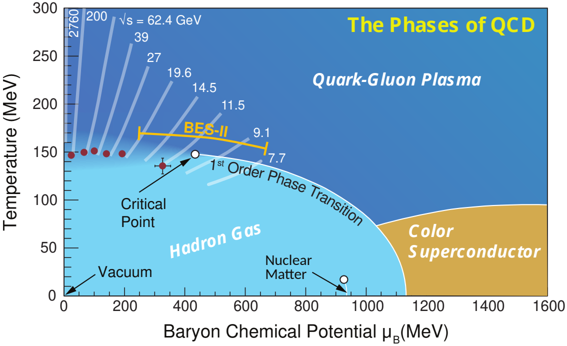
Relativistic Hydrodynamic Fluctuations

M. Stephanov

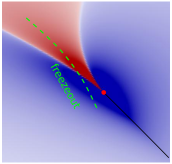
with X. An, G. Basar and H.-U. Yee, 1902.09517

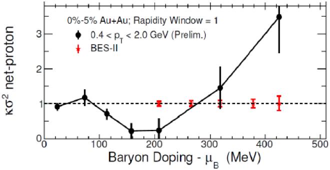
Critical point: intriguing hints

Where on the QCD phase boundary is the CP?



Equilibrium κ_4 vs T and μ_B:





"intriguing hint" (2015 LRPNS)

Motivation for phase II of BES at RHIC and BEST topical collaboration.

Theory/experiment gap: predictions assume equilibrium, but

Non-equilibrium physics is essential near the critical point.

Challenge: develop hydrodynamics *with fluctuations* capable of describing *non-equilibrium* effects on critical-point signatures.

Also note:
Fluctuations are the first step to extend hydro to *smaller systems*.

Stochastic hydrodynamics

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- Usually treated in linear order in fluctuations.
  Non-linearities + point-like noise ⇒ UV divergences.
  In numerical simulations – cutoff dependence.

Deterministic approach

- Variables are one- and two-point functions:
  $\psi = \langle \breve{\psi} \rangle$ and $G = \langle \breve{\psi}\breve{\psi} \rangle - \langle \breve{\psi} \rangle \langle \breve{\psi} \rangle$ – equal-time correlator

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi, G]; \qquad \text{(conservation)}$$

$$\partial_t G = \text{Relaxation}[G \to G^{(\mathrm{eq})}(\psi)]$$

- In Bjorken flow by Akamatsu *et al*, Martinez-Schaefer.
  For arbitrary relativistic flow – by An *et al* (this talk).
  Earlier, in *nonrelativistic* context, – by Andreev in 1970s.

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- Advantage: deterministic equations.

  "Infinite noise" causes UV renormalization of EOS and transport coefficients – can be taken care of *analytically* *(1902.09517)*

General deterministic formalism

An, Basar, Yee, MS, 1902.09517

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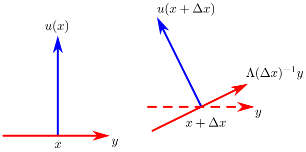
- Important issue in *relativistic* hydro – “equal-time” in the definition of

  $$G(x, y) = \langle\, \phi(x + y/2)\, \phi(x - y/2)\, \rangle.$$

  Addressed by constructing “confluent” derivative.



- Renormalization performed *analytically*, giving cutoff-independent equations.

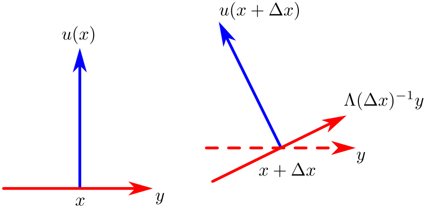
Equal time

We want evolution equation for equal time correlator $G = \langle \phi(t, \boldsymbol{x}_+) \phi(t, \boldsymbol{x}_-) \rangle$. But what does "equal time" mean?

"Equal time" in $\langle \phi(x_+) \phi(x_-) \rangle$ depends on the choice of frame.

The most natural choice is local $u(x)$ (with $x = (x_+ + x_-)/2$).

