Fluctuation-dissipation relation and fluctuation theorem for causal hydrodynamic fluctuations

K. Murase, Annals Phys. **411**, 167969 (2019); KM in preparation

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Koichi Murase (Peking University)

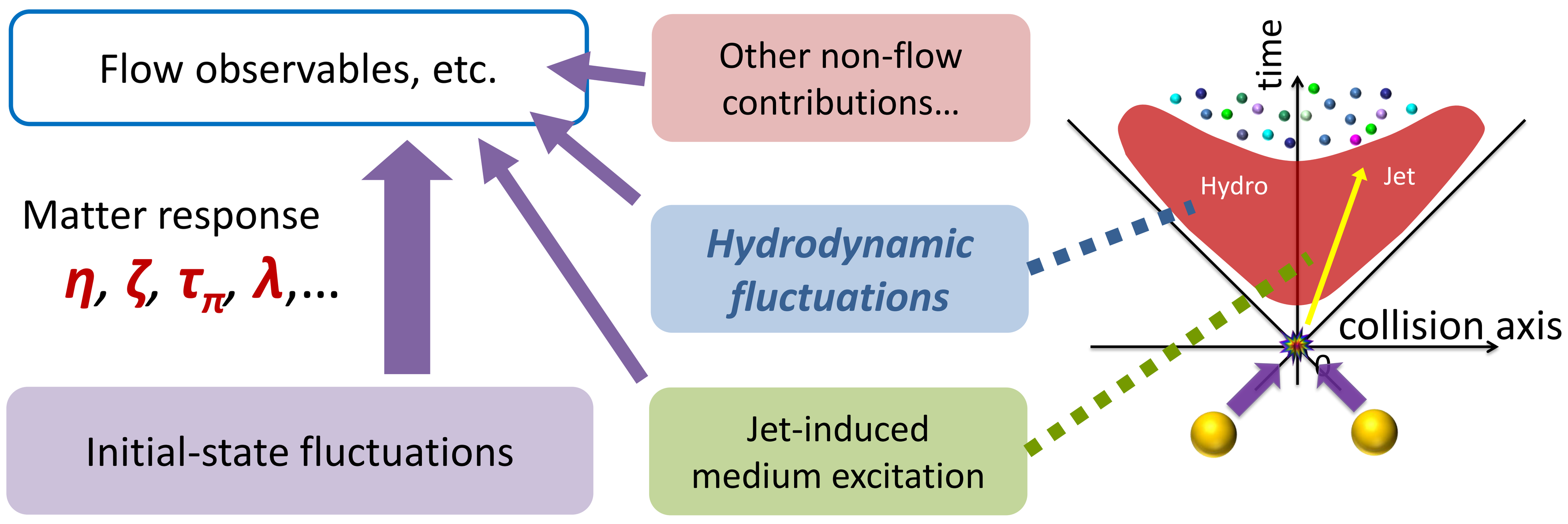
murase@pku.edu.cn
phys.murase@gmail.com

Abstract: To implement *fluctuating hydrodynamics* into dynamical models based *on causal dissipative hydrodynamics*, the *fluctuation-dissipation relation* (*FDR*) in inhomogeneous and non-static background should be considered. **FDR is modified** depending on the adopted constitutive equations. **Without the modification, *fluctuation theorem* (*FT*) is violated.**

1. Background and Motivation

Event-by-event fluctuations

To extract the properties of the matter created in high-energy nuclear collisions, flow observables are important experimental observables to which different kinds of event-by-event fluctuations contribute.



For precise determination of matter properties, all kinds of fluctuations have to be implemented and investigated in dynamical models. Also, for the critical point search especially non-trivial behavior of hydrodynamic fluctuations play an important role.

Hydrodynamic fluctuations: the thermal fluctuations of hydrodynamics

Hydrodynamic fluctuations can be introduced as a noise term in the constitutive equation (*CE*) in the framework of *fluctuating hydrodynamics*. The behavior is determined by *fluctuation-dissipation relation* (*FDR*).

