Search for the QCD phase transition

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Student Lecture
Quark Matter 2019

“A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it.”

A. Einstein
• Phase Transitions
• Cumulants: What are they and why are they useful
• Some preliminary experimental results and what they could mean
  - Some tricky experimental issues
  - Comparing data with Theory
  - Cumulants and correlations
• Spinodal instability
• Summary
An old question

Fermi 1953

tera Kelvin

Matter in unusual conditions
discussed for many years ....
gets more colorful ...
What we know about the Phase Diagram

Lattice QCD:
T_c \sim 155 \text{ MeV}
pseudo-critical line up to O(\mu^2)
pressure (EoS) up to O(\mu^6)

155\text{MeV}  

T

Nuclear Liquid-Gas

\sim 920 \text{ MeV}

\mu

Theory, Measurements
What we “hope” for

T

155 MeV

Cross over transition

μ

~920 MeV

Nuclear
Liquid-Gas

NB: critical point of water is at T=647K and p=22.06 MPa
Is there a critical point?
Google finds everything...
Phase Transitions

Examples:
Water - vapor (liquid - gas)
Water - ice
Ferromagnet

Order parameter: Tells in which phase the system is
Examples?

Control parameter: Moves system from one phase to another
Examples?

Phase co-existence: Two or more phases can exist together
Examples?
Phase Co-Existence

Water-vapor co-existence
a.k.a your water kettle

Ferro-magnet
Weiss domains
Free Energy:
\[ \Omega = \Omega(T, \mu; \Phi) \]

**Φ**: Order parameter

What we are used to:
One minimum
Free Energy:

\[ \Omega = \Omega(T, \mu; \Phi) \]

\( \Phi \): Order parameter

1\textsuperscript{st} order phase co-existence

Diagram showing the relationship between temperature (T), chemical potential (\( \mu \)), and free energy (\( \Omega \)).
Free Energy:

\[ \Omega = \Omega(T, \mu; \Phi) \]

\( \Phi \): Order parameter

In “dense” phase (close to transition)
Free Energy

Free Energy:
\[ \Omega = \Omega(T, \mu; \Phi) \]
\( \Phi \): Order parameter

In “dilute” phase (close to transition)
At the critical point

Free Energy:
\[ \Omega = \Omega(T, \mu; \Phi) \]

\( \Phi \): Order parameter
Free Energy:
\[ \Omega = \Omega(T, \mu; \Phi) \]
\(\Phi\): Order parameter
Free Energy:

\[ \Omega = \Omega(T, \mu; \Phi) \]

\( \Phi \): Order parameter
Looking for signs of a transition

Cross over transition

Nuclear Liquid-Gas

\( T \)

\( 155 \text{ MeV} \)

\( \sim 920 \text{ MeV} \)

\( \mu \)

\( \mu_c \)
Cumulants and phase structure

What we always see....

What it really means....

“\( T_c \) \sim 160 \text{ MeV}
Derivatives

0th order

1st order

3rd order

5th order
How to measure derivatives

At $\mu = 0$:

$$Z = \text{tr} \ e^{-\hat{E}/T+\mu/T\hat{N}_B}$$

$$\langle E \rangle = \frac{1}{Z} \text{tr} \ \hat{E} \ e^{-\hat{E}/T+\mu/T\hat{N}_B} = -\frac{\partial}{\partial 1/T} \ln(Z)$$

$$\langle (\delta E)^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = \left(-\frac{\partial}{\partial 1/T}\right)^2 \ln(Z) = \left(-\frac{\partial}{\partial 1/T}\right) \langle E \rangle$$

$$\langle (\delta E)^n \rangle = \left(-\frac{\partial}{\partial 1/T}\right)^{n-1} \langle E \rangle$$

Cumulants of Energy measure the temperature derivatives of the EOS
Cumulants of Baryon number measure the chem. pot. derivatives of the EOS
Cumulants of (Baryon) Number

\[ K_n = \frac{\partial^n}{\partial(\mu/T)^n} \ln Z = \frac{\partial^{n-1}}{\partial(\mu/T)^{n-1}} \langle N \rangle \]

\[ K_1 = \langle N \rangle, \quad K_2 = \langle N - \langle N \rangle \rangle^2, \quad K_3 = \langle N - \langle N \rangle \rangle^3 \]

Cumulants scale with volume (extensive): \( K_n \sim V \)

Volume not well controlled in heavy ion collisions

Cumulant Ratios:

\[ \frac{K_2}{\langle N \rangle}, \quad \frac{K_3}{K_2}, \quad \frac{K_4}{K_2} \]
Measuring cumulants (derivatives)

\[ K_2 = \langle N - \langle N \rangle \rangle^2 = \sum_{N} P(N)(N - \langle N \rangle)^2 \]

\[ K_3 = \langle N - \langle N \rangle \rangle^3 = \sum_{N} P(N)(N - \langle N \rangle)^3 \]

\[ P(N) = \frac{N_{\text{events}}(N)}{N_{\text{events}}(\text{total})} \]
Simple model

