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Hydrodynamics far from equilibrium: a concrete example

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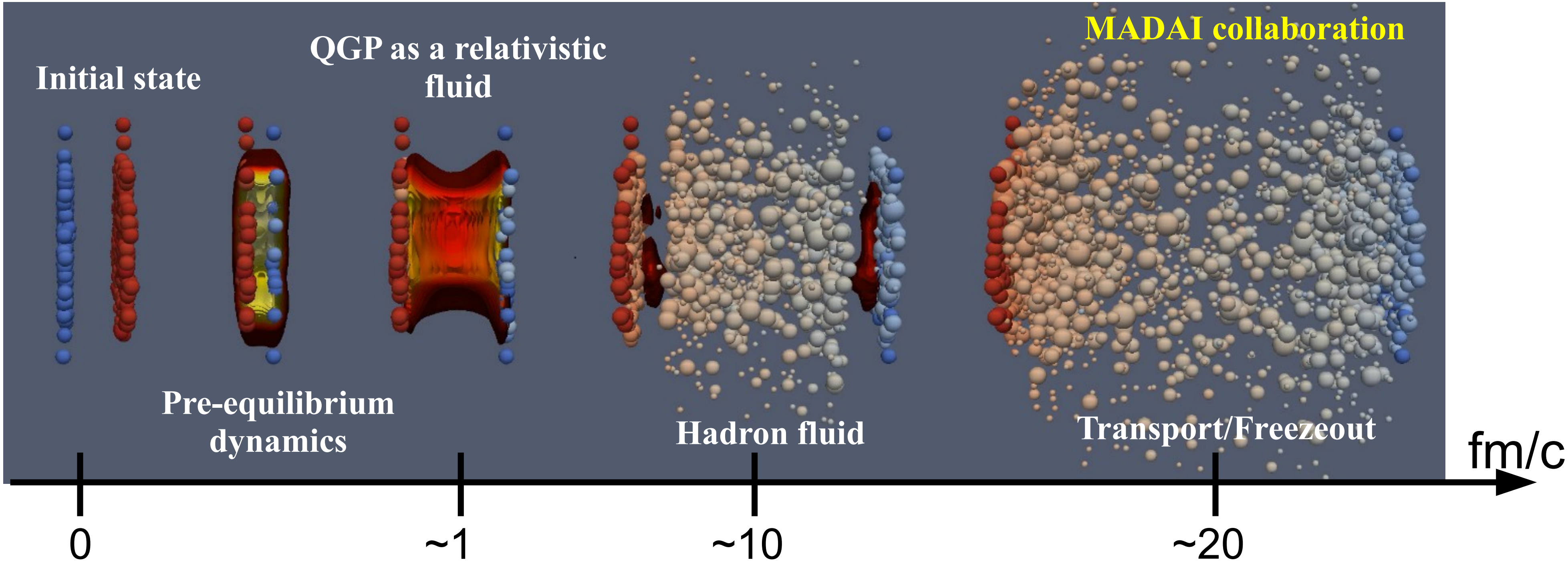
What you will see:

- ✔ Motivation: why fluid-dynamical descriptions work?
- ✔ Derivation of fluid dynamics using method of moments
- ✔ Can we have hydrodynamic behavior far from equilibrium?

Empirical: fluid-dynamical models of heavy ion collisions work well at RHIC and LHC energies



Main assumption: fluid dynamics is applied on *very small* time scales ~1 fm

Does this make sense?

Validity of fluid dynamics *traditionally* associated with:

➛ *"proximity" to (local) equilibrium*

➛ *"small" gradients* $K_N \sim \frac{\ell}{L} \ll 1$

Do these things occur early in Heavy Ion Collisions?

No reason to believe that they do.

Then why does hydro work?
What assumptions are really required?

We can study this problem in Kinetic theory

$$\boxed{k^\mu \partial_\mu f_{\mathbf{k}} = C[f]}$$

Boltzmann eq.

?????