Derivatives wrt x at "y-fixed" should take this into account:



using $\Lambda(\Delta x) u(x + \Delta x) = u(x)$:

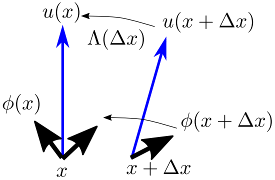
$$\Delta x \cdot \bar{\nabla} G(x, y) \equiv$$
$$G(x + \Delta x, \Lambda(\Delta x)^{-1} y) - G(x, y)\,.$$

not $G(x + \Delta x, y) - G(x, y)\,.$

Confluent derivative, connection and correlator

Confluent derivative: ($\bar{\nabla} u = 0$)

$$\Delta x \cdot \bar{\nabla}\phi = \Lambda(\Delta x)\phi(x+\Delta x) - \phi(x)$$



Confluent two-point correlator:

$$\bar{G}(x,y) = \Lambda(y/2)\, G(x,y)\, \Lambda(-y/2)^T$$

(boost to $u(x)$ – rest frame at midpoint)

$$\bar{\nabla}_\mu \bar{G}_{AB} = \partial_\mu \bar{G}_{AB} - \bar{\omega}^{C}_{\mu A}\bar{G}_{CB} - \bar{\omega}^{C}_{\mu B}\bar{G}_{AC} - \mathring{\omega}^{b}_{\mu a}\, y^a \frac{\partial}{\partial y^b}\bar{G}_{AB}\,.$$

Connection $\bar{\omega}$ makes sure that only the change of ϕ_A *relative* to local rest frame u is counted.

Connection $\mathring{\omega}$ corrects for a possible rotation of the local basis triad e_a defining local coordinates y^a. The derivative is independent of e_a.

We then define the Wigner transform $W_{AB}(x,q)$ of $\bar{G}_{AB}(x,y)$.

Scales

- b: hydro cell size. We coarse grain operators over scale $b \gg \ell_{\mathrm{mic}} \sim c_s/T$ to leave only slow modes for which quantum fluctuatuations are negligible compared to thermal, i.e., $\hbar\omega \ll kT$.
- L: hydrodynamic gradients scale. Must be $L \gg b$.

back

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- ℓ_*: equilibration length (characteristic scale of y): diffusion length during evolution time (typically $\tau_{\text{ev}} \sim L/c_s$)

$$\ell_* \sim \sqrt{\gamma\tau_{\text{ev}}} \sim \sqrt{\gamma L/c_s} \qquad q_* \equiv 1/\ell_*$$

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- $\ell_{\text{mic}} \ll L$ implies the hierarchy (and power-counting scheme):

$$\ell_{\text{mic}} \ll b < \ell_* \ll L \quad \text{or} \quad T/c_s \gg \Lambda > q_* \gg k \quad (\gamma q^2 \sim c_s k)$$

$q \gg k$ – similar to kinetic theory (where Wigner function $\equiv$ p.d.f.)

Matrix equation and diagonalization

After many nontrivial cancellations we find evolution eq.:

$$u\cdot\bar{\nabla}W \;=\; \underbrace{-i[\mathbb{L},W]}_{\text{oscillation}}\;\underbrace{-\{\mathbb{Q},W-W^{(0)}\}}_{\text{relaxation to } W^{(0)}}\;\underbrace{+\mathcal{K}\circ W}_{\text{background}}$$

expand

$$\text{Ideal hydro} \to \mathbb{L}\sim c_s q,$$

$$\text{Noise/dissipation} \to \mathbb{Q}\sim\gamma q^2, \quad \text{and} \quad \text{Background} \to \mathcal{K}\sim\partial_\mu u_\nu, \nabla\,.$$

W is relaxing to equilibrium $W\to W^{(0)}$ at a rate $2\gamma q^2$ disturbed by background hydrodynamic gradients of order k.