Change degrees of freedom at phase transition

\[ \langle N \rangle = \text{dof}(\mu) \, e^{\mu/T} \int d^3p e^{-E/T} \]

Degrees of freedom

\[ \langle N \rangle = \text{dof}(\mu) \, e^{\mu/T} \int d^3p e^{-E/T} \]

Change degrees of freedom at phase transition
\( \frac{K_2}{\langle N \rangle} \)

\( \frac{K_3}{K_2} \)

\( \frac{K_4}{K_2} \)

\( \langle N \rangle e^{\mu/T} \)

\( \mu - \mu_c \)
Close to $\mu=0$

$$F = F(r), \quad r = \sqrt{T^2 + a\mu^2}$$

$a \sim$ curvature of critical line

$$\frac{\partial^2}{\partial \mu^2} F(T, \mu)|_{\mu=0} = \frac{a}{T} \frac{\partial}{\partial T} F(T, \mu = 0) \sim \langle E \rangle$$

Needs higher order cumulants (derivatives) at $\mu \sim 0$
Lattice at $\mu=0$

Equation of state

S. Borsanyi et al, JHEP 1011 (2010) 077

$$\frac{\partial^2}{\partial \mu^2} F(T, \mu) \bigg|_{\mu=0} = \frac{a}{T} \frac{\partial}{\partial T} F(T, \mu = 0) \sim \langle E \rangle$$
Cumulants: a closer look

\[ Z = \text{tr} \ e^{-\hat{E}/T + \mu/T \hat{N}_B} \]

\[ K_n = \frac{\partial^n}{\partial(\mu/T)^n} \ln Z = \frac{\partial^{n-1}}{\partial(\mu/T)^{n-1}} \langle N \rangle \quad \text{Cumulants are extensive: } K_n \sim V \]

\[ K_2 = \langle N - \langle N \rangle \rangle^2 = \int d^3x d^3y \langle \delta \rho(x) \delta \rho(y) \rangle ; \quad \delta \rho(x) = \rho(x) - \bar{\rho} \]

Susceptibility:

\[ \chi(2)_{i,j} = \frac{1}{VT^3} \int d^3x d^3y \langle \delta \rho_i(x) \delta \rho_j(y) \rangle = \frac{1}{T^3} \bar{\rho}_i \delta_{i,j} + \frac{1}{T^3} \int d^3r C_{i,j}(r) \]

Correlation function (in configuration space!):\n
\[ C_{i,j}(\vec{r}) = \langle \delta \rho_i(\vec{r}) \delta \rho_j(0) \rangle - \bar{\rho}_i \delta_{i,j} \delta(\vec{r}) \sim \frac{\exp[-r/\xi_{i,j}]}{r} \]

Correlation length (in configuration space!):\n
\[ \xi_{i,j} \]

Relation to cumulant:\n
\[ K_2 = VT^3 \chi(2)_{i,i} \]
Correlation length

\[ C(r) \sim \frac{\exp[-r/\xi]}{r} \]

Static correlation function; “Yukawa” potential with mass:

\[ m \sim \frac{1}{\xi} \]

\[ \chi \sim \int C(r) d^3r \sim \xi^2 \sim \frac{1}{m^2} \]

Critical point (second order)

Cross over

\[ m_\sigma \rightarrow 0, \quad \xi \rightarrow \infty \]

\[ m_\sigma, \xi \quad \text{finite} \]
Critical point

- Second order phase transition
- Fluctuations at all length scales
  - Critical opalescence
Higher moments (cumulants) and $\xi$

Consider probability distribution for the order-parameter field:

$$P[\sigma] \sim \exp \{-\Omega[\sigma]/T\},$$

$$\Omega = \int d^3 x \left[ \frac{1}{2} (\nabla \sigma)^2 + \frac{m_2^2}{2} \sigma^2 + \frac{\lambda_3}{3} \sigma^3 + \frac{\lambda_4}{4} \sigma^4 + \ldots \right]. \quad \Rightarrow \quad \xi = m_{\sigma}^{-1}$$

Moments (connected) of $q = 0$ mode $\sigma_V \equiv \int d^3 x \sigma(x)$:

$$\kappa_2 = \langle \sigma_V^2 \rangle = VT \xi^2; \quad \kappa_3 = \langle \sigma_V^3 \rangle = 2VT^2 \lambda_3 \xi^6; \quad \kappa_4 = \langle \sigma_V^4 \rangle_c \equiv \langle \sigma_V^4 \rangle - 3\langle \sigma_V^2 \rangle^2 = 6VT^3 \left[ 2(\lambda_3 \xi)^2 - \lambda_4 \right] \xi^8.$$

Tree graphs. Each propagator gives $\xi^2$.