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \cdots$$
$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots$$

2^{nd}- order hydro

In particular, we can use Israel-Stewart´s approach

Israel-Stewart theory: basic ideas

Israel-Stewart theory: *14-moment approximation*

$$\boxed{f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}}$$

equilibrium *non-equilibrium*

1 – ***Truncated*** Taylor
series in momentum : $\phi_{\mathbf{k}} = \varepsilon + \varepsilon_{\mu} k^{\mu} + \varepsilon_{\mu\nu} k^{\mu} k^{\nu}$

- degrees of freedom *reduced* by the ***explicit truncation*** of expansion
- *14* fields left

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Israel-Stewart theory: *14-moment approximation*

$$\boxed{f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}} \qquad \phi_{\mathbf{k}} = \varepsilon + \varepsilon_{\mu}k^{\mu} + \varepsilon_{\mu\nu}k^{\mu}k^{\nu}$$

equilibrium *non-equilibrium*

2 – Expansion coefficients mapped to conserved currents via ***matching conditions***

5 eqs.

$$\begin{aligned} u_{\mu}N^{\mu} &= n_0 \\ u_{\mu}T^{\mu\nu} &= \varepsilon_0 u^{\nu} \end{aligned}$$

Definition of equilibrium

9 eqs.

$$\begin{aligned} \pi^{\mu\nu} &= \Delta^{\mu\nu}_{\alpha\beta}T^{\alpha\beta} \\ n^{\mu} &= \Delta^{\mu}_{\alpha}N^{\alpha} \\ \Pi &= -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} \end{aligned}$$

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Israel-Stewart theory: *14-moment approximation*

$$\boxed{f_{\mathbf{k}} = \underbrace{f_{0\mathbf{k}}}_{\text{equilibrium}} + \underbrace{f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}}_{\text{non-equilibrium}}} \qquad \phi_{\mathbf{k}} = \varepsilon + \varepsilon_{\mu} k^{\mu} + \varepsilon_{\mu\nu} k^{\mu} k^{\nu}$$

2 – <u>Expansion coefficients mapped to conserved currents via ***matching conditions***</u>

Shear only: $$f_{\mathbf{k}} = f_{0\mathbf{k}} + \frac{1}{2I_{42}} \pi^{\mu\nu} k_{\mu} k_{\nu}$$

$$I_{42} = \frac{1}{15} \int \frac{d^3k}{(2\pi)^3 k^0} |\mathbf{k}|^4 f_{0\mathbf{k}} \xrightarrow[\textit{classical limit}]{} (\varepsilon+P)T^2$$

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Israel-Stewart theory: *14-moment approximation*

$$\boxed{f_{\mathbf{k}} = \underbrace{f_{0\mathbf{k}}}_{\text{equilibrium}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\underbrace{\phi_{\mathbf{k}}}_{\text{non-equilibrium}}} \qquad \phi_{\mathbf{k}} = \varepsilon + \varepsilon_{\mu}k^{\mu} + \varepsilon_{\mu\nu}k^{\mu}k^{\nu}$$

3 – <u>Equations of motion taken from the ***second moment*** of the Boltzmann equation</u>

$$\Delta^{\lambda\rho}_{\mu\nu}\left(\partial_{\alpha}\int_{K} k^{\alpha}k^{\mu}k^{\nu} f_{\mathbf{k}} = \int_{K} C[f]\, k^{\mu}k^{\nu}\right) \longleftrightarrow \text{shear}$$

$$u_{\nu}\Delta^{\lambda}_{\mu}\left(\partial_{\alpha}\int_{K} k^{\alpha}k^{\mu}k^{\nu} f_{\mathbf{k}} = \int_{K} C[f]\, k^{\mu}k^{\nu}\right) \longleftrightarrow \text{diffusion}$$

$$u_{\mu}u_{\nu}\left(\partial_{\alpha}\int_{K} k^{\alpha}k^{\mu}k^{\nu} f_{\mathbf{k}} = \int_{K} C[f]\, k^{\mu}k^{\nu}\right) \longleftrightarrow \text{bulk}$$

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Final Equations of motion

GSD, Niemi, Molnar, Rischke, PRD 85, 114047 (2012)

$$\begin{aligned}
\dot{\Pi} &= -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \ell_{\Pi n} \partial \cdot n - \tau_{\Pi n} n \cdot \dot{u} - \delta_{\Pi\Pi} \Pi \theta \\
&\quad - \lambda_{\Pi n} n \cdot \nabla \alpha_0 + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu} \ , \qquad (20) \\
\dot{n}^{\langle\mu\rangle} &= -\frac{n^\mu}{\tau_n} + \beta_n \nabla^\mu \alpha_0 - n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta - \ell_{n\Pi} \nabla^\mu \Pi \\
&\quad + \ell_{n\pi} \Delta^{\mu\nu} \partial_\lambda \pi_\nu^\lambda + \tau_{n\Pi} \Pi \dot{u}^\mu - \tau_{n\pi} \pi_\nu^\mu \dot{u}^\nu \\
&\quad - \lambda_{nn} n^\nu \sigma_\nu^\mu + \lambda_{n\Pi} \Pi \nabla^\mu \alpha_0 - \lambda_{n\pi} \pi^{\mu\nu} \nabla_\nu \alpha_0 \ , \qquad (21) \\
\dot{\pi}^{\langle\mu\nu\rangle} &= -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\alpha^{\langle\mu} \omega^{\nu\rangle\alpha} - \tau_{\pi n} n^{\langle\mu} \dot{u}^{\nu\rangle} \\
&\quad + \ell_{\pi n} \nabla^{\langle\mu} n^{\nu\rangle} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi_\alpha^{\langle\mu} \sigma^{\nu\rangle\alpha} \\
&\quad + \lambda_{\pi n} n^{\langle\mu} \nabla^{\nu\rangle} \alpha_0 + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} \ . \qquad (22)
\end{aligned}$$

