Kinetic theory of massive spin-1/2 particles from the Wigner-function formalism


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Why magneto-hydrodynamics (MHD)?

- Early stage of non-central heavy-ion collisions: large orbital angular momenta and strong electromagnetic fields.

  Figure from V. Roy, S. Pu, L. Rezzolla, and D. H. Rischke, PRC96 (2017) 054909

- For massive spin-0 particles, second-order dissipative MHD has already been studied.

- But all elementary matter particles are fermions...
Spin effects in heavy-ion collisions

- Chiral vortical effect (CVE): charge currents induced by vorticity.
- Chiral magnetic effect (CME): charge currents induced by magnetic fields.

Has been studied in massless case.

- Y. Hidaka, S. Pu, and D.-L. Yang, PRD 95 (2017) 091901;

Similar effects for massive particles?
Towards MHD with spin

What we want: kinetic theory and fluid dynamics for massive spin-1/2 particles in inhomogeneous electromagnetic fields.

J.-H. Gao, and Z.-T. Liang, PRD 100 (2019) 056021

Starting point: quantum field theory, Dirac equation.
Strategy: use Wigner functions to derive kinetic theory.
Goal: determine fluid-dynamical equations of motion from resulting Boltzmann equation.
Quantum analogue of classical distribution function.
Contains information about quantum state of system.
Off-equilibrium: two-point function depends not only on relative coordinate $y$, but also on central coordinate $x$.

Wigner transformation of two-point function:


$$W(x, p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p\cdot y} \langle \bar{\Psi}(x + \frac{y}{2}) \psi(x - \frac{y}{2}) \rangle,$$
Wigner functions

- Quantum analogue of classical distribution function.
- Contains information about quantum state of system.
- Off-equilibrium: two-point function depends not only on relative coordinate $y$, but also on central coordinate $x$.
- **Wigner transformation** of two-point function:

  $W(x, p) = \int \frac{d^4 y}{(2\pi \hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \langle \Psi(x + \frac{y}{2}) U(x, x - \frac{y}{2}) \bar{\Psi}(x - \frac{y}{2}) : \rangle$,

  with **gauge link**

  $U(b, a) \equiv P \exp \left( -\frac{i}{\hbar} \int_a^b dz^\mu A_\mu(z) \right)$

  to ensure gauge invariance.
Calculating the Wigner function

- In general: result of calculation of Wigner function directly from definition is not on-shell.
- Momentum variable of directly calculated Wigner function is physical (kinetic) momentum only at zeroth order in $\hbar$/gradients.
- Dirac equation implies transport equation for Wigner function.
- Idea: Find general solutions for this transport equation by expanding in powers of $\hbar$.
- Decompose $W$ in transport equation into generators of Clifford algebra:

$$W = \frac{1}{4} \left( \mathcal{F} + i \gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} S_{\mu\nu} \right).$$

- Equations for $\mathcal{F}$ (scalar, "distribution function") and $S_{\mu\nu}$ (tensor, "dipole moment") decouple from rest.
- Determine $\mathcal{V}_\mu$ ("vector current"), $\mathcal{A}_\mu$ ("polarization"), $\mathcal{P}$ from $S_{\mu\nu}, \mathcal{F}$.
- Results will hold up to order $O(\hbar)$.
- Notation: $W = W^{(0)} + \hbar W^{(1)} + O(\hbar^2)$. 
General results

\[ F = m \left[ V \, \delta(p^2 - m^2) - \frac{\hbar}{2} F^{\mu \nu} \Sigma^{(0)}_{\mu \nu} A^{(0)} \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \]

\[ S_{\mu \nu} = m \left[ \tilde{\Sigma}_{\mu \nu} \delta(p^2 - m^2) - \hbar F_{\mu \nu} V^{(0)} \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \]

\[ \mathcal{P} = \frac{\hbar}{4m} \epsilon^{\mu \nu \alpha \beta} \nabla_\mu \left[ p_\nu \Sigma^{(0)}_{\alpha \beta} A^{(0)} \delta(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \]

\[ \mathcal{V}_\mu = \delta(p^2 - m^2) \left[ p_\mu V + \frac{\hbar}{2} \nabla_\nu \Sigma^{(0)}_{\mu \nu} A^{(0)} \right] - \hbar \left[ \frac{1}{2} p_\mu F^{\alpha \beta} \Sigma^{(0)}_{\alpha \beta} + \Sigma^{(0)}_{\mu \nu} F^{\nu \alpha} p_\alpha \right] A^{(0)} \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2), \]

\[ \mathcal{A}_\mu = -\frac{1}{2} \epsilon_{\mu \nu \alpha \beta} p^\nu \tilde{\Sigma}^{\alpha \beta} \delta(p^2 - m^2) + \hbar \tilde{F}_{\mu \nu} p^\nu V^{(0)} \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2), \]

\[ \tilde{\Sigma}^{(0)}_{\mu \nu} = \Sigma^{(0)}_{\mu \nu} A^{(0)}, \]

\[ p_\nu \tilde{\Sigma}^{\mu \nu} = \hbar \nabla_\mu V^{(0)}, \]

\[ \nabla^\mu \equiv \partial^\mu - F^{\mu \nu} \partial_\nu. \]
Boltzmann equation for massive spin-1/2 particles