Scaling requires “running”: $\lambda_3 = \tilde{\lambda}_3 T (T \xi)^{-3/2}$ and $\lambda_4 = \tilde{\lambda}_4 (T \xi)^{-1}$, i.e.,

$$\kappa_3 = \langle \sigma_V^3 \rangle = 2VT^{3/2} \tilde{\lambda}_3 \xi^{4.5}; \quad \kappa_4 = 6VT^2 \left[ 2(\tilde{\lambda}_3)^2 - \tilde{\lambda}_4 \right] \xi^7.$$
Finite size scaling

Second order (critical point)

\[ \xi \sim V^{2/3}, \quad \chi \sim V^{4/3} \]

(mean field)

Cross over

\[ \xi = \text{const}, \quad \chi = \text{const} \]

NB: 1st order: \( \chi \sim V \)
QCD at $\mu=0$ is cross-over


Figure 3: Continuum extrapolated susceptibilities

$\chi$ finite

$T^4/(m^2 \Delta \chi)$

$\chi$ infinite

large volume

small volume

$1/(T_c^3 V)$

For true phase transitions the infinite volume extrapolation should be consistent with zero, whereas for an analytic crossover the infinite volume extrapolation gives a non-vanishing value. The continuum-extrapolated susceptibilities show no phase-transition-like volume dependence, though the volume changes by a factor of five. The $V \to \infty$ extrapolated value is $22(2)$ which is $11\sigma$ away from zero. For illustration, we fit the expected asymptotic behaviour for first-order and $O(4)$ (second order) phase transitions shown by dotted and dashed lines, which results in chance probabilities of $10^{-19}$ ($7 \times 10^{-13}$), respectively. Error bars are s.e.m with systematic estimates.
The phase diagram

Increase chemical potential by lowering the beam energy

In reality, we add baryons (nucleons) from target and projectile to mid-rapidity
What to expect from experiment?

Below “$T_c$”

Above “$T_c$”
Expectation from Calculations

Characteristic “Oscillating pattern” is expected for the QCD critical point but the exact shape depends on the location of freeze-out with respect to the location of CP

- M. Stephanov, *PRL*107, 052301(2011)
- V. Skokov, Quark Matter 2012

N. Xu, CPOD 2016
K₄/K₂ follows expectation, K₃/K₂ no so much..... URQMD totally fails to get trend for K₄/K₂!
The measurement process

Figure 3: (Color online) Energy dependence of efficiency corrected cumulant ratios $\kappa_2 = \frac{C_4}{C_2}$ and $S_2 = \frac{C_3}{C_2}$ of net-proton distributions in Au+Au collisions at different centralities ($0\sim 5\%$, $5\sim 10\%$, $30\sim 40\%$, $70\sim 80\%$).

Unity at 7.7 GeV. The $S_2$ at $0\sim 5\%$ centrality bin shows a large drop at 7.7 GeV. One may note that we only have statistical errors shown in the figure, which are still large due to limited statistics. The systematical errors, which are dominated by the efficiency correction and the particle identification, are being studied.

Large acceptance is crucial for fluctuations of conserved quantities in heavy-ion collisions to probe the QCD phase transition and critical point. The signals for the phase transition and/or CP will be suppressed with small acceptance. In the Fig. 4, we show the energy dependence of efficiency corrected $\kappa_2$ and $S_2$ of net-proton distributions with various $p_T$ and rapidity range for $0\sim 5\%$ most central Au+Au collisions. The Skellam baseline assumes the protons and anti-protons distribute as independent Poisson distributions. It is constructed from the efficiency-corrected mean values of the protons and anti-protons. It is expected to represent the thermal statistical fluctuations of the net-proton number [24]. The $\kappa_2$ and $S_2$ are to be unity for Skellam baseline as well as in the Hadron Resonance Gas model. In the two upper panels of Fig. 4, when we gradually enlarge the $p_T$ or rapidity acceptance, the values of $\kappa_2$ show a small changes close to unity at energies above 39 GeV, while below 39 GeV, more pronounced structure is observed for a larger $p_T$ or rapidity acceptance. In the two lower panels of Fig. 4, when we enlarge the $p_T$ or rapidity acceptance, the $S_2$ shows strong suppression with respect to unity and monotonically decrease with energy. In contrast to $\kappa_2$, the significant increase above unity at 7.7 GeV is not observed in $S_2$, but shows strong suppression below unity. The published results are shown as solid red triangles in the figure.

