

Kinetic theory of massive spin-1/2 particles from the Wigner-function formalism

N. Weickgenannt, X.-l. Sheng, E. Speranza, Q. Wang, and D.H. Rischke, PRD 100 (2019) 056018

Quark Matter 2019, Wuhan

November 6, 2019

- Early stage of non-central **heavy-ion collisions**: large orbital angular momenta and strong electromagnetic fields.

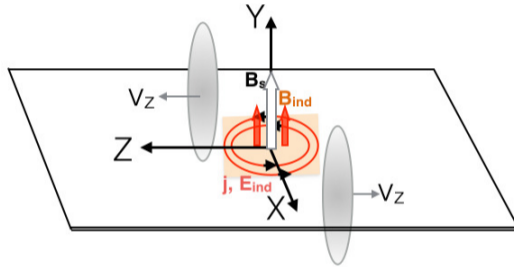


Figure from V. Roy, S. Pu, L. Rezzolla, and D. H. Rischke, PRC96 (2017) 054909

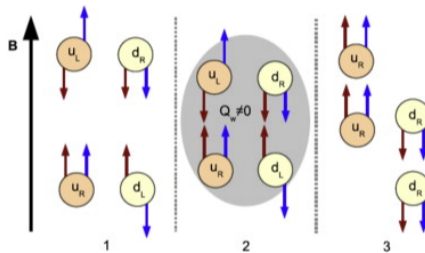
- For massive **spin-0** particles, second-order dissipative MHD has already been studied.

G.S. Denicol, X.-G. Huang, E. Molnar, G.M. Monteiro, H. Niemi, J. Noronha, D.H. Rischke, and Q. Wang, PRD 98 (2018) 076009

G.S. Denicol, E. Molnar, H. Niemi, and D.H. Rischke, PRD 99 (2019) 056017

- But all elementary matter particles are fermions...

- Chiral vortical effect (CVE): charge currents induced by **vorticity**.
- Chiral magnetic effect (CME): charge currents induced by **magnetic fields**.



Dmitri E. Kharzeev, Larry D. McLerran, and Harmen J. Warringa, NPA 803 (2008)

- Has been studied in **massless** case.
 - J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601;
 - Y. Hidaka, S. Pu, and D-L. Yang, PRD95 (2017) 091901;
 - A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010
- Similar effects for **massive** particles?

- **What we want:** kinetic theory and fluid dynamics for **massive spin-1/2** particles in inhomogeneous electromagnetic fields.

J.-H. Gao, and Z.-T. Liang, *PRD* **100** (2019) 056021

K. Hattori, Y. Hidaka, and D.-L. Yang, *arXiv:1903.01653* (2019)

Z. Wang, X. Guo, S. Shi, and P. Zhuang, *PRD* **100** (2019) 014015

- **Starting point:** quantum field theory, Dirac equation.
- **Strategy:** use Wigner functions to derive kinetic theory.
- **Goal:** determine fluid-dynamical equations of motion from resulting Boltzmann equation.

- Quantum analogue of classical distribution function.
- Contains information about quantum state of system.
- Off-equilibrium: two-point function depends not only on relative coordinate y , but also on central coordinate x .
- Wigner transformation of two-point function:

H.-Th. Elze, M. Gyulassy, and D. Vasak, *Ann. Phys.* **173** (1987) 462

$$W(x, p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \langle : \bar{\Psi}(x + \frac{y}{2}) \Psi(x - \frac{y}{2}) : \rangle,$$

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H.-Th. Elze, M. Gyulassy, and D. Vasak, *Ann. Phys.* **173** (1987) 462

$$W(x, p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \langle : \bar{\Psi}(x + \frac{y}{2}) U(x + \frac{y}{2}, x) U(x, x - \frac{y}{2}) \Psi(x - \frac{y}{2}) : \rangle,$$

with gauge link

$$U(b, a) \equiv P \exp \left(-\frac{i}{\hbar} \int_a^b dz^\mu A_\mu(z) \right)$$

to ensure gauge invariance.