- By Taylor expansion of $\delta$-function:

$$F = \frac{2}{(2\pi\hbar)^3} m \sum_s \delta(p^2 - m^2 - \hbar S \Sigma^{(0)}_{\mu\nu} F^{\mu\nu}) f_s$$

Modified on-shell condition.

$f_s$ distribution functions for spin-up ($s = +$) and spin-down ($s = -$) particles, $V = f_+ + f_-$ and $A^{(0)} = f_+^{(0)} - f_-^{(0)}$.

- Generalized Boltzmann equation

$$\sum_s \delta(p^2 - m^2) \left\{ p^\mu \partial_{x^\mu} f_s + \partial_{p^\mu} \left[ F^{\mu\nu} p_{\nu} + \frac{\hbar}{4} s \Sigma^{(0)}_{\nu\rho}(\partial^\mu F_{\nu\rho}) \right] f_s \right\} = 0$$

- Force on particle: first Mathisson-Papapetrou-Dixon (MPD) equation

$\rightarrow$ Particle with classical dipole moment $\Sigma^{(0)}_{\mu\nu}$ in electromagnetic field:

W. Israel, General Relativity and Gravitation 9 (1978) 451

$$m \frac{d}{d\tau} p^\mu = F^{\mu\nu} p_{\nu} + \frac{\hbar}{4} s \Sigma^{(0)}_{\nu\rho}(\partial^\mu F_{\nu\rho}).$$

$\tau$: worldline parameter, $\frac{d}{d\tau} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}^\mu \frac{\partial}{\partial p^\mu}$. 
Kinetic equation for dipole moment

- $\bar{\Sigma}_{\mu\nu}$ determined by kinetic equation for dipole moment:

$$\delta(p^2 - m^2) \left[ p \cdot \nabla \bar{\Sigma}_{\mu\nu} - \bar{\Sigma}^{\lambda\nu} F_{\lambda}^\mu + \bar{\Sigma}^{\lambda\mu} F_{\lambda}^\nu + \frac{1}{2} (\partial_{\alpha} F^{\mu\nu}) \partial_{\rho\alpha} V^{(0)} \right] = 0. $$

- To zeroth order:

$$m \frac{d}{d\tau} \Sigma_{\mu\nu}^{(0)} = \Sigma_{\lambda\nu}^{(0)} F_{\lambda}^\mu - \Sigma_{\lambda\mu}^{(0)} F_{\lambda}^\nu. $$

- Recover second MPD equation for dipole-moment tensor $\Sigma_{\mu\nu}^{(0)}$!

  W. Israel, General Relativity and Gravitation 9 (1978) 451

- Equivalent to Bargmann-Michel-Telegdi (BMT) equation

  V. Bargmann, L. Michel, and V.L. Telegdi, PRL 2 (1959) 435

$$m \frac{d}{d\tau} n_{\mu}^{(0)} = F_{\mu\nu} n_{\nu}^{(0)},$$

with classical spin vector

$$n_{\mu}^{(0)} = -\frac{1}{2m} \epsilon_{\mu\nu\alpha\beta} p_{\nu} \Sigma_{\alpha\beta}^{(0)}.$$
Massless limit

- Non-relativistic dipole-moment tensor connected to spin three-vector $n^k$:
  \[ \Sigma^{ij} = \epsilon^{ijk} n^k. \]

- For massive particles: define spin in rest frame.
  U. Heinz, PLB 144 (1984) 228
  \[ \Sigma^{\mu\nu} = -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta. \]

- For massless particles: define spin in arbitrary frame with four-velocity $u^\mu$.
  Spin vector $n^\mu$ is always parallel to momentum.
  \[ \Sigma_{u}^{\mu\nu} = -\frac{1}{p \cdot u} \epsilon^{\mu\nu\alpha\beta} u_\alpha p_\beta. \]

- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \rightarrow \Sigma_{u}^{\mu\nu}$.

- Result agrees with previously known massless solution!
  Y. Hidaka, S. Pu, and D.-L. Yang, PRD 95 (2017) 091901
In global equilibrium: Analytic solution for Boltzmann equation.