The efficiency-corrected net-charge results are shown in Fig. 5. We did not observe non-monotonic behavior for $S_2$ and $\kappa_2$ within current statistics for net-charge. The expectation from...
Or in the real world.....
Modeling the detector (multiplicities only)

Detector maps TRUE number of particles onto OBSERVED number of particles

\[ p(n) = \sum_{N} B(n, N, \epsilon, \ldots) P(N) \]

\[ p_n = B_{n,N} P_N \]

B is matrix which controls the mapping
Figure 3. The need for unfolding. The left panel shows a measured spectrum in a limited region of phase space superimposed with the true distribution that caused the entries in one single measured bin (exemplarily at multiplicity 30 indicated by the line). Clearly the shape of this true distribution depends on the shape of the multiplicity distribution given by the model used (a suggestive example is if the true spectrum stopped at a multiplicity of 40: the true distribution that contributed to the measured multiplicity of 30 would clearly be different, still events at a multiplicity of 30 would be measured). Inversely, in the right panel, the true distribution is shown superimposed with the measured distribution caused by events with the true multiplicity 30 (exemplarily). The shape of this measured distribution still depends on the detector simulation, i.e., the transport code and reconstruction, but not on the multiplicity distribution given by the model (only events with multiplicity 30 contribute to the shown measured distribution).

3.1.2. Unfolding of Multiplicity Distributions

Given a vector $T$ representing the true spectrum, the measured spectrum $M$ can be calculated using the detector response matrix $R$:

$$M = RT.$$  

(34)

The aim of the analysis is to infer $T$ from $M$. Simple weighting, i.e., assuming that a measured multiplicity $m$ is caused 'mostly' by a true multiplicity $t$, would not be correct. This is illustrated in Figure 3. Analogously, adding for each measured $m$ multiplicity the corresponding row of the detector response matrix to the true distribution is also incorrect. This is model-dependent and thus may produce an incorrect result. On the other hand the measured spectrum which is the result of a given true multiplicity is only determined by the detector simulation and is independent of the assumed spectrum.

Given a measured spectrum, the true spectrum is formally calculated as follows:

$$T = R^{-1}M.$$  

(35)

To get TRUE $P(N)$ we need to invert matrix $B$ so that

$$P(N) = \sum_n B^{-1}(N, n, \epsilon, \ldots)p(n)$$

This is called UNFOLDING.

In practice simple inverting does not work!
Or in the real world…
Example: Binomial

\[
p_n = \begin{pmatrix}
p(0) \\
p(1) \\
p(2) \\
p(3) \\
p(4)
\end{pmatrix} = \begin{pmatrix}
1 & 1 - \epsilon & (1 - \epsilon)^2 & (1 - \epsilon)^3 & (1 - \epsilon)^4 \\
0 & \epsilon & 2\epsilon(1 - \epsilon) & 3\epsilon(1 - \epsilon)^2 & 4\epsilon(1 - \epsilon)^3 \\
0 & 0 & \epsilon^2 & 3\epsilon^2(1 - \epsilon) & 6\epsilon^2(1 - \epsilon)^2 \\
0 & 0 & 0 & \epsilon^3 & 4\epsilon^3(1 - \epsilon) \\
0 & 0 & 0 & 0 & \epsilon^4
\end{pmatrix}
\begin{pmatrix}
P(0) \\
P(1) \\
P(2) \\
P(3) \\
P(4)
\end{pmatrix}
\]

Binomial probability \( \epsilon < 1 \) is often called “efficiency”

Theoretically: \( n_{Obs.} \leq N_{True} \Rightarrow B \) is triangular

\( B_{n,N} \) almost singular! \( \text{STAR: } 0.6<\epsilon<0.8 \)

In Practice: Who knows… is the detector even “binomial”
Binomial allows to invert (at least for cumulants)

\[
\frac{K_4}{K_2} = 5
\]

STAR acceptace (protons)

\[
\frac{K_4}{K_2} = 1
\]

\[
\frac{K_4}{K_2} = -1
\]

\[
\frac{K_4}{K_2} = -5
\]

Fraction of BARYONS observed
Is $B(n,N)$ binomial?

Efficiency depends on multiplicity!
Binomial distributions and real detectors

The most obvious correction:
Multiplicity dependence of efficiency

$$\epsilon(N) = \epsilon_0 + \epsilon'(N - \langle N \rangle)$$

More details:
A. Bzdak, R. Holzmann et al.
arXiv:1603.09057
Other models for $B(n,N)$

**Hypergeometric**

<table>
<thead>
<tr>
<th>Hypergeometric</th>
<th>$\alpha = 0.6$</th>
<th>$\alpha = 1.0$</th>
<th>$\alpha = 2.0$</th>
<th>$\alpha = 5.0$</th>
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</thead>
<tbody>
<tr>
<td>$K_3/K_2$</td>
<td>1.16</td>
<td>1.12</td>
<td>1.07</td>
<td>1.03</td>
</tr>
<tr>
<td>$K_4/K_2$</td>
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<td>0.88</td>
<td>0.98</td>
<td>1.00</td>
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<tr>
<td>$K_5/K_2$</td>
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<td>1.68</td>
<td>1.23</td>
<td>1.05</td>
</tr>
<tr>
<td>$K_6/K_2$</td>
<td>-3.99</td>
<td>-1.38</td>
<td>0.31</td>
<td>0.89</td>
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</tbody>
</table>

