle \ldots \rangle_\epsilon = \frac{\int d^2x \ldots T^{\tau\tau}(\vec{x})}{\int d^2x T^{\tau\tau}(\vec{x})}$$

$$\langle \ldots \rangle_u = \frac{\int d^2x \ldots U(\vec{x})}{\int d^2x T^{\tau\tau}(\vec{x})}$$

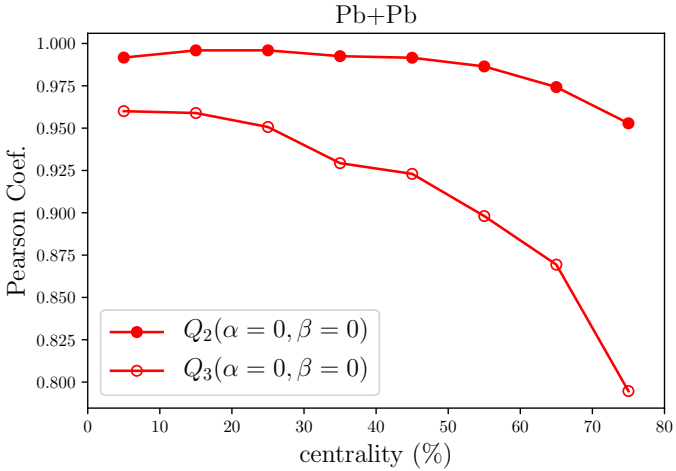
$$\langle \ldots \rangle_c = \frac{\int d^2x \ldots C(\vec{x})}{\int d^2x T^{\tau\tau}(\vec{x})}$$

Numerical validation

Does it work?

- Check with numerical simulations
  1. Adjustable toy model (see backup slides)
  2. Realistic simulations — IP-Glasma + MUSIC + UrQMD
- Quantify the quality of estimator statistically
- $Q_n =$ linear correlation between V_n and $V_n^{(\mathrm{est})}$
  - $Q_n = 1 \implies$ perfect estimator
  - $Q_n = 0 \implies$ no (linear) correlation

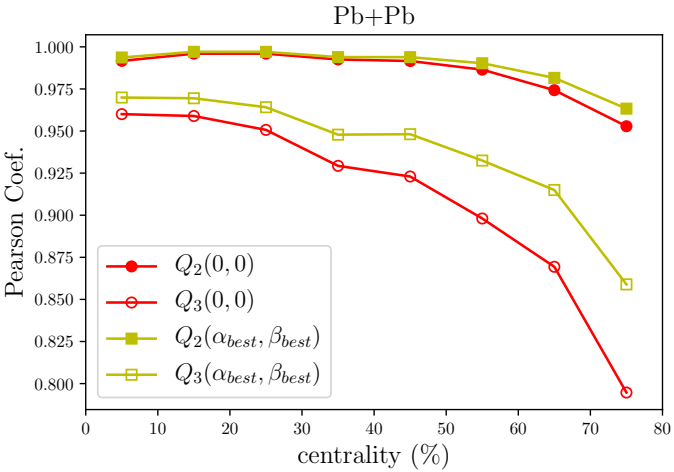
Numerical validation



- Traditional $\mathcal{E}_n$ a good estimator $\implies$ $T^{\mu\nu}$ contributions small*

  (*Note: Pb-Pb, zero initial viscous tensor)
- Accounting for $T^{\mu\nu}$ improves estimator
- (α, β) independent of harmonic, small dependence on centrality

Numerical validation



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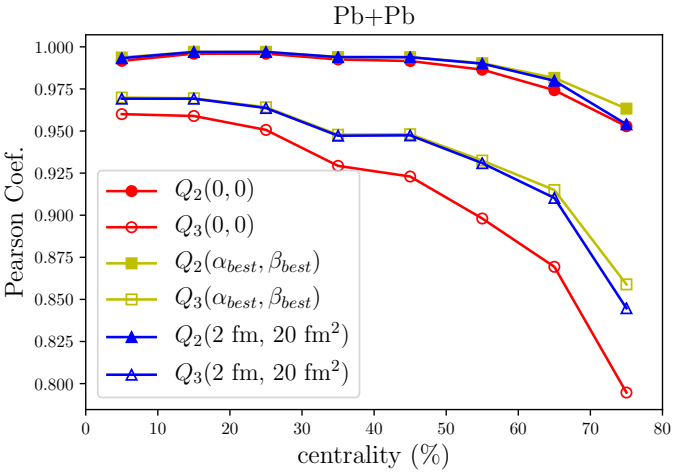

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  (*Note: Pb-Pb, zero initial viscous tensor)
- Accounting for $T^{\mu\nu}$ improves estimator
- (α, β) independent of harmonic, small dependence on centrality

Summary

- Cumulant expansion
  - Assuming hierarchy of scales, can make direct connection between final and initial state, isolate effects of system evolution
  - Can be systematically improved — higher cumulants (smaller length scales) and non-linear terms can be relevant.
- Can generalize to include effects of initial $T^{\mu\nu}$
  - Determine the relevant aspects
  - Interplay between spatial and momentum components
- Hybrid hydrodynamic simulations give validation to proposed framework.