Vector current is explicitly calculated as:

\[
\mathcal{V}^\mu = \frac{2}{(2\pi \hbar)^3} \sum_s \left[ \delta(p^2 - m^2) \left( p^\mu - m^2 \right) \sqrt{s} \tilde{\omega}^{\mu\nu} n^{(0)}_\nu \partial_\beta \cdot \pi \right] \\
+ \hbar s \tilde{F}^{\mu\nu} n^{(0)}_\nu \delta'(p^2 - m^2) + \frac{\delta(p^2 - m^2)}{2m} \epsilon^{\nu\mu\alpha\beta} p_\alpha \nabla_\nu n^{(0)}_\beta \right] f^{(0)}_s,
\]

with zeroth-order equilibrium distribution function

\[
f^{(0)}_s = \left[ \exp(\beta \cdot \pi - \beta \mu_s) + 1 \right]^{-1},
\]

where \( \pi^\mu \) canonical momentum, \( \beta^\mu \) thermal fluid velocity, \( \beta \) inverse temperature, \( \mu_s \) chemical potential.

Analogue of chiral vortical effect (CVE) for massive particles.

D. T. Son and P. Surowka, PRL 103 (2009) 0906.5044

Analogue of chiral magnetic effect (CME).


Thermal vorticity tensor: \( \omega_{\mu\nu} \equiv \frac{1}{2} \left( \partial_\mu \beta_\nu - \partial_\nu \beta_\mu \right) \).

Dual thermal vorticity tensor: \( \tilde{\omega}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta} \).
Obtain expression for axial-vector current:

\[ A^\mu = \frac{2}{(2\pi \hbar)^3} \sum_s \left[ \delta(p^2 - m^2) \left( s m n^{(0)}_\nu - \frac{\hbar}{2\pi} \hat{\omega}^\mu_\nu p_\nu \partial_\beta \pi \right) + \hbar \hat{F}^\mu_\nu p_\nu \delta'(p^2 - m^2) \right] f_s^{(0)} - \frac{\hbar}{2} \epsilon^{\mu\nu\alpha\beta} p_\nu \Xi_{\alpha\beta} \delta(p^2 - m^2). \]

- Classical spin precession.
- Analogue of axial chiral vortical effect (ACVE).
- Analogue of chiral separation effect (CSE).

Fluid-dynamical equations with spin 1

- Particle current:
  \[ J^\mu = \int \, d^4p \, \nu^\mu. \]
  Not parallel to the fluid velocity!
  \[ \partial_\mu J^\mu = 0. \]
  Conserved!

- Canonical energy-momentum tensor (matter part):
  \[ T^{\mu\nu}_{\text{mat}} = \int \, d^4p \, p^\nu \nu^\mu. \]
  Not symmetric!
  \[ \partial_\mu T^{\mu\nu}_{\text{mat}} = F^{\nu\mu} J_\mu. \]
  Conserved in combination with electromagnetic and interaction part.
Fluid-dynamical equations with spin II

- Canonical spin tensor (matter part):

\[ S^{\lambda,\mu\nu}_{\text{mat}} = -\frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \int d^4 p \, A_\rho, \]

\[ \hbar \partial_\lambda S^{\lambda,\mu\nu}_{\text{mat}} = T^{\nu\mu}_{\text{mat}} - T^{\mu\nu}_{\text{mat}}. \]

Not conserved!

Spin angular momentum and orbital angular momentum are converted into one another.

→ Consideration of spin leads to additional fluid-dynamical equation of motion.


W. Florkowski, F. Becattini, and E. Speranza, APB 49 (2018) 1409;


Derived transport equation for distribution function and polarization for massive spin-1/2 particles in inhomogeneous electromagnetic fields.

- Recovered classical equations of motion.
- Solution agrees with previously known massless solution in massless limit.
- Derived explicit expressions for currents in global equilibrium.
- Found analogues of CVE, CME, ACVE, and CSE for massive particles.
- Derived fluid-dynamical equations of motion.
Outlook

- Solve kinetic equations.
- Include collisions.
  - Boltzmann equation without assuming equilibrium.
  - Non-local collision term for spin-orbit interaction.
- Derive equations of motion for dissipative quantities.
  - Method of moments.
Back-up
Conventions and Definitions

- Natural units, $c = k_B = 1$, but keep $\hbar$ explicitly.
- To simplify notation: only write positive-energy parts of solutions.
- Polarization direction $n^\mu$: space-like unit vector parallel to axial-vector current.
- Spin quantization direction: unit vector, purely spatial in particle rest frame.
  - “spin up”, $s = +$: projection of spin onto quantization direction positive.
  - “spin down”, $s = -$: projection of spin onto quantization direction negative.
  Here: chosen to be identical to polarization direction.

