Using dimensional analysis as a measure of fine tuning

Stefan Recksiegel (TUM) Tangier, September 2019

1. Dimensional Analysis
   - Fractal Dimensions
   - How long is the coastline of Britain ?
   - Fine tuning in a Toy Model
2. Physics
   - Examples: SM with 4 generations and Littlest Higgs with T-parity
   - Sommerfeld enhancement in Dark Matter annihilation
3. Conclusions

Dimensional Analysis:
Fractal Dimensions and the coastline of Britain

Fractal Dimensions

Haussdorff dimension

A geometric shape has **Haussdorff dimension** d if the relationship between its mass m and length L is $\boldsymbol{m \propto L^d}$

This coincides with the **“normal life” understanding** of dimensionality for **integer** d.

Fractal Dimensions

Haussdorff dimension

A geometric shape has **Haussdorff dimension** d if the relationship between its mass m and length L is $\boldsymbol{m \propto L^d}$

This coincides with the **“normal life” understanding** of dimensionality for **integer** d. For **Fractals**, d is not an integer.

Fractal Dimensions

Haussdorff dimension

A geometric shape has **Haussdorff dimension** d if the relationship between its mass m and length L is $\boldsymbol{m \propto L^d}$

This coincides with the **"normal life" understanding** of dimensionality for **integer** d. For **Fractals**, d is not an integer.

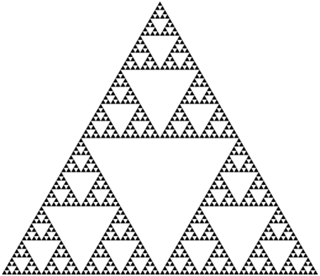
E.g. **Sierpínski triangle**

Fractal Dimensions

Haussdorff dimension

A geometric shape has **Haussdorff dimension** d if the relationship between its mass m and length L is $\boldsymbol{m \propto L^d}$

This coincides with the **“normal life” understanding** of dimensionality for **integer** d. For **Fractals**, d is not an integer.



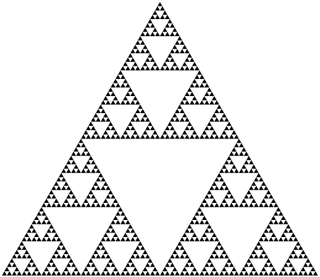
E.g. **Sierpínski triangle**:
mass triples when size doubles
$\rightarrow d = log(3)/log(2) \approx 1.585$

Fractal Dimensions

Haussdorff dimension

A geometric shape has **Haussdorff dimension** d if the relationship between its mass m and length L is $\boldsymbol{m \propto L^d}$

This coincides with the **"normal life" understanding** of dimensionality for **integer** d. For **Fractals**, d is not an integer.



E.g. **Sierpínski triangle**:
mass triples when size doubles
$\rightarrow d = log(3)/log(2) \approx 1.585$

How long is the coastline of Britain ?

How long is the coastline of Britain ?

That depends on the **length of the ruler** that we use!

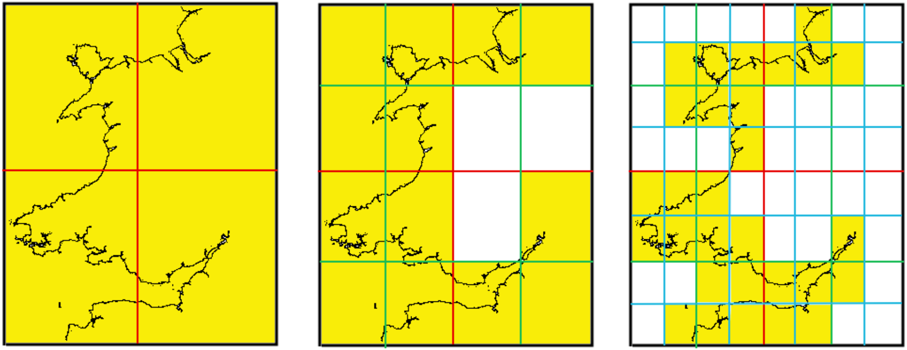
How long is the coastline of Britain ?

That depends on the **length of the ruler** that we use!

How long is the coastline of Britain ?

Unit **200km**: ca. 2400km, Unit **50km**: ca. 3400km.