Acknowledgements

FAPESP
SÃO PAULO RESEARCH FOUNDATION

CNPq
Conselho Nacional de Desenvolvimento Científico e Tecnológico

CAPES

Grants:
16/24029-6, 17/05685-2, 18/24720-6

Extra Slides

Toy model tests

$$T^{00}(\vec{x}) = Ae^{-\frac{r^2}{2\sigma^2}(1+a_n \cos n\phi)}$$

$$U(\vec{x}) = T^{0x} + iT^{0y} = |U|e^{\phi_u}$$

$$|U| = rBe^{-\frac{r^2}{2\rho^2}(1-b_n \cos n\phi)}$$

$$\phi_u = \phi - c_n \sin n\phi$$

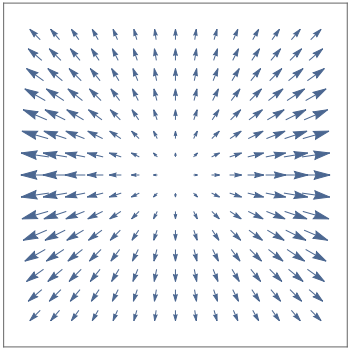
$$C(\vec{x}) = \frac{1}{2}(T^{xx} - T^{yy}) + iT^{xy} = |C|e^{i2\phi_c}$$

$$|C| = r^2 Pe^{-\frac{r^2}{2\rho^2}(1-p_n \cos n\phi)}$$

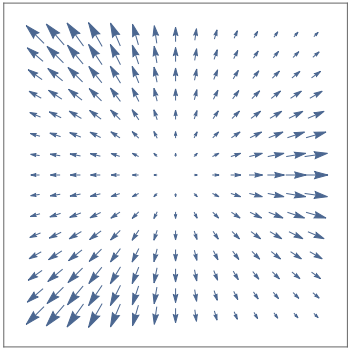
$$\phi_c = \phi - q_n \sin n\phi$$

Toy model momentum density

$$|U| = rBe^{-\frac{r^2}{2\rho^2}(1-b_n \cos n\phi)}$$



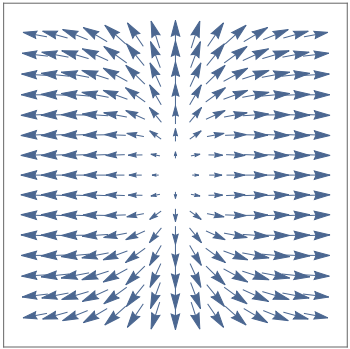
$b_2 \neq 0$



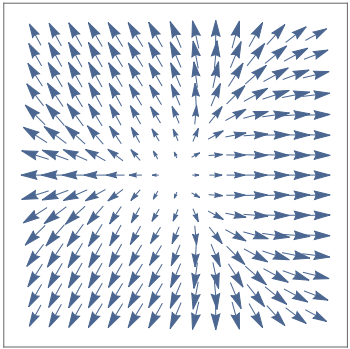
$b_3 \neq 0$

Toy model momentum density

$$\phi_u = \phi - c_n \sin n\phi$$

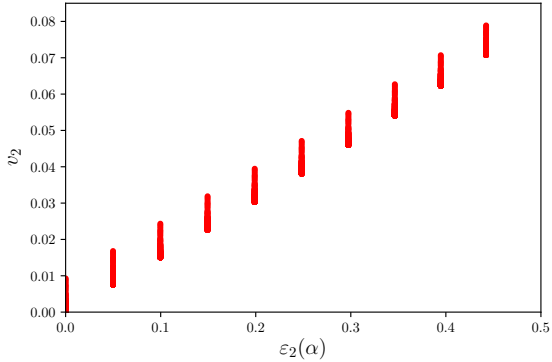


$c_2 \neq 0$



$c_3 \neq 0$

Toy Model Testing



$$v_2 = \kappa_2 \varepsilon_2(\alpha = 0.0 \text{ fm}, \beta = 0.0 \text{ fm}^2)$$