- In general: result of calculation of Wigner function directly from definition is **not on-shell**.
- Momentum variable of directly calculated Wigner function is physical (kinetic) momentum **only at zeroth order in \hbar /gradients**.
- Dirac equation implies transport equation for Wigner function.
- Idea: **Find general solutions for this transport equation by expanding in powers of \hbar** .
- Decompose W in transport equation into generators of Clifford algebra:

$$W = \frac{1}{4} \left(\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu} \right).$$

- Equations for \mathcal{F} (scalar, “distribution function”) and $\mathcal{S}_{\mu\nu}$ (tensor, “dipole moment”) decouple from rest.
- Determine \mathcal{V}_μ (“vector current“), \mathcal{A}_μ (“polarization“), \mathcal{P} from $\mathcal{S}_{\mu\nu}, \mathcal{F}$.
- Results will hold up to order $\mathcal{O}(\hbar)$.
- Notation: $W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2)$.

$$\begin{aligned}
 \mathcal{F} &= m \left[V \delta(p^2 - m^2) - \frac{\hbar}{2} F^{\mu\nu} \Sigma_{\mu\nu}^{(0)} A^{(0)} \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\
 \mathcal{S}_{\mu\nu} &= m \left[\bar{\Sigma}_{\mu\nu} \delta(p^2 - m^2) - \hbar F_{\mu\nu} V^{(0)} \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\
 \mathcal{P} &= \frac{\hbar}{4m} \epsilon^{\mu\nu\alpha\beta} \nabla_\mu \left[p_\nu \Sigma_{\alpha\beta}^{(0)} A^{(0)} \delta(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\
 \mathcal{V}_\mu &= \delta(p^2 - m^2) \left[p_\mu V + \frac{\hbar}{2} \nabla^\nu \Sigma_{\mu\nu}^{(0)} A^{(0)} \right] \\
 &\quad - \hbar \left[\frac{1}{2} p_\mu F^{\alpha\beta} \Sigma_{\alpha\beta}^{(0)} + \Sigma_{\mu\nu}^{(0)} F^{\nu\alpha} p_\alpha \right] A^{(0)} \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2), \\
 \mathcal{A}_\mu &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} p^\nu \bar{\Sigma}^{\alpha\beta} \delta(p^2 - m^2) + \hbar \tilde{F}_{\mu\nu} p^\nu V^{(0)} \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2),
 \end{aligned}$$

$$\bar{\Sigma}^{(0)\mu\nu} = \Sigma^{(0)\mu\nu} A^{(0)},$$

$$p_\nu \bar{\Sigma}^{\mu\nu} = \hbar \nabla^\mu V^{(0)},$$

$$\nabla^\mu \equiv \partial_x^\mu - F^{\mu\nu} \partial_{p\nu}.$$

- By Taylor expansion of δ -function:

$$\mathcal{F} = \frac{2}{(2\pi\hbar)^3} m \sum_s \delta(p^2 - m^2 - \hbar \frac{s}{2} \Sigma_{\mu\nu}^{(0)} F^{\mu\nu}) f_s$$

Modified on-shell condition.

f_s distribution functions for spin-up ($s = +$) and spin-down ($s = -$) particles, $V = f_+ + f_-$ and $A^{(0)} = f_+^{(0)} - f_-^{(0)}$.

- Generalized Boltzmann equation

$$\sum_s \delta(p^2 - m^2) \left\{ p^\mu \partial_{x^\mu} f_s + \partial_{p^\mu} \left[F^{\mu\nu} p_\nu + \frac{\hbar}{4} s \Sigma^{(0)\nu\rho} (\partial^\mu F_{\nu\rho}) \right] f_s \right\} = 0$$

- Force on particle: first Mathisson-Papapetrou-Dixon (MPD) equation
 → Particle with classical dipole moment $\Sigma^{(0)\mu\nu}$ in electromagnetic field:

W. Israel, *General Relativity and Gravitation* 9 (1978) 451

$$m \frac{d}{d\tau} p^\mu = F^{\mu\nu} p_\nu + \frac{\hbar}{4} s \Sigma^{(0)\nu\rho} (\partial^\mu F_{\nu\rho}).$$

τ : worldline parameter, $\frac{d}{d\tau} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}^\mu \frac{\partial}{\partial p^\mu}$.

