Analytical solutions and attractors of viscous hydrodynamics for Bjorken flow

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Abstract

Causal higher-order theories of relativistic viscous hydrodynamics for Bjorken flow is considered and the associated dynamical attractor is studied. We demonstrate the presence of attractor even in the minimal causal Maxwell-Cattaneo theory and find analytical solutions for the energy density and shear stress for higher-order theories. By studying these solutions we characterize and uniquely determine the attractors. We find exponential decay of initial state memory for small Knudsen numbers, while power-law decay of deviations from attractor for large Knudsen numbers.

1. Introduction

Relativistic viscous hydrodynamics has been very successful in explaining a wide range of collective phenomena observed in heavy-ion collisions. Based on the paradigm that hydrodynamics requires local thermodynamic equilibrium to be applicable, this successful hydrodynamic description led to the belief that these collisions create a nearly thermalized medium close to local thermal equilibrium. However, numerical dissipative relativistic fluid dynamics provides evidence of large deviations from local thermal equilibrium. This "unreasonable effectiveness" of hydrodynamics has generated much recent interest in the very foundations of fluid dynamics, culminating in the formulation of a new "far-from-local-equilibrium fluid dynamics" paradigm [1, 2]. The present work is a contribution to this ongoing discussion, adding new analytic results for the heavily studied simple case of (0+1)-dimensional Bjorken expansion.

2. Attractor in Minimal-causal theory (Maxwell-Cattaneo)

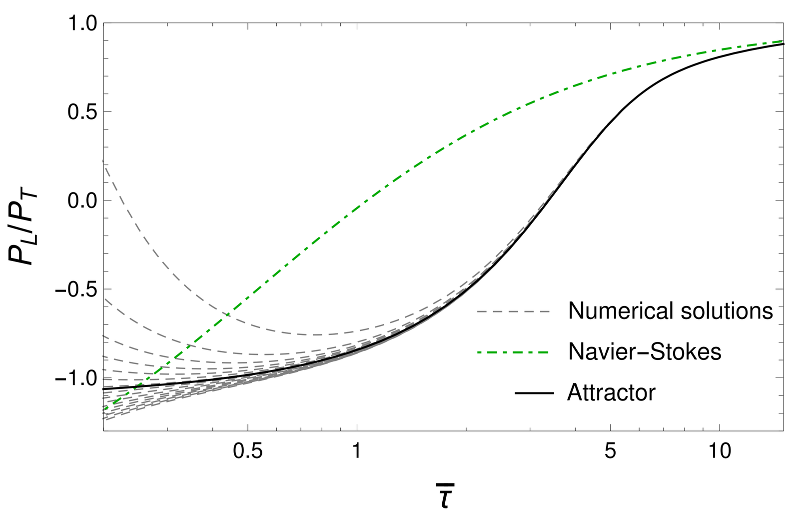
Shear evolution equation for minimal causal Maxwell-Cattaneo theory [3]: $\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$.

Energy density and shear evolution equations for a transversally homogeneous and longitudinally boost-invariant conformal system reduces to:

$$\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}\left(\frac{4}{3}\epsilon - \pi\right), \qquad \frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \frac{4}{3}\frac{\beta_\pi}{\tau}. \tag{1}$$

Where $\beta_\pi \equiv \eta/\tau_\pi = 4\epsilon/15$. Eq.(1) in terms of the normalized shear stress $\bar{\pi} = \pi/(\epsilon+P)$ and rescaled time variable $\bar{\tau} \equiv \tau/\tau_\pi$:

$$\frac{d\bar{\tau}}{d\tau} = \left(\frac{\bar{\pi}+2}{3}\right)\frac{\bar{\tau}}{\tau}, \qquad \left(\frac{\bar{\pi}+2}{3}\right)\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{1}{\bar{\tau}}\left(\frac{4}{15} + \frac{4}{3}\bar{\pi} - \frac{4}{3}\bar{\pi}^2\right). \tag{2}$$



Attractor in Maxwell-Cattaneo:

$$\frac{P_L}{P_T} = \frac{P-\pi}{P+\pi/2} = \frac{1-4\bar{\pi}}{1+2\bar{\pi}}$$

Numerical solutions for a broad range of initial conditions join the attractor at $\bar{\tau} \sim 2$, but that start to agree with the NS solution only for $\bar{\tau} \gtrsim 20$.