$$T^{00} = Ae^{-\frac{r^2}{2\sigma^2}(1+a_2 \cos 2\phi)}$$

$$|U| = rBe^{-\frac{r^2}{2\rho^2}(1-b_2 \cos 2\phi)}$$

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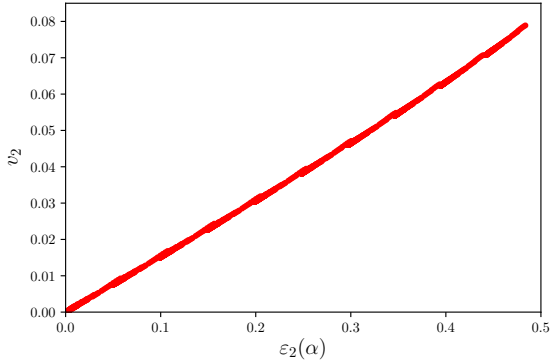
$$\phi_c = \phi - q_2 \sin 2\phi$$

$B\ (fm^{-5})$	a_2	b_2	c_2
$\{0.0 - 4.0\}$	$\{0.0 - 0.45\}$	$\{0.0 - 0.6\}$	$\{0.0 - 0.4\}$

$P\ (fm^{-6})$	p_2	q_2
0	0	0

Toy Model Testing



$$v_2 = \kappa_2 \varepsilon_2(\alpha = 9.0 \text{ fm}, \beta = 0.0 \text{ fm}^2)$$

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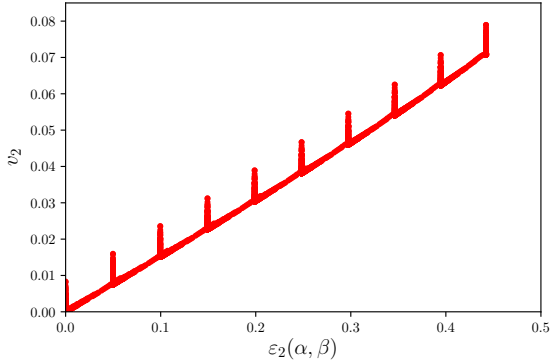

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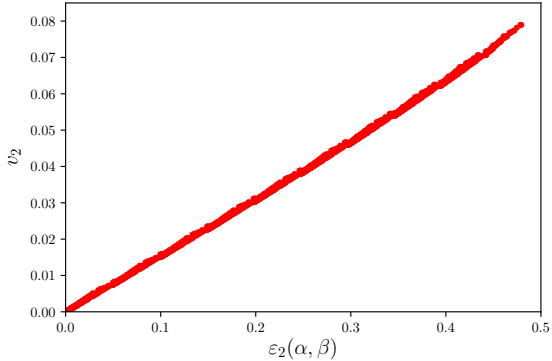
Toy Model Testing



$$v_2 = \kappa_2 \varepsilon_2(\alpha = 9.0 \text{ fm}, \beta = 5.0 \text{ fm}^2)$$

$$T^{00} = Ae^{-\frac{r^2}{2\sigma^2}(1+a_2 \cos 2\phi)}$$

$$|U| = rBe^{-\frac{r^2}{2\rho^2}(1-b_2 \cos 2\phi)}$$

$$\phi_u = \phi - c_2 \sin 2\phi$$

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$\{0.0 - 4.0\}$	$\{0.0 - 0.45\}$	$\{0.0 - 0.6\}$	$\{0.0 - 0.4\}$

$P\ (fm^{-6})$	p_2	q_2
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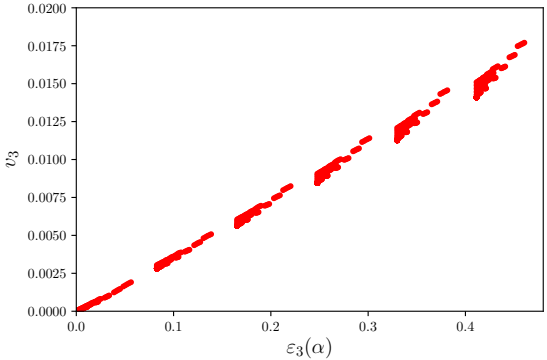
Toy Model Testing



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$$\phi_u = \phi - c_3 \sin 3\phi$$

$$|C| = r^2 Pe^{-\frac{r^2}{2\rho^2}(1-p_3 \cos 3\phi)}$$

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$\{0.0 - 4.0\}$	$\{0.0 - 0.45\}$	$\{0.0 - 0.6\}$	$\{0.0 - 0.4\}$

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