- $\bar{\Sigma}_{\mu\nu}$ determined by kinetic equation for dipole moment:

$$\delta(p^2 - m^2) \left[p \cdot \nabla \bar{\Sigma}^{\mu\nu} - \bar{\Sigma}^{\lambda\nu} F_{\lambda}^{\mu} + \bar{\Sigma}^{\lambda\mu} F_{\lambda}^{\nu} + \frac{1}{2} (\partial_{x\alpha} F^{\mu\nu}) \partial_{p\alpha} V^{(0)} \right] = 0.$$

- To zeroth order:

$$m \frac{d}{d\tau} \Sigma^{(0)\mu\nu} = \Sigma^{(0)\lambda\nu} F_{\lambda}^{\mu} - \Sigma^{(0)\lambda\mu} F_{\lambda}^{\nu}.$$

- Recover second MPD equation for dipole-moment tensor $\Sigma_{\mu\nu}^{(0)}$!

W. Israel, *General Relativity and Gravitation* 9 (1978) 451

- Equivalent to Bargmann-Michel-Telegdi (BMT) equation

V. Bargmann, L. Michel, and V.L. Telegdi, *PRL* 2 (1959) 435

$$m \frac{d}{d\tau} n^{(0)\mu} = F^{\mu\nu} n_{\nu}^{(0)},$$

with classical spin vector

$$n^{(0)\mu} = -\frac{1}{2m} \epsilon^{\mu\nu\alpha\beta} p_{\nu} \Sigma_{\alpha\beta}^{(0)}.$$

- Non-relativistic dipole-moment tensor connected to spin three-vector n^k :

$$\Sigma^{ij} = \epsilon^{ijk} n^k.$$

- For **massive** particles: define spin in **rest frame**.

U. Heinz, PLB 144 (1984) 228

$$\Sigma^{\mu\nu} = -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta.$$

- For **massless** particles: define spin in **arbitrary frame with four-velocity u^μ** . Spin vector n^μ is always parallel to **momentum**.

J.-Y. Chen, D.T. Son, and M. Stephanov, PRL 115 (2015) 021601

$$\Sigma_u^{\mu\nu} = -\frac{1}{p \cdot u} \epsilon^{\mu\nu\alpha\beta} u_\alpha p_\beta.$$

- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \rightarrow \Sigma_u^{\mu\nu}$.
- Result agrees with previously known massless solution!

Y. Hidaka, S. Pu, and D.-L. Yang, PRD 95 (2017) 091901

A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010

J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, PRD 98 (2018) 036019

- In global equilibrium: Analytic solution for Boltzmann equation.
- Vector current is explicitly calculated as:

$$\mathcal{V}^\mu = \frac{2}{(2\pi\hbar)^3} \sum_s \left[\delta(p^2 - m^2) \left(p^\mu - m\hbar \frac{s}{2} \tilde{\omega}^{\mu\nu} n_\nu^{(0)} \partial_{\beta \cdot \pi} \right) + \hbar s \tilde{F}^{\mu\nu} n_\nu^{(0)} \delta'(p^2 - m^2) + \hbar \frac{s}{2m} \delta(p^2 - m^2) \epsilon^{\nu\mu\alpha\beta} p_\alpha \nabla_\nu n_\beta^{(0)} \right] f_s^{(0)},$$

with zeroth-order equilibrium distribution function

$$f_s^{(0)} = [\exp(\beta \cdot \pi - \beta \mu_s) + 1]^{-1},$$

where π^μ canonical momentum, β^μ thermal fluid velocity, β inverse temperature, μ_s chemical potential.

- Analogue of chiral vortical effect (CVE) for massive particles.

D. T. Son and P. Surowka, PRL 103 (2009) 0906.5044

- Analogue of chiral magnetic effect (CME).

D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, NPA 803 (2008) 0711.0950

- Thermal vorticity tensor: $\omega_{\mu\nu} \equiv \frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$.
- Dual thermal vorticity tensor: $\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta}$.