Constitutive equation (e.g. Pressure)

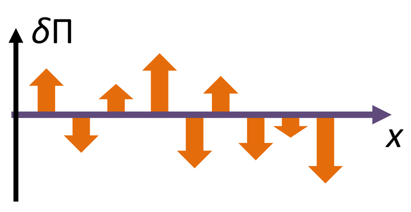
$$P + \Pi = P(e,n) - \zeta\theta \quad + \delta\Pi,$$

EoS+Viscosity — Noise term

Stochastic differential equation ~ *the Langevin equation*

Typical structure of FDR

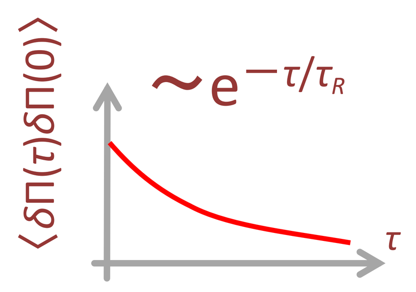
$$\langle \delta\Pi(x)\delta\Pi(y)\rangle = 2T\zeta\delta^{(4)}(x-y)$$

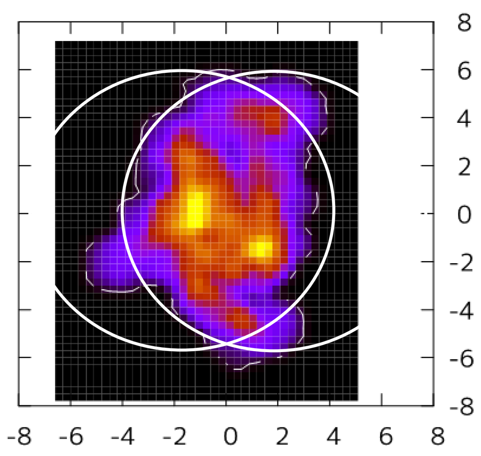


Hydrodynamic fluctuation arises locally in each space-time points

Causal viscous hydrodynamics (2nd-order hydro) used in dynamical models

FDR becomes non-local.
Noise is "colored"





When the system is non-static and inhomogeneous as in high-energy nuclear collisions, we cannot define a single temperature, etc. for the FDR.

? How the FDR for 2nd-order hydrodynamics should be modified in non-static and inhomogeneous matter? ?

2. Method: FDR in inhomogeneous systems

Step 1. Define (linear-response) CE in inhomogeneous background

$$L_x\,\Gamma(x) = M_x\,\kappa(x)F(x) + \xi(x)$$

Γ: $\pi^{\mu\nu}$, Π, ν^μ

Γ: dissipative current — κ: 1st order transport coefficient — ξ: noise term

L_x, M_x: linear operators — F: thermodynamic force

κ: 2η, ζ, λ

F: $\sigma^{\mu\nu}$, θ, $T\nabla^\mu(\mu/T)$

Step 2. Solve CE to obtain the integral form

$$\Gamma(x) = \int dy\, G(x,y)\kappa(y)F(y) + \delta\Gamma(x)$$

δΓ: integral form noise

$$\xi(x) = L_x\delta\Gamma(x)$$

Obtain explicit form of *memory function* $G(x,y) = L_x^{-1} M_x\, \delta(x-y) = \ldots$

Step 3. Use FDR based on non-equilibrium statistical operator method

"Integral-form FDR"

$$\langle \delta\Gamma(x)\delta\Gamma(y)\rangle\Theta(x^0-y^0) = G(x-y)T(y)\kappa(y)$$

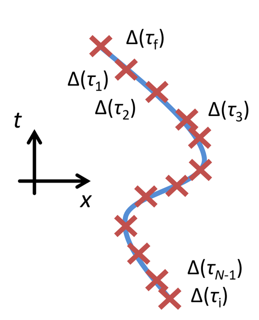
Obtain autocorrelations for integral form noise δΓ(x)

D. N. Zubarev, Nonequilibrium Statistical Thermodynamics (Plenum, New York, 1974).
A. Hosoya, M.-a. Sakagami, and M. Takao, Annals Phys. 154, 229 (1984), and many...

Step 4. Obtain noise autocorrelation in original CE

"Differential-form FDR"

$$\langle \xi(x)\xi(y)\rangle = L_x L_y \langle \delta\Gamma(x)\delta\Gamma(y)\rangle = \ldots$$



3. Result: e.g. Simplified Israel-Stewart case

Input CE: simplified IS eqs.

$$(1+\tau_\Pi \mathcal{D})\Pi = -\zeta\theta + \xi_\Pi, \quad \text{Bulk pressure}$$
$$(1+\tau_\pi \mathcal{D})\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \xi_\pi^{\mu\nu}, \quad \text{Shear stress}$$
$$(\delta_{ij} + \tau_{ij}\mathcal{D})\nu_j^\mu = \kappa_{ij} T\nabla^\mu \frac{\mu_j}{T} + \xi_i^\mu. \quad \text{Diffusion current}$$

Defs: substantial time derivatives
$$\mathrm{D} = u^\mu\partial_\mu,$$
$$\mathcal{D}\pi^{\mu\nu} = \Delta^{\mu\nu}{}_{\alpha\beta}\,\mathrm{D}\,\pi^{\alpha\beta},$$
$$\mathcal{D}\nu_i^\mu = \Delta^\mu{}_\alpha\,\mathrm{D}\,\nu_i^\alpha.$$