**Beta Binomial**

<table>
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<tr>
<th>Beta-binomial</th>
<th>$\alpha = 30$</th>
<th>$\alpha = 60$</th>
<th>$\alpha = 150$</th>
<th>$\alpha = 1000$</th>
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<tbody>
<tr>
<td>$K_3/K_2$</td>
<td>1.28</td>
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<td>$K_4/K_2$</td>
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<td>1.07</td>
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<td>$K_5/K_2$</td>
<td>-1.11</td>
<td>1.15</td>
<td>1.63</td>
<td>1.16</td>
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<tr>
<td>$K_6/K_2$</td>
<td>5.71</td>
<td>-0.44</td>
<td>1.80</td>
<td>1.32</td>
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Insights from Theory
Compare Data with Lattice QCD and other field theoretical models
Compare Data with Lattice QCD and other field theoretical models

• Lattice cannot calculate hadron abundances
• Cumulants are well defined quantities
• Compare cumulants !?
  - Baryon number conservation
  - Experiment measures protons not all baryons
  - Volume is not fixed in experiment
  - Experiment has finite momentum space coverage (usually)
Compare Data with Lattice QCD

For example: Wuppertal-Budapest (arXiv:1305.5161)
(similar from Hot QCD)
Baryon number conservation

**Lattice** works in grand-canonical ensemble: Baryon number conserved only on average

**Experiment:** Baryon number is conserved event-by-event

No physics other than baryon number conservation

Bzdak et al. arXiv:1203.4529

\[ R_{4,2} = \frac{K_4}{K_2} \]

\[ R_{6,2} = \frac{K_6}{K_2} \]
Protons vs Baryons

Fast isospin exchange
a.k.a lots of pions:
protons and neutrons follow binomial distribution

\[ P(N_p) = \frac{B!}{N_p!(B-N_p)!} p^{N_p} (1-p)^{B-N_p} \]

with \( p \approx 0.5 \)

(Kitazawa, Asakawa arXiv:1107.2755)
Finite acceptance

Example: “Charge” susceptibility

\[ \chi_Q = \int d^3x \langle \rho(x) \rho(0) \rangle = \int d^3p \langle \tilde{\rho}(p) \tilde{\rho}(0) \rangle \]

Equivalence of *Integrated* coordinate space and momentum space correlation function

Experiment almost never integrates ALL of momentum space!

Lattice (hopefully) does integrate over all coordinate space
Correlations: Lattice vs Data

\[
\langle n(y_1)(n(y_2) - \delta(y_1 - y_2)) \rangle = \langle n(y_1) \rangle \langle n(y_2) \rangle [1 + C(y_1, y_2)]
\]

\[
C(y_1, y_2) \sim \exp \left( -\frac{(y_1 - y_2)^2}{2\sigma^2} \right)
\]

\[
\frac{\langle (\delta N)^2 \rangle}{\langle N \rangle} = 1 + \langle N \rangle \int_{\Delta/2}^{\Delta/2} C(y_1, y_2) dy_1 dy_2
\]

\[
\frac{\langle (\delta N)^2 \rangle}{\langle N \rangle} = 1.5
\]

"Lattice result"

"Charge conservation"
Dependence on Rapidity window

- Kurtosis depends strongly on Rapidity window
- Comparison with Lattice:
  - Lattice catches the full correlation length
  - Need to expand rapidity window until signal saturates (after correcting for charge conservation)

Any comparison of Lattice to Data needs to assure that cumulants reach asymptotic value in experiment.
So far this has NOT ben established for proton cumulants
K₄/K₂ follows expectation, K₃/K₂ no so much…..
URQMD totally fails to get trend for K₄/K₂!
Further insights: Correlations

Cumulants

\[ K_n = \frac{\partial^n}{\partial(\mu/T)^n} \ln Z \]

\[ K_2 = \langle N - \langle N \rangle \rangle^2 = \langle (\delta N)^2 \rangle \]

\[ \rho_2(p_1, p_2) = \rho_1(p_1)\rho_1(p_2) + C_2(p_1, p_2). \]

\[ K_3 = \langle (\delta N)^3 \rangle \]

\[ \rho_3(p_1, p_2, p_3) = \rho_1(p_1)\rho_1(p_2)\rho_1(p_3) + \rho_1(p_1)C_2(p_2, p_3) + \rho_1(p_2)C_2(p_1, p_3) + \rho_1(p_3)C_2(p_1, p_2) + C_3(p_1, p_2, p_3) \]

From Cumulants to Correlations
(no anti-protons)

Defining integrated correlations function

\[ C_n = \int dp_1 \ldots dp_n C_n(p_1, \ldots, p_n) \]

Simple Algebra leads to relation between correlations \( C_n \) and \( K_n \)

\[ C_2 = -K_1 + K_2, \]
\[ C_3 = 2K_1 - 3K_2 + K_3, \]
\[ C_4 = -6K_1 + 11K_2 - 6K_3 + K_4, \]

or vice versa

\[ K_2 = \langle N \rangle + C_2 \]
\[ K_3 = \langle N \rangle + 3C_2 + C_3 \]
\[ K_4 = \langle N \rangle + 7C_2 + 6C_3 + C_4 \]
Significant four particle correlations!