- Obtain expression for axial-vector current:

$$\mathcal{A}^\mu = \frac{2}{(2\pi\hbar)^3} \sum_s \left[\delta(p^2 - m^2) \left(s m n^{(0)\mu} - \frac{\hbar}{2} \tilde{\omega}^{\mu\nu} p_\nu \partial_{\beta \cdot \pi} \right) + \hbar \tilde{F}^{\mu\nu} p_\nu \delta'(p^2 - m^2) \right] f_s^{(0)} - \frac{\hbar}{2} \epsilon^{\mu\nu\alpha\beta} p_\nu \Xi_{\alpha\beta} \delta(p^2 - m^2).$$

- Classical spin precession.
- Analogue of axial chiral vortical effect (ACVE).
- Analogue of chiral separation effect (CSE).

D. E. Kharzeev, J. Liao, S. A. Voloshin, and G. Wang, *Prog. Part. NP* 88 (2016), 1511.04050

- Particle current:

$$J^\mu = \int d^4 p \mathcal{V}^\mu.$$

Not parallel to the fluid velocity!

$$\partial_\mu J^\mu = 0.$$

Conserved!

- Canonical energy-momentum tensor (matter part):

$$T_{mat}^{\mu\nu} = \int d^4 p p^\nu \mathcal{V}^\mu.$$

Not symmetric!

$$\partial_\mu T_{mat}^{\mu\nu} = F^{\nu\mu} J_\mu.$$

Conserved in combination with electromagnetic and interaction part.

- Canonical spin tensor (matter part):

$$S_{mat}^{\lambda, \mu\nu} = -\frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \int d^4p \mathcal{A}_\rho,$$

$$\hbar \partial_\lambda S_{mat}^{\lambda, \mu\nu} = T_{mat}^{\nu\mu} - T_{mat}^{\mu\nu}.$$

Not conserved!

Spin angular momentum and orbital angular momentum are converted into one another.

→ Consideration of spin leads to **additional fluid-dynamical equation of motion**.

W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, PRC 97 (2018) 041901;

W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, PRD 97 (2017) 116017;

W. Florkowski, F. Becattini, and E. Speranza, APB 49 (2018) 1409;

F. Becattini, W. Florkowski, and E. Speranza, PLB 789 (2019) 419-425;

K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, H. Taya, PLB 795 (2019) 100-106

- Derived transport equation for distribution function and polarization for massive spin-1/2 particles in inhomogeneous electromagnetic fields.
- Recovered classical equations of motion.
- Solution agrees with previously known massless solution in massless limit.
- Derived explicit expressions for currents in global equilibrium.
- Found analogues of CVE, CME, ACVE, and CSE for massive particles.
- Derived fluid-dynamical equations of motion.

- Solve kinetic equations.
- Include **collisions**.
 - Boltzmann equation without assuming equilibrium.
 - **Non-local collision term** for spin-orbit interaction.
- Derive equations of motion for **dissipative quantities**.
 - Method of moments.

Back-up

- Natural units, $c = k_B = 1$, but keep \hbar explicitly.
- To simplify notation: only write positive-energy parts of solutions.
- **Polarization direction n^μ** : space-like unit vector parallel to axial-vector current.
- **Spin quantization direction**: unit vector, purely spatial in particle rest frame.
“spin up”, $s = +$: projection of spin onto quantization direction positive.
“spin down”, $s = -$: projection of spin onto quantization direction negative.
Here: chosen to be **identical to polarization direction**.

$$\bar{u}_s \gamma^\mu \gamma^5 u_s = 2ms n^\mu$$

- Dipole-moment tensor:

$$\begin{aligned} s\Sigma^{\mu\nu} &= \frac{1}{2m} \bar{u}_s \frac{i}{2} [\gamma^\mu, \gamma^\nu] u_s \\ &= -s \frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta \end{aligned}$$

- Spin tensor:

rank-3 tensor $S^{\lambda,\mu\nu}$ such that total angular momentum

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$$

- Distribution function f_{rs} is Hermitian matrix in spin space.
- Can be diagonalized by Unitary transformation:

$$f_{rs} = D_{rr'} \tilde{f}_{s'} \delta_{r's'} D_{s's}^\dagger.$$

- Redefine spinors

$$\tilde{u}_s \equiv \sum_{s'} u_{s'} D_{s's}.$$

- Define

$$sn^\mu \equiv \tilde{u}_s \gamma^\mu \gamma^5 u_s.$$

- Only diagonal part contributes!