Result (FDR)

Complicated expression due to the tensor structure…

$$\langle \xi_\Pi(x)\xi_\Pi(x')\rangle = \left(2 + \tau_\Pi \mathrm{D}\ln\frac{T\zeta}{\tau_\Pi} - \tau_\Pi\theta\right) T\zeta\delta^{(4)}(x-x'),$$
$$\langle \xi_\pi^{\mu\nu}(x)\xi_\pi^{\alpha\beta}(x')\rangle = 2\left[\left(2 + \tau_\pi \mathrm{D}\ln\frac{T\eta}{\tau_\pi} - \tau_\pi\theta\right)\Delta^{\mu\nu\alpha\beta} + \tau_\pi\mathcal{D}\Delta^{\mu\nu\alpha\beta}\right] T\eta\delta^{(4)}(x-x'),$$
$$\langle \xi_i^\mu(x)\xi_j^\alpha(x')\rangle = -2T\kappa_{ij}\Delta^{\mu\alpha}\delta^{(4)}(x-x')$$
$$- \Delta^{\mu\alpha}[K_{ij}^{\mathrm{A}}(x)\mathcal{D} - K_{ij}^{\mathrm{A}}(x')\mathcal{D}']\delta^{(4)}(x-x')$$
$$+ \sum_{k=1}^{n}\left\{-\Delta^{\mu\alpha}\left[\tau_{ik}\mathrm{D}T\kappa_{kj} - (\mathrm{D}\tau_{ik})T\kappa_{kj} - \tau_{ik}\theta T\kappa_{kj}\right]^{\mathrm{S}} - K_{ij}^{\mathrm{S}}\mathcal{D}\Delta^{\mu\alpha}\right\}\delta^{(4)}(x-x'),$$

where $K_{ij}^{\mathrm{S/A}}(x) = \sum_{k=1}^n T(x)(\tau_{ik}(x)\kappa_{kj}(x) \pm \tau_{jk}(x)\kappa_{ki}(x))/2$, and $[\circ_{ij}]^{\mathrm{S}} = (\circ_{ij} + \circ_{ji})/2$.

Essential structure is…

$$\langle \xi(x)\xi(x')\rangle = \left(2 + \tau_R \mathrm{D}\ln\frac{T\kappa}{\tau_R} - \tau_R\theta\right) T\kappa\delta^{(4)}(x-x').$$

✓ Modification to FDR ∝ Relaxation time τ_R and expansion θ

Important in **HIC** with the short-scale (∼τ_R) dynamics of rapidly expanding system

4. Fluctuation theorem & FDR modification

Fluctuation theorem

Fluctuation theorem (*FT*), known in the non-equilibrium statistical mechanics, describes the entropy production distribution in non-equilibrium processes. FT contains FDR as a special case, i.e., FT is a more general tool for non-equilibrium processes.

Typical structure of FT

D. J. Evans, E. G. D. Cohen, G. P. Morriss, Phys. Rev. Lett. 71, 2401–2404 (1993)

$$\ln\frac{\Pr(\delta S = \alpha)}{\Pr^\dagger(\delta S^\dagger = -\alpha)} = \alpha$$

δS: entropy production
Pr: probability density
†: for inverse process

➡ How about FT in causal hydro?

E-by-E 0+1D simulations of Bjorken flow (expanding system)

Setup: Dynamical eqs.

$$\frac{de}{d\tau} = -\frac{e+p}{\tau}\left(1 - \frac{\pi - \Pi}{sT}\right)$$
$$\left(\tau_\pi\frac{d}{d\tau} + 1\right)\pi = \frac{4\eta}{3\tau} + \xi_\pi,$$
$$\left(\tau_\Pi\frac{d}{d\tau} + 1\right)\Pi = -\frac{\zeta}{\tau} + \xi_\Pi.$$

FDR: Case 1: Without modification

$$\langle \xi_\pi(\tau)\xi_\pi(\tau')\rangle = \frac{8T\eta\delta(\tau-\tau')}{3\tau\Delta\eta_s\Delta x\Delta y},$$
$$\langle \xi_\Pi(\tau)\xi_\Pi(\tau')\rangle = \frac{2T\zeta\delta(\tau-\tau')}{\tau\Delta\eta_s\Delta x\Delta y}.$$