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Main point of this talk

- *All previous steps can be applied* ***assuming the form:*** $f_{0\mathbf{k}}\left(\lambda, u_\mu k^\mu / \Lambda\right)$

scalars

$$\boxed{f_{\mathbf{k}} \quad = \quad f_{0\mathbf{k}} + \delta f_{\mathbf{k}}}$$

isotropic, non-equilibrium

correction, anisotropic

- This derivation does *not* require proximity to equilibrium

14-moment approx.: shear term only

$$\boxed{f_{\mathbf{k}} = f_{0\mathbf{k}} + \frac{1}{2I_{42}}\pi^{\mu\nu}k_\mu k_\nu}$$

$$I_{42} = \frac{1}{15}\int \frac{d^3k}{(2\pi)^3\, k^0}\left|\mathbf{k}\right|^4 f_{0\mathbf{k}}$$

Matching conditions

$$f_{\mathbf{k}} = f_{0\mathbf{k}}\left(\lambda, u_\mu k^\mu/\Lambda\right) + \delta f_{\mathbf{k}}$$

5 parameters – can be associated with velocity, temperature and chemical potential

$$\left\{\begin{aligned} \Lambda &= \Lambda\left(T,\mu\right) \\ \lambda &= \lambda\left(T,\mu\right) \end{aligned}\right.$$

Equations of motion: *<u>ultrarelativistic</u>* gas of *hard spheres*

We recover the usual equation for the shear stress:

$$\boxed{\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}}\sigma\pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - 2\sigma_\lambda^{\langle\mu}\pi^{\nu\rangle\lambda} - \frac{4}{3}\pi^{\mu\nu}\theta}$$

Transport coefficients:

functional dependence on $f_{0\mathbf{k}}$

$$\frac{1}{\tau_\pi} = \left(1 + 4\frac{P_0 I_{40}}{n_0 I_{50}}\right)\frac{1}{3\lambda_{\mathrm{mfp}}}$$

$$\eta = \frac{4I_{40}\varepsilon^2}{3n_0 I_{50} + 12 P_0 I_{40}}\lambda_{\mathrm{mfp}}$$

Thermodynamic integrals: $I_{nq} = \frac{(-1)^q}{(2q+1)!!}\int dK\, E_{\mathbf{k}}^{n-2q}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^q f_{0\mathbf{k}}$