- From Dirac equation: transport equation for Wigner function:

H.-Th. Elze, M. Gyulassy, and D. Vasak, *Ann. Phys.* **173** (1987) 462

$$(\gamma_\mu K^\mu - m)W(X, p) = 0,$$

with

$$\begin{aligned} K^\mu &\equiv \Pi^\mu + \frac{1}{2}i\hbar\nabla^\mu, \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta)F^{\mu\nu}\partial_{p\nu}, \\ \Pi^\mu &\equiv p^\mu - \hbar\frac{1}{2}j_1(\Delta)F^{\mu\nu}\partial_{p\nu}, \end{aligned}$$

$\Delta = \frac{1}{2}\hbar\partial_p \cdot \partial_x$ with ∂_x only acting on $F^{\mu\nu}$ and $j_0(r) = \sin(r)/r$, $j_1(r) = [\sin(r) - r\cos(r)]/r^2$ spherical Bessel functions.

- Exact quantum kinetic equation for Wigner function for massive spin 1/2-particles and inhomogeneous fields!
- Only assumption: vanishing collision kernel, external classical gauge fields.

- $p \cdot u$ related to rest-frame energy $\sqrt{p^2} \rightarrow \delta$ -function!
- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \rightarrow \Sigma_u^{\mu\nu}$.
- Attention: $\delta(p^2 - m^2)/m \rightarrow \delta(p^2)/(p \cdot u)$.
- Find general solution for constraint on $\bar{\Sigma}^{\mu\nu}$.
- Define right- and left-handed currents $J_\mu^\pm \equiv \frac{1}{2}(\mathcal{V}_\mu^{m=0} \pm \mathcal{A}_\mu^{m=0})$
- Result

$$J_\mu^\pm = \left[p_\mu \delta(p^2) \pm \frac{1}{2} \hbar \epsilon_{\mu\nu\alpha\beta} p^\nu F^{\alpha\beta} \delta'(p^2) \pm \frac{1}{2} \hbar \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha u^\beta}{p \cdot u} \delta(p^2) \nabla^\nu \right] f_\pm.$$

agrees with previously known massless solution!

Y. Hidaka, S. Pu, D.-L. Yang, PRD 95 (2017) 091901; A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010; J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, PRD 98 (2018) 036019

- Equilibrium distribution function:

$$f_s^{eq} = (e^{g_s} + 1)^{-1},$$

with g linear combination of conserved quantities charge, momentum, and angular momentum:

$$g_s = \beta \pi \cdot U - \beta \mu_s + \frac{\hbar}{4} s \Sigma^{\mu\nu} \partial_\mu (\beta U_\nu).$$

Here, $\pi_\mu \equiv p_\mu + A_\mu$ is canonical momentum, U_μ is fluid velocity, $\beta \equiv \frac{1}{T}$ is inverse temperature, and μ_s is chemical potential.

- To zeroth order

$$f_s^{(0)} = (e^{g_{s0}} + 1)^{-1},$$

with

$$g_{s0} = \beta (\pi \cdot U - \mu_s).$$

- "Homogeneous" part of the Boltzmann equation fulfilled if:

$$\begin{aligned}\mu_s &= \text{const}, \\ \partial_\nu \beta_\mu + \partial_\mu \beta_\nu &= 0,\end{aligned}$$

- "Inhomogeneous" part of Boltzmann equation:
additional conditions to make global equilibrium possible, e.g.

$$\begin{aligned}\mu_{s=+} - \mu_{s=-} &= 0 \text{ or} \\ \partial_{x^\alpha} F_{\mu\nu} &= 0.\end{aligned}$$