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The slowest W modes ($\omega \ll c_s|q|$) are 4 “diagonal” ones in the basis of ideal hydro modes – sound-sound, shear-shear – and can be isolated by time-averaging over faster modes. see equations

Sound-sound and phonon kinetic equation

$$\underbrace{\left[(u+v)\cdot\bar{\nabla}+f\cdot\frac{\partial}{\partial q}\right]W_+}_{\mathcal{L}_+[W_+]}=-\gamma_L q^2(W_+-\underbrace{Tw}_{W^{(0)}})+\underbrace{\mathcal{K}'}_{\sim\,\partial_\mu u_\nu,\ a_\mu}W_+$$

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Several nontrivial observations:

- A phonon $E = c_s(x)|\boldsymbol{q}|$ in an inhomogenious flow:

$$v = c_s\hat{q}_\perp,$$

$$\underbrace{f_\mu}_{\text{force}} = \underbrace{-E(a_\mu + 2v^\nu\omega_{\nu\mu})}_{\text{inertial + Coriolis}} \underbrace{-q_{\perp\nu}\partial_{\perp\mu}u^\nu}_{\text{“Hubble”}} \underbrace{-\bar{\nabla}_{\perp\mu}E}_{\text{potential}}.$$

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- Normalizing $N_+ = W_+/(wc_s|q|)$ eliminates $\mathcal{K}'$ terms:

$$\mathcal{L}_+[N_+] = -\gamma_L q^2(N_+ - \underbrace{T/E}_{E\to 0\text{ of eqlbm. BE dist.}})$$

The relaxation term is completely local.

Renormalization

Expand $T^{\mu\nu}$ in fluctuations and average over noise.

Expansion of $\langle T^{\mu\nu}\rangle$ contains $\langle\phi(x)\phi(x)\rangle = G(x,0) = \int \frac{d^3q}{(2\pi)^3}\, W(x,q)$.

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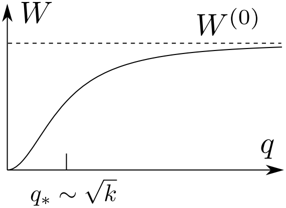

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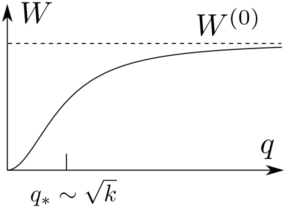

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$$G(x,0) \quad \sim \quad \underbrace{\Lambda^3}_{\text{ideal (EOS)}} \quad + \quad \underbrace{\Lambda\,\partial u}_{\text{visc. terms}} \quad + \quad \underbrace{\widetilde{G}}_{\text{finite “}\partial^{3/2}\text{”}}$$

Renormalized equations

Reorganizing expansion of $\langle T^{\mu\nu}\rangle$ by absorbing *local* cutoff-dependent terms into EOS and visc. coeffs.:

$$\langle T^{\mu\nu}(x)\rangle = (\epsilon u^\mu u^\nu + p(\epsilon)\Delta^{\mu\nu} + \Pi^{\mu\nu})_R + \frac{1}{w}\underbrace{\left[\left(\dot{c}_s \widetilde{G}_{ee}(x) - c_s^2 \widetilde{G}_\lambda^\lambda(x)\right)\Delta^{\mu\nu} + \widetilde{G}^{\mu\nu}(x)\right]}_{\text{local in } \tilde{G}\text{, but not in } u,\ \epsilon \text{ -- } \partial^{3/2}}$$

we obtain finite (cutoff independent) system of deterministic equations:

$$\left\{\begin{array}{rcl} \partial_\mu \langle T^{\mu\nu}\rangle & = & 0\,; \\ u\cdot\bar{\nabla}W & = & \text{Relaxation}[W \to W^{(0)}(\epsilon, u)]\,. \end{array}\right.$$

describing evolution of hydrodynamic variables and their fluctuations.

Outlook (to-do-list)

- In progress *(Xin An's talk on Wednesday):*
  - Add baryon *charge*.
  - Merge with Hydro+. Unify critical and non-critical fluctuations.
- Add higher-order correlators for *non-gaussian* fluctuations.
- Connect *fluctuating* hydro with freezeout kinetics and implement in full hydrodynamic code and event generator.

  Compare with experiment.
- First-order transition in fluctuating hydrodynamics?