Four particle correlation dominate $K_4$ for central collisions at 7.7 GeV

$K_2 = \langle N \rangle + C_2$

$K_3 = \langle N \rangle + 3C_2 + C_3$

$K_4 = \langle N \rangle + 7C_2 + 6C_3 + C_4$
Correlations

Based on prelim. STAR data

Dip at 19.6 GeV from NEGATIVE $C_2$!
Energy dependence

Based on prelim. STAR data

Note: anti-protons are non-negligible above 19.6 GeV
Rapidity dependence

\[ C_k(\Delta Y) = \int_{\Delta Y} dy_1 \ldots dy_k \rho_1(y_1) \ldots \rho_1(y_k) c_k(y_1, \ldots, y_k) \]

Assume: \[ \rho_1(y) \sim const. \]

Short range correlations:

\[ c_k(y_1, \ldots, y_k) \sim \delta(y_1 - y_2) \ldots \delta(y_{k-1} - y_k) \]

\[ C_k(\Delta Y) \sim \Delta Y \rightarrow K_k \sim \Delta Y \]

Long range correlations:

\[ c_k(y_1, \ldots, y_k) = const. \]

\[ C_k(\Delta Y) \sim (\Delta Y)^k \sim \langle N \rangle^k \]

\[ \Rightarrow K_n = K_n(\langle N \rangle) \]
Long range correlations

\[ C_k = \langle N \rangle^k c_k \]
\[ c_k = \text{const.} \implies K_n = K_n \left( \langle N \rangle \right) \]
Can we understand these correlations?

- Two particle correlations can be understood by simple Glauber model + Baryon number conservation

Four particle correlations are orders of magnitudes larger in the data
Also seen in URQMD calculations by He et al. PLB774 (2017) 623

Need to assume the ~40% of protons come from 8-nucleon cluster in order to get magnitude right!
The same as Fig. 6 and just replace the X-axis with the corresponding mean proton number ($\langle N \rangle$).

Fig. 8. Energy dependence of proton (baryon) cumulants and correlation functions in 0–5% most central Au+Au collisions at $\sqrt{s_{NN}} = 7.7$ to 200 GeV from UrQMD model (black circles) and STAR preliminary data [15,37,56]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

To understand the contributions to the cumulants from different physics effects, we decompose the various order cumulants into multi-particle correlation functions based on the equations (2) and (3). It means that each cumulant in the first column is just equal to the sum of the results in the second and the third columns. It is easily noticed that the strong suppression observed in various order proton (baryon) cumulants from UrQMD at low energies are mainly caused by the negative two-proton correlation functions ($c_2$), which is due to the anti-correlation between proton (baryon) caused by the BNC effects. The results for the three and four-particle correlation functions for protons (baryons) in the UrQMD model show a flat energy dependence and close to zero.
Latest STAR result on net-proton cumulants

X. Luo, NPA 956 (2016) 75

K₄/K₂ above baseline K₃/K₂ below baseline
Shape of probability distribution

\[ K_3 < \langle N \rangle \]
\[ K_4 > \langle N \rangle \]

\[ K_3 = \langle N - \langle N \rangle \rangle^3 \]
\[ K_4 = \langle N - \langle N \rangle \rangle^4 - 3 \langle N - \langle N \rangle \rangle^2 \]
Simple two component model

Weight of small component: \( \sim 0.3\% \)

FIG. 1. The multiplicity distribution \( P(N) \) at \( p_s = 7.7 \text{ GeV} \) in the two component model given by Eq. (1) constructed with (a) efficiency unfolded values for \( h_{Ni} \), \( C_3 \) and \( C_4 \) and (b) with imposed efficiency of 0.65.

It is worth noting that \( C_6 / C_5 \approx C_5 / C_4 \approx C_4 / C_3 \) in agreement with the discussion presented in the previous Section. We note that the resulting \( C_2 \approx 3.85 \) is slightly more negative than the data. However, as pointed out, e.g., in [45], the second order factorial cumulant receives sizable positive contribution from participant fluctuations \( C_2'' \approx 3 \) whereas the correction to \( C_3 \) and \( C_4 \) are small. In view of the sizable errors in the preliminary STAR data we consider the present fit as satisfactory.