3. Higher-order theories

The energy density and shear evolution equations for Müller-Israel-Stewart (MIS)[4, 5], Denicol-Niemi-Molnar-Rischke (DNMR) [6] and third-order [7] theories for Bjorken flow can be brought into the generic form:

$$\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}\left(\frac{4}{3}\epsilon - \pi\right), \quad \frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \frac{1}{\tau}\left[\frac{4}{3}\beta_\pi - \left(\lambda + \frac{4}{3}\right)\pi - \chi\frac{\pi^2}{\beta_\pi}\right]. \tag{3}$$

Eq.(3) in terms of the normalized shear stress $\bar{\pi} = \pi/(\epsilon+P)$ and $\bar{\tau} \equiv \tau/\tau_\pi$:

$$\frac{d\bar{\tau}}{d\tau} = \left(\frac{\bar{\pi}+2}{3}\right)\frac{\bar{\tau}}{\tau}, \qquad \left(\frac{\bar{\pi}+2}{3}\right)\frac{d\bar{\pi}}{d\bar{\tau}} = -\bar{\pi} + \frac{1}{\bar{\tau}}\left(a - \lambda\,\bar{\pi} - \gamma\,\bar{\pi}^2\right). \tag{4}$$

Theory	β_π	a	λ	χ	γ
MIS	$4P/5$	$4/15$	0	0	$4/3$
DNMR	$4P/5$	$4/15$	$10/21$	0	$4/3$
Third-order	$4P/5$	$4/15$	$10/21$	$72/245$	$412/147$

The coefficients β_π, a, λ, χ, and γ for three theories– MIS, DNMR and Third-order.

4. Approximate analytical solutions

Eq. (3) in terms of temperature $T(\tau)$ and inverse Reynolds number $\bar{\pi}(\tau)$:

$$\frac{dT}{d\tau} = \frac{T}{3\tau}(\bar{\pi} - 1), \qquad \frac{d\bar{\pi}}{d\tau} = -\frac{\bar{\pi}}{\tau_\pi} + \frac{1}{\tau}\left(a - \lambda\bar{\pi} - \gamma\bar{\pi}^2\right). \tag{5}$$

Eqs. (5) are coupled as $T\tau_\pi = 5\bar{\eta} = \text{const}$.

We obtain analytical solutions for the following three approximations:

- **Constant relaxation time [8]:** $\tau_\pi(\tau) = \text{const}$.
- **Relaxation time from ideal hydrodynamics:**

Temperature from ideal fluid law: $T_{\rm id} = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3} \implies \tau_\pi(\tau) = b\tau^{1/3}$.

- **Relaxation time from Navier-Stokes evolution:**

Temperature from NS: $T_{\rm NS} = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3}\left[1 + \frac{2\bar{\eta}}{3\tau_0 T_0}\left\{1 - \left(\frac{\tau_0}{\tau}\right)^{2/3}\right\}\right]$

$$\implies \tau_\pi(\tau) = \frac{\tau^{1/3}}{d - \frac{2}{15}\tau^{-2/3}}.$$

Analytical solutions for $\bar{\pi}(w)$ and $\epsilon(w)$:

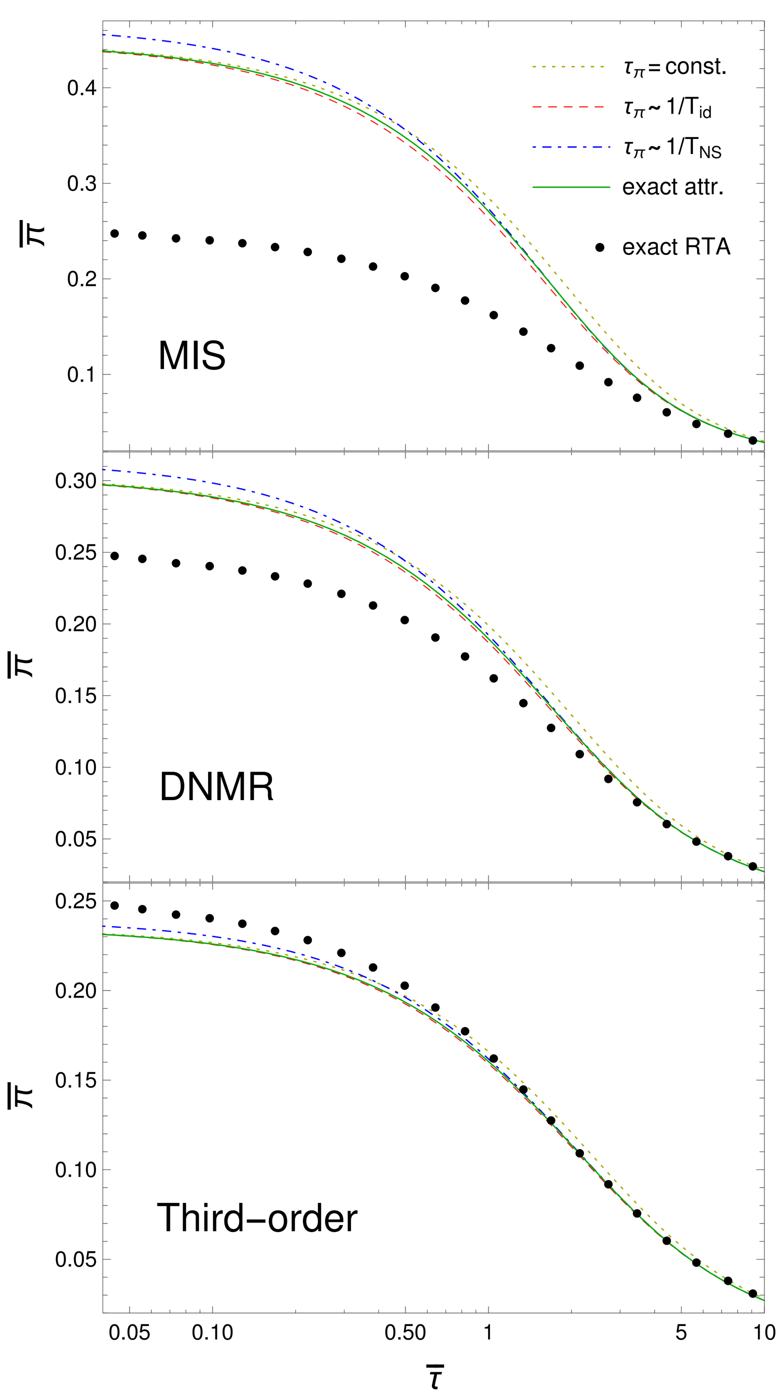
$$\bar{\pi}(w) = \frac{(k+m+\frac{1}{2})M_{k+1,m}(w) - \alpha\,W_{k+1,m}(w)}{\gamma|\Lambda|\left[M_{k,m}(w) + \alpha\,W_{k,m}(w)\right]}, \tag{6}$$

$$\epsilon(w) = \epsilon_0\left(\frac{w_0}{w}\right)^{\frac{4}{3}\left(|\Lambda| - \frac{k}{\gamma}\right)} e^{-\frac{2}{3\gamma}(w-w_0)}\left(\frac{M_{k,m}(w) + \alpha\,W_{k,m}(w)}{M_{k,m}(w_0) + \alpha\,W_{k,m}(w_0)}\right)^{\frac{4}{3\gamma}} \tag{7}$$

$T(\tau)$	w	Λ	k	m
const.	$\bar{\tau}$	-1	$-\frac{1}{2}(\lambda+1)$	$\frac{1}{2}\sqrt{4a\gamma+\lambda^2}$
ideal	$\frac{3}{2}\bar{\tau}$	$-\frac{3}{2}$	$-\frac{1}{4}(3\lambda+2)$	$\frac{3}{4}\sqrt{4a\gamma+\lambda^2}$
NS	$\frac{3}{2}\left(\bar{\tau}+\frac{a}{2}\right)$	$-\frac{3}{2}$	$-\frac{1}{4}\left(3\left(\lambda-\frac{a}{2}\right)+2\right)$	$\frac{3}{4}\sqrt{4a\gamma+\left(\lambda-\frac{a}{2}\right)^2}$

Arguments and parameters of Eqs. (6-7). α encodes the initial condition $\bar{\pi}_0$.

5. Analytical attractors



- **Uniquely determining attractor:**

We propose the quantity–

$$\psi(\alpha_0) \equiv \lim_{\bar{\tau}\to 0}\frac{\partial\bar{\pi}}{\partial\alpha}\bigg|_{\alpha=\alpha_0} \tag{8}$$

diverges at α_0 which corresponds to attractor. $\alpha_0 = 0$ for cases studied here. Therefore attractor solution:

$$\bar{\pi}_{\rm attr}(w) = \frac{k+m+\frac{1}{2}}{\gamma|\Lambda|}\frac{M_{k+1,m}(w)}{M_{k,m}(w)}. \tag{9}$$

- **Convergence of IC at large $\bar{\tau}$:**

From Sol. (6) obtained, separation between two curves:

$$\delta\bar{\pi}(\bar{\tau}) \approx w^{2k}e^{-|\Lambda|w} \tag{10}$$

Λ is the Lyapunov exponent. We see exponential decay for large $\bar{\tau}$.

- **Convergence of IC at small $\bar{\tau}$:**

For large Knudsen number:

$$\delta\bar{\pi}(\bar{\tau}) \approx w^{-2m} \tag{11}$$

We see power law decay for small $\bar{\tau}$ [9].

Approximate solutions obtained are surprisingly accurate and they yield valuable insights into the details of initial state memory loss.

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