More

Linearized fluctuation equations

$$u\cdot\partial\phi_A = -\big(\mathbb{L}+\mathbb{Q}+\mathbb{K}\big)_{AB}\phi^B - \xi_A\,,$$

where

$$\mathbb{L}\equiv\begin{pmatrix}0 & c_s\partial_{\perp\nu}\\ c_s\partial_{\perp\mu} & 0\end{pmatrix},\quad \mathbb{Q}\equiv\begin{pmatrix}0 & 0\\ 0 & -\gamma_\eta\Delta_{\mu\nu}\partial_\perp^{\,2}-(\gamma_\zeta+\frac{1}{3}\gamma_\eta)\partial_{\perp\mu}\partial_{\perp\nu}\end{pmatrix}$$

$$\mathbb{K}\equiv\begin{pmatrix}(1+c_s^2+\dot c_s)\theta & 2c_s a_\nu\\ \frac{1+c_s^2-\dot c_s}{c_s}a_\mu & -u_\mu a_\nu+\partial_{\perp\nu}u_\mu+\Delta_{\mu\nu}\theta\end{pmatrix},\quad \xi\equiv(0,\Delta_{\mu\kappa}\partial_\lambda\breve S^{\lambda\kappa})$$

$$\langle\xi_A(x_+)\xi_B(x_-)\rangle = 2Tw\mathbb{Q}^{(y)}_{AB}\delta^3(y_\perp)\,.$$

$$\begin{aligned}u\cdot\partial G_{AB}(x,y) = &-\Big(\mathbb{L}^{(y)}+\frac{1}{2}\mathbb{L}+\mathbb{Q}^{(y)}+\mathbb{K}+\mathbb{Y}\Big)_{AC}G^C{}_B(x,y)\\ &-\Big(-\mathbb{L}^{(y)}+\frac{1}{2}\mathbb{L}+\mathbb{Q}^{(y)}+\mathbb{K}+\mathbb{Y}\Big)_{BC}G_A{}^C(x,y)\\ &+2Tw\mathbb{Q}^{(y)}_{AB}\delta^3(y_\perp),\end{aligned}$$

Correlation matrix evolution equation

back

$$u\cdot\bar{\nabla}W(x;q) = -\left[i\mathbb{L}^{(q)}+\mathbb{K}^{(a)},W\right]-\left\{\frac{1}{2}\bar{\mathbb{L}}+\mathbb{Q}^{(q)}+\mathbb{K}^{(s)},W\right\}+\theta W+2Tw\mathbb{Q}^{(q)}+(\partial_{\perp\lambda}u_\mu)q^\mu\frac{\partial W}{\partial q_\lambda}$$
$$+\frac{1}{2}a_\lambda\left\{\left(1-\frac{\dot{c}_s}{c_s^2}\right)\mathbb{L}^{(q)},\frac{\partial W}{\partial q_\lambda}\right\}+\frac{\partial}{\partial q_\lambda}\left(\{\Omega_\lambda^{(s)},W\}+[\Omega_\lambda^{(a)},W]-\frac{1}{4}[\mathbb{H}_\lambda,[\mathbb{L}^{(q)},W]]\right),$$

where

$$\mathbb{L}^{(q)}\equiv c_s\begin{pmatrix}0 & q_\nu\\ q_\mu & 0\end{pmatrix},\quad \bar{\mathbb{L}}\equiv c_s\begin{pmatrix}0 & \bar{\nabla}_{\perp\nu}\\ \bar{\nabla}_{\perp\mu} & 0\end{pmatrix},\quad \mathbb{Q}^{(q)}\equiv\begin{pmatrix}0 & 0\\ 0 & \gamma_\eta\Delta_{\mu\nu}q^2+\left(\gamma_\zeta+\frac{1}{3}\gamma_\eta\right)q_\mu q_\nu\end{pmatrix},$$

$$\mathbb{K}^{(s)}\equiv\begin{pmatrix}(1+c_s^2+\dot{c}_s)\,\theta & \frac{1}{2c_s}(1+2c_s^2)\,a_\nu\\ \frac{1}{2c_s}(1+2c_s^2)\,a_\mu & \Delta_{\mu\nu}\theta+\theta_{\mu\nu}\end{pmatrix},\quad \mathbb{K}^{(a)}\equiv\begin{pmatrix}0 & -\frac{1-c_s^2-\dot{c}_s}{2c_s}a_\nu\\ \frac{1-c_s^2-\dot{c}_s}{2c_s}a_\mu & -\omega_{\mu\nu}\end{pmatrix},$$