The resulting probability distribution, \( P(N) \), Eq. (1), is shown in the left panel of Fig. 1. Even though the component centered at \( N \sim 25 \) has a very small probability \( \rho \sim 0.3\% \) it gives rise to a shoulder at low \( N \) which should be visible in the multiplicity distribution. However, this would require an unfolding of the measured distribution [27] in order to remove the effect of a finite detection efficiency. Assuming a binomial model for the efficiency with a constant detection probability of \( \rho = 0.65 \), which roughly corresponds to that of the STAR measurement, the observed multiplicity distribution of the two component model is shown in the right panel of Fig. 1. In this case the small component \( \rho \) is barely visible. This observation is consistent with the fact that the efficiency uncorrected cumulants measured by STAR are more or less consistent with a Poisson (or binomial to be more precise) expectation.

\[
\begin{align*}
K_2 & = h_{Ni} + C_2, \\
K_5 & = h_{Ni} + 5C_2 + 10C_3 + 10C_4 + 5C_5 + C_6,
\end{align*}
\]

I think we should add a few sentences about centrality but we need to be careful about this sudden jump of \( C_3 \) in 5-10%. The STAR data is not really good for a quantitative discussion...

(VK: why not the centrality dependence? )

(VK: Maybe we can just mumble about the fact that \( C_3 \) is already very small at larger centrality and thus the whole approach is questionable...? )
Simple two component model

Difficult to see in the real data with efficiency $\epsilon = 0.65$

![Graph showing the multiplicity distribution $P(N)$ at $p_s = 7.7$ GeV in the two component model given by Eq. (1) constructed with (a) efficiency unfolded values for $h_{N_i}$, $C_3$, and $C_4$, and (b) with imposed efficiency of 0.65.](image)

The cumulant ratios are read (CHECK PLEASE)

$$K_5/K_2 \approx 3.4$$
$$K_6/K_2 \approx 3.12$$

It is worth noting that $C_6/C_5 \approx C_5/C_4 \approx C_4/C_3$ in agreement with the discussion presented in the previous Section. We note that the resulting $C_2 \approx 3.85$ is slightly more negative than the data. However, as pointed out, e.g., in [45], the second order factorial cumulant receives sizable positive contribution from participant fluctuations $C_2'$ whereas the correction to $C_3$ and $C_4$ are small. In view of the sizable errors in the preliminary STAR data we consider the present fit as satisfactory.

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I think we should add a few sentences about centrality but we need to be careful about this sudden jump of $C_3$ in 5-10%. The STAR data is not really good for a quantitative discussion...

\[ K_2 = h_{N_i} + C_2, \]
\[ K_5 = h_{N_i} + 5C_2 + 15C_3 + 35C_4 + 105C_5 \]
\[ K_6 = h_{N_i} + 31C_2 + 90C_3 + 315C_4 + 105C_5 + C_6. \]
Two component model

\[ P(N) = (1 - \alpha)P(a)(N) + \alpha P(b)(N) \]

\[ \bar{N} = \langle N(a) \rangle - \langle N(b) \rangle \]

\[ C_2 = C_2^{(a)} - \alpha \{ \bar{C}_2 - (1 - \alpha)\bar{N}^2 \} \]

\[ C_3 = C_3^{(a)} - \alpha \{ \bar{C}_3 + (1 - \alpha) [(1 - 2\alpha)\bar{N}^3 - 3\bar{N}\bar{C}_2] \} \]

\[ C_4 = C_4^{(a)} - \alpha \{ \bar{C}_4 - (1 - \alpha) [(1 - 6\alpha + 6\alpha^2)\bar{N}^4 - 6(1 - 2\alpha)\bar{N}^2\bar{C}_2 + 4\bar{N}\bar{C}_3 + 3(\bar{C}_2)^2] \} \]

\[ \bar{C}_n = C_n^{(a)} - C_n^{(b)} , \]

For Poisson, \( C(a), C(b) = 0 \)

Fit to STAR data: \( \langle N(a) \rangle \sim 40, \langle N(b) \rangle \sim 25, \alpha \sim 0.003 \)
Two component model

\[ P(N) = (1 - \alpha)P(a)(N) + \alpha P(b)(N) \]
\[ \tilde{N} = \langle N(a) \rangle - \langle N(b) \rangle > 0 \]

For \( P(a), P(b) \) Poisson, or (to good approximation) Binomial
\[ C_n = (-1)^n K_n^B \tilde{N}^n \quad n \geq 2 \]

\( K_n^B \) : Cumulant of Bernoulli distribution
\[ \alpha \ll 1, K_n^B = \alpha \Rightarrow C_n \simeq \alpha (-1)^n \tilde{N}^n \]

\[ \Rightarrow |C_n| \sim \langle N \rangle^n \] as seen by STAR (i.e. “infinite” correlation length)

\[
\begin{align*}
\frac{C_4}{C_3} &= \frac{C_5}{C_4} = \frac{C_{n+1}}{C_n} = -\tilde{N} \\
\end{align*}
\]

predict: \( \tilde{N} \simeq 15 \)

Clear and falsifiable prediction: \( C_5 \approx -2650 \quad C_6 \approx 41000 \)
This model can be tested RIGHT NOW!