$\Delta\eta_s\Delta x\Delta y$: considered fluid element volume

Case 2: With modification

Modification

$$\langle \xi_\pi(\tau)\xi_\pi(\tau')\rangle = \frac{8T\eta\delta(\tau-\tau')}{3\tau\Delta\eta_s\Delta x\Delta y}\left(1 - \frac{T\eta}{2}\partial_\mu\frac{\tau_\pi u^\mu}{T\eta}\right),$$
$$\langle \xi_\Pi(\tau)\xi_\Pi(\tau')\rangle = \frac{2T\zeta\delta(\tau-\tau')}{\tau\Delta\eta_s\Delta x\Delta y}\left(1 - \frac{T\zeta}{2}\partial_\mu\frac{\tau_\Pi u^\mu}{T\zeta}\right).$$

Check: ***Steady-state FT*** (a version of FT)

T. Hirano, R. Kurita, K. Murase, Nucl. Phys. A 984 (2019) 44-67

$$\ln\frac{\Pr(\bar\sigma = \alpha)}{\Pr(\bar\sigma = -\alpha)} = \alpha t + O(\tau_R/t)$$

σ: entropy production rate $\bar\sigma = \dfrac{s(\tau) - s(\tau_0)}{\tau - \tau_0}$

⟷ *When the distribution is Gaussian*

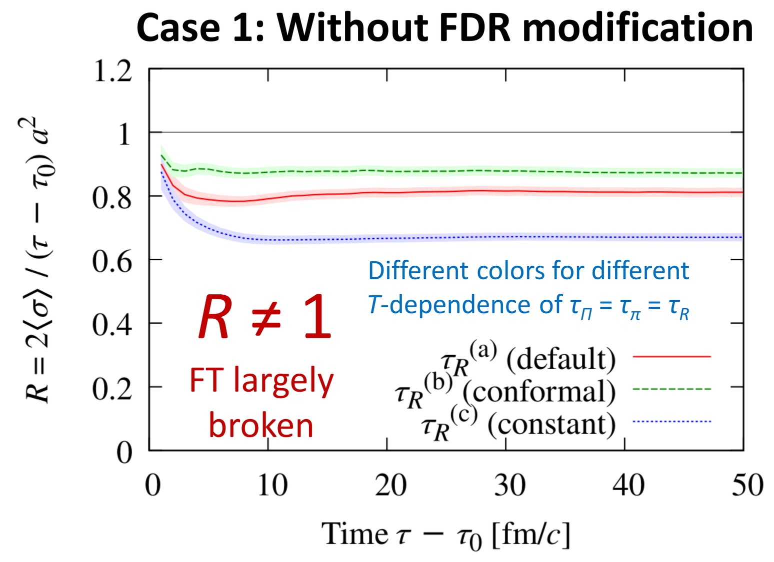
Calculate this quantity for FT

$$R := \frac{2\langle\bar\sigma\rangle}{a^2\cdot(\tau-\tau_0)} \to 1.$$

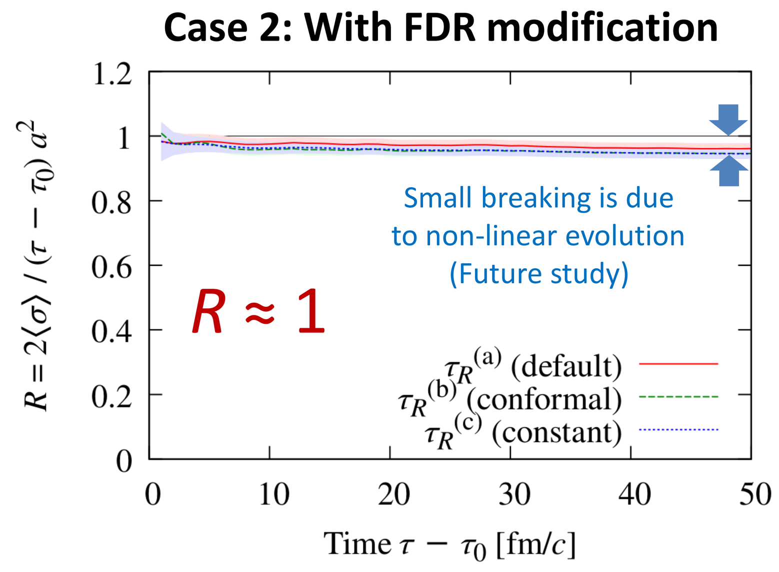
a^2: variance $a^2 = \langle\bar\sigma^2\rangle - \langle\bar\sigma\rangle^2$.

Result

Case 1: Without FDR modification



Case 2: With FDR modification



✓ FDR modification is needed to satisfy FT

5. Summary

- FDR has modification for causal (2nd-order) hydrodynamics
- Without FDR modification, FT breaks down
- **FDR modification has to be implemented in dynamical models with hydrodynamic fluctuations in causal hydrodynamics**