$$\Omega_\lambda^{(s)}\equiv\frac{c_s^2}{2}\begin{pmatrix}2\omega_{\kappa\lambda}q^\kappa & 0\\ 0 & \omega_{\mu\lambda}q_\nu+\omega_{\nu\lambda}q_\mu\end{pmatrix},\quad \Omega_\lambda^{(a)}\equiv\frac{c_s^2}{2}\begin{pmatrix}0 & 0\\ 0 & \omega_{\mu\lambda}q_\nu-\omega_{\nu\lambda}q_\mu\end{pmatrix},$$

$$\mathbb{H}_\lambda\equiv c_s\begin{pmatrix}0 & \partial_\nu u_\lambda\\ \partial_\mu u_\lambda & 0\end{pmatrix},$$

$$\theta^{\mu\nu}=\frac{1}{2}\left(\partial_\perp^\mu u^\nu+\partial_\perp^\nu u^\mu\right),\quad \theta=\theta_\mu^\mu,\quad \omega_{\mu\nu}=\frac{1}{2}(\partial_{\perp\mu}u_\nu-\partial_{\perp\nu}u_\mu).$$

Wigner function equations

go back

Sound-sound

$$(u \pm c_s\hat{q}) \cdot \bar{\nabla} W_\pm - \left(\pm \left(c_s - \frac{\dot{c}_s}{c_s} \right) |q| a_\mu + (\partial_{\perp\mu} u_\nu) q^\nu + 2c_s^2 q^\lambda \omega_{\lambda\mu} \right) \frac{\partial W_\pm}{\partial q_\mu}$$
$$= -\gamma_L q^2 (W_\pm - Tw) - \left((1 + c_s^2 + \dot{c}_s)\theta + \theta_{\mu\nu}\hat{q}^\mu \hat{q}^\nu \pm \frac{1 + 2c_s^2}{c_s} \hat{q} \cdot a \right) W_\pm \,,$$

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Shear-shear

$$u\cdot\bar{\nabla}\widehat{W} = -2q^2\gamma_\eta(\widehat{W} - Tw\widehat{\mathbb{1}}) + (\partial_{\perp\mu}u_\nu)q^\nu\nabla^\mu_{(q)}\widehat{W} - \left\{\widehat{K},\widehat{W}\right\} + \left[\widehat{\Omega},\widehat{W}\right]\,,$$

where

$$\widehat{K}^{ij} \equiv \frac{1}{2}\theta\,\delta^{ij} + \theta^{\mu\nu}t^{(i)}_\mu t^{(j)}_\nu, \quad \text{and} \quad \widehat{\Omega}^{ij} \equiv \omega^{\mu\nu}t^{(i)}_\mu t^{(j)}_\nu, \quad i=1,2;$$

go back

Large q behavior of W

The part which does not lead to UV divergences:

$$\widetilde{W} = W - W^{(0)} - W^{(1)}$$

The equilibrium part (the divergent integral renormalizes EOS):

$$W^{(0)}_{\pm} = Tw \quad \text{and} \quad W^{(0)}_{T_i,T_j} = Tw\delta_{ij}.$$

The first background gradient correction
(integral renormalizes viscosities):

$$\begin{aligned} W^{(1)}_{\pm}(x,q) &= \frac{Tw}{\gamma_L q^2}\left((c_s^2 - \dot{c}_s)\theta - \theta_{\mu\nu}\hat{q}^{\mu}\hat{q}^{\nu}\right) \,, \\ W^{(1)}_{T_iT_j}(x,q) &= \frac{Tw}{\gamma_\eta q^2}\left(c_s^2\theta\,\delta^{ij} - \theta^{\mu\nu}t^{(i)}_{\mu}t^{(j)}_{\nu}\right) \,. \end{aligned}$$

back