Model prediction:

\[ C_5 = -2645 \pm 0.14, \quad C_6 = 40900 \pm 0.18, \]
\[ C_7 = -615135 \pm 0.26, \quad C_8 = 8520220 \pm 0.42, \]

\[ C_5 = -307 \pm 0.31, \quad C_6 = 3085 \pm 0.41, \]
\[ C_7 = -30155 \pm 0.61, \quad C_8 = 271492 \pm 1.06, \]

Based on 144393 events (same as STAR 0-5% at 7.7 GEV)
Speculation

FIG. 3. Probability distribution at various points close to the co-existence line for the van der Waals model.

This is shown in Fig. 3. The multiplicity distribution extracted from the STAR cumulants, van der Waals model in a finite volume to calculate the multiplicity distributions for various

It is noteworthy that two event classes distribution looks very similar to that of a system

central 5% correspond to 150k events \[38\]. Given

The present STAR dataset for

factorial cumulants should increase leading to a probability distribution which should exhibit

\[N<0\]

\[1.8/T\]

\[0.9\]

\[1.1\]

\[0.8\]

\[1.2\]

\(\mu/\mu_c\)

\(T/T_c\)

\(P(N)\)

Bzdak et al, arXiv:1804.04463
Free Energy

\[ \Omega = \Omega(T, \mu; \Phi) \]

\( \Phi \): Order parameter
Simple two component model

Analyse data for $N_p < 20$
- Is flow etc different?
- “Inspect by eye (<1% of all events)
Co-existence region

System should spent long time in spinodal region

Spinodal instability: Mechanical instability

\[
\frac{\partial P}{\partial \rho} < 0
\]

Exponential growth of clumping

Non-equilibrium phenomenon!
Phase-transition dynamics: Density clumping

Phase transition \(\Rightarrow\) \[
\begin{align*}
\text{Phase coexistence: surface tension} \\
\text{Phase separation: instabilities}
\end{align*}
\]

Introduce a gradient term:

\[ p(r) = p_0(\varepsilon(r), \rho(r)) - C\rho(r)\nabla^2\rho(r) \]

Insert the modified pressure into existing ideal finite-density fluid dynamics code

Use UrQMD for pre-equilibrium stage to obtain fluctuating initial conditions

Simulate central Pb+Pb collisions at \(\approx 3\ \text{GeV/A}\) beam kinetic energy on fixed target, using an Equation of State either with a phase transition or without (Maxwell partner):

**With** phase transition:

**Without** phase transition:

Density enhancement:

Evolution of density moments

\[ \langle \rho^N \rangle = \frac{1}{A} \int \rho(r)^N \rho(r) d^3r \]

J. Steinheimer & J. Randrup, PRL 109, 212301(2012)
PRL 87, 054903 (2013)

\( E_{\text{Lab}} = 3\ \text{GeV} \)
Consider two Equations of State

Steinheimer et al,
PQM ("liquid-gas")

“QCD”
Time evolution

Oscillation of nearly stable droplets for “liquid-gas” EoS

Higher pressure leads to faster evolution of “QCD” EoS.

Cluster a.k.a. nuclei

Even if total baryon number does not fluctuate the baryon density does.

Therefore measure production of NUCLEI: d, $^3$He, $^4$He, $^7$Li....

$$\langle d \rangle \sim \langle \rho^2 \rangle \quad \langle ^3He \rangle \sim \langle \rho^3 \rangle \quad \langle ^7Li \rangle \sim \langle \rho^7 \rangle$$

Extracts higher moments of the baryon density at freeze out

Nice Idea, but...
“Cluster” formation

“QCD” EoS

\[
\left( \frac{S}{B} \right)_{\text{hadron-gas}} < \left( \frac{S}{B} \right)_{\text{QGP-liquid}}
\]

Clumping in coordinate space is compensated by dilution in momentum space \(\rightarrow\) tiny effect

“Deep” learning fails as well....

Coordinate space 😊

Momentum space 😞

Steinheimer et al., arXiv:1906.06562
Summary

• Cumulants measure derivatives of the free energy (equation of state)
  - Sensitive to “wiggles” a.k.a. “remnants” of phase transition
• Experiments are difficult: detector needs to be understood well
• Careful when comparing theory with measured cumulants
• Correlations a.k.a. factorial cumulants provide complementary insights
  - strong four particle correlation at low energies
• Don’t forget the first order phase transition
  - Spinodal instability
• Very active field, both in experiment and theory

VERY INTERESTING TIMES AHEAD