$$\bar{u}_s \gamma^\mu \gamma^5 u_s = 2ms n^\mu$$
Spin tensor vs. dipole-moment tensor

- **Dipole-moment tensor:**

\[
\begin{align*}
    s \Sigma^{\mu\nu} &= \frac{1}{2m} \bar{u}_s \frac{i}{2} [\gamma^\mu, \gamma^\nu] u_s \\
    &= -s \frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta
\end{align*}
\]

- **Spin tensor:**

rank-3 tensor \( S^{\lambda,\mu\nu} \) such that total angular momentum

\[
J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\mu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}
\]
Distribution function $f_{rs}$ is Hermitian matrix in spin space. Can be diagonalized by Unitary transformation:

$$f_{rs} = D_{rr'} \tilde{f}_{s'} \delta_{r's'} D_{s's}^\dagger.$$ 

Redefine spinors

$$\tilde{u}_s \equiv \sum_{s'} u_{s'} D_{s's}.$$ 

Define

$$sn^{\mu} \equiv \tilde{u}_s \gamma^{\mu} \gamma^5 u_s.$$ 

Only diagonal part contributes!
From Dirac equation: transport equation for Wigner function:

\[ (\gamma^\mu K^\mu - m) W(X, p) = 0, \]

with

\[ K^\mu \equiv \Pi^\mu + \frac{1}{2} i \hbar \nabla^\mu, \]
\[ \nabla^\mu \equiv \partial^\mu - j_0(\Delta) F^{\mu\nu} \partial_{\nu}, \]
\[ \Pi^\mu \equiv p^\mu - \hbar \frac{1}{2} j_1(\Delta) F^{\mu\nu} \partial_{\nu}, \]

\[ \Delta = \frac{1}{2} \hbar \partial_p \cdot \partial_x \text{ with } \partial_x \text{ only acting on } F^{\mu\nu} \text{ and } j_0(r) = \frac{\sin(r)}{r}, \]
\[ j_1(r) = [\sin(r) - r \cos(r)]/r^2 \]
spherical Bessel functions.

Exact quantum kinetic equation for Wigner function for massive spin 1/2-particles and inhomogeneous fields!

Only assumption: vanishing collision kernel, external classical gauge fields.
Massless limit: details

- $p \cdot u$ related to rest-frame energy $\sqrt{p^2} \rightarrow \delta$-function!
- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \rightarrow \Sigma^{\mu\nu}_{u}$.
- Attention: $\delta(p^2 - m^2)/m \rightarrow \delta(p^2)/(p \cdot u)$.
- Find general solution for constraint on $\bar{\Sigma}^{\mu\nu}$.
- Define right- and left-handed currents $J_{\mu}^{\pm} \equiv \frac{1}{2}(V_{\mu}^{m=0} \pm A_{\mu}^{m=0})$
- Result
  
  $$J_{\mu}^{\pm} = \left[ p_{\mu} \delta(p^2) \pm \frac{1}{2} \hbar \epsilon_{\mu\nu\alpha\beta} p^{\nu} F^{\alpha\beta} \delta'(p^2) \pm \frac{1}{2} \hbar \epsilon_{\mu\nu\alpha\beta} \frac{p^{\alpha} u^{\beta}}{p \cdot u} \delta(p^2) \nabla^{\nu} \right] f_{\pm}$$

agrees with previously known massless solution!

Equilibrium distribution function:

\[ f_s^{eq} = (e^{g_s} + 1)^{-1}, \]

with \( g \) linear combination of conserved quantities charge, momentum, and angular momentum:

\[ g_s = \beta \pi \cdot U - \beta \mu_s + \frac{\hbar}{4} s \sum_{\mu \nu} \partial_\mu (\beta U_\nu). \]

Here, \( \pi_\mu \equiv p_\mu + A_\mu \) is canonical momentum, \( U_\mu \) is fluid velocity, \( \beta \equiv \frac{1}{T} \) is inverse temperature, and \( \mu_s \) is chemical potential.

To zeroth order

\[ f_s^{(0)} = (e^{g_{s0}} + 1)^{-1}, \]

with

\[ g_{s0} = \beta (\pi \cdot U - \mu_s). \]
Equilibrium conditions

- "Homogeneous" part of the Boltzmann equation fulfilled if:
  \[ \mu_s = \text{const}, \]
  \[ \partial_\nu \beta_\mu + \partial_\mu \beta_\nu = 0, \]

- "Inhomogeneous" part of Boltzmann equation:
  additional conditions to make global equilibrium possible, e.g.
  \[ \mu_{s=+} - \mu_{s=-} = 0 \text{ or } \]
  \[ \partial_{x\alpha} F_{\mu\nu} = 0. \]