Equations of motion: *<u>ultrarelativistic</u>* gas of *hard spheres*

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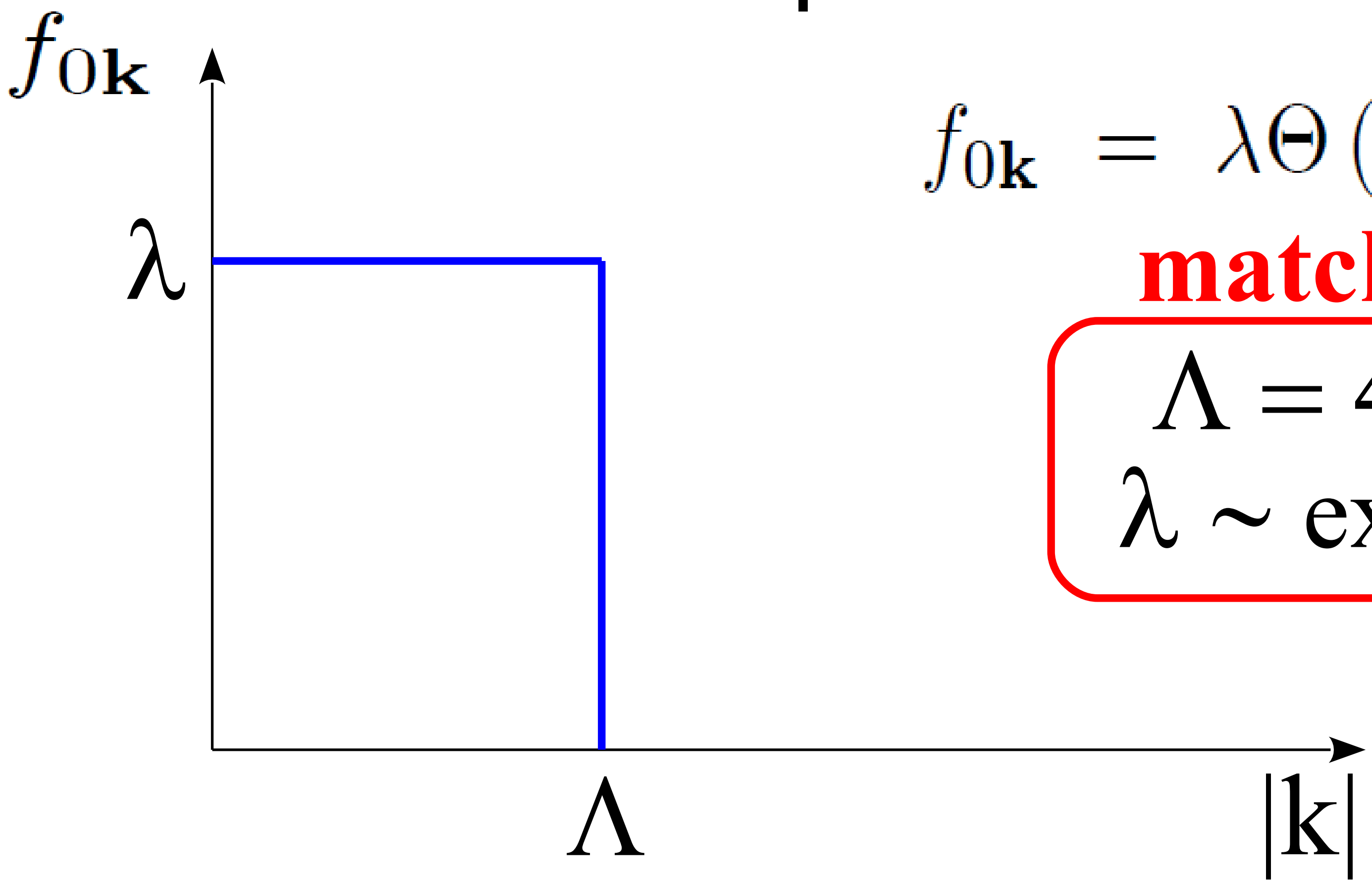
"Equilibrium" Transport coefficients:

$$\tau_\pi = \frac{9}{5}\ell_{\mathrm{mfp}}$$

$$\tau_\pi = \frac{6\eta}{\varepsilon + P} \qquad \eta = \frac{6}{5}\frac{T}{\sigma}$$

<u>Coefficients derived by Israel-Stewart</u>

Example of non-equilibrium state: “over-occupied” state



$$f_{0\mathbf{k}} \;=\; \lambda\Theta\left(\Lambda - E_{\mathbf{k}}\right)$$

matching

$$\boxed{\begin{array}{l}\Lambda = 4\mathrm{T} \\ \lambda \sim \exp(\alpha)\end{array}}$$

Equations of motion: *<u>ultrarelativistic</u>* gas of *hard spheres*

We recover the usual equation for the shear stress:

$$\boxed{\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}}\sigma\pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - 2\sigma_\lambda^{\langle\mu}\pi^{\nu\rangle\lambda} - \frac{4}{3}\pi^{\mu\nu}\theta}$$

Over-occupied Transport coefficients:

$$\tau_\pi = \frac{18}{13}\ell_{\mathrm{mfp}}$$

$$\eta = \frac{84}{65}\frac{T}{\sigma_T}$$

- qualitatively the same
- appears to be slightly more viscous

Equations of motion: *ultrarelativistic* gas of *hard spheres*

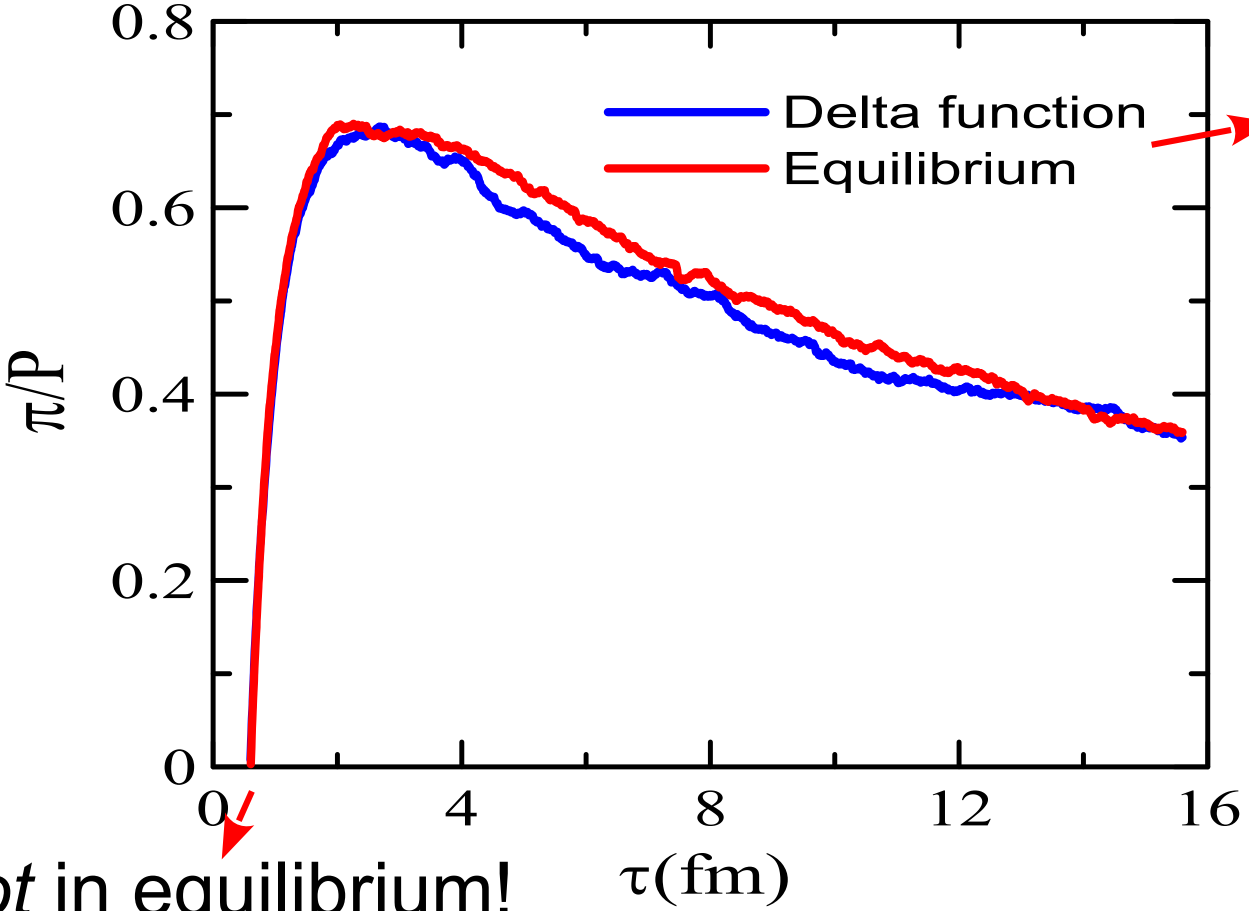
We recover the usual equation for the shear stress:

$$\boxed{\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}}\sigma\pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - 2\sigma_\lambda^{\langle\mu}\pi^{\nu\rangle\lambda} - \frac{4}{3}\pi^{\mu\nu}\theta}$$

$f_{0\mathbf{k}}$	$\lambda\exp\left(-E_\mathbf{k}/\Lambda\right)$	$\lambda\Theta\left(\Lambda - E_\mathbf{k}\right)$	$\lambda\delta\left(E_\mathbf{k} - \Lambda\right)$
τ_π	$\frac{9}{5}\ell_{\mathrm{mfp}}$	$\frac{18}{13}\ell_{\mathrm{mfp}}$	$\frac{9}{7}\ell_{\mathrm{mfp}}$
η	$\frac{6}{5}\frac{T}{\sigma_T}$	$\frac{84}{65}\frac{T}{\sigma_T}$	$\frac{9}{7}\frac{T}{\sigma_T}$

Coefficients do not change much with $f_{0\mathbf{k}}$. Can we see this?

Boltzmann eq. + Bjorken flow: *ultrarelativistic* gas of *hard spheres*



not in equilibrium!

Initial conditions, fixed energy

shear viscosity

$$\frac{\eta}{n} \approx 6$$

Evolution of shear stress does not see this non-equilibrium effect

Conclusions

local equilibrium  **hydrodynamics**

- The applicability of fluid-dynamical models of heavy ion collisions cannot be easily justified

- The derivation of hydrodynamics is more general than previously considered: **"hydrodynamic equations" can be obtained even far from equilibrium.**

- May be relevant for the initial stages of heavy ion collisions