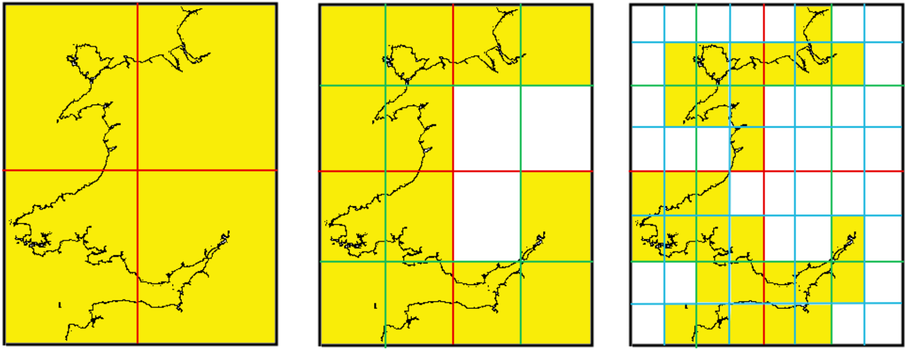
The Box Counting algorithm



1/1 4/4 13/16 28/64

2D: For solid objects, the fill ratio will approach a constant, for a line, it will approach $1/n$ ($n \times n$ boxes). For a fractal, . . .

The Box Counting algorithm

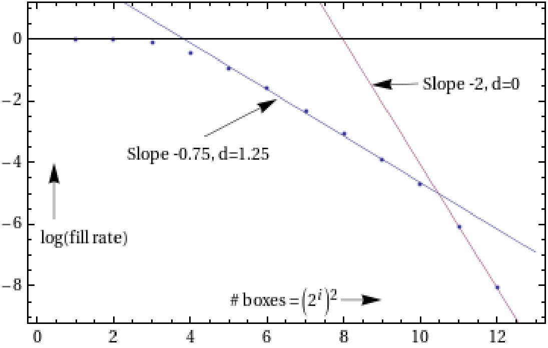


1/1 4/4 13/16 28/64

2D: For **solid** objects, the **fill ratio** will approach a **constant**, for a **line**, it will approach $\mathbf{1/n}$ ($n \times n$ boxes). For a fractal, ...

. . . , we can make a **logarithmic plot** of the fill ratio:

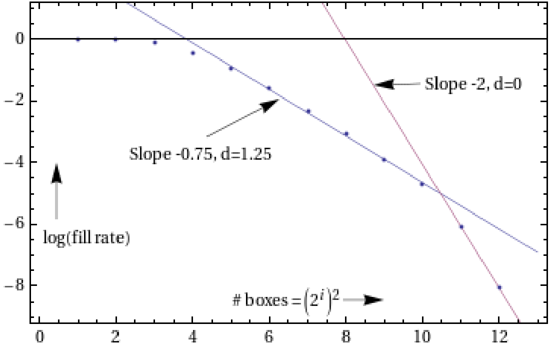
. . . , we can make a **logarithmic plot** of the fill ratio:



The **dimension** of the **British coastline** is **1.25**.

Benoît Mandelbrot, 1967

. . . , we can make a **logarithmic plot** of the fill ratio:



The **dimension** of the **British coastline** is **1.25**.

Benoît Mandelbrot, 1967

"Fine tuning" in a Toy Model

Let us look at two functions, one *well behaved* one and one that requires **fine tuning** in x to bring the **"observable"** (i.e. $f(x)$) into a rather narrow "experimental window" of 0.5 ± 0.005:

$$\sin(x) \qquad \text{and} \qquad \sin(1/x)$$

Traditional definition of fine tuning: Barbieri-Guidice NPB 306:63(1988)

$$\text{BGfine}_{O(x)} = \left| \frac{x}{O^2} \frac{\partial O^2(x)}{\partial x} \right|$$

"Fine tuning" in a Toy Model

Let us look at two functions, one *well behaved* one and one that requires **fine tuning** in x to bring the **"observable"** (i.e. $f(x)$) into a rather narrow "experimental window" of 0.5 ± 0.005:

$$\sin(x) \qquad \text{and} \qquad \sin(1/x)$$

Traditional definition of fine tuning: Barbieri-Guidice NPB 306:63(1988)

$$\text{BGfine}_{O(x)} = \left| \frac{x}{O^2} \frac{\partial O^2(x)}{\partial x} \right|$$

Analytically very easy to calculate, but which value of x to choose?

"Fine tuning" in a Toy Model

Let us look at two functions, one *well behaved* one and one that requires **fine tuning** in x to bring the **"observable"** (i.e. $f(x)$) into a rather narrow "experimental window" of 0.5 ± 0.005:

$$\sin(x) \qquad \text{and} \qquad \sin(1/x)$$

Traditional definition of fine tuning: Barbieri-Guidice NPB 306:63(1988)

$$\mathrm{BGfine}_{O(x)} = \left| \frac{x}{O^2} \frac{\partial O^2(x)}{\partial x} \right|$$

Analytically very easy to calculate, but which value of x to choose?

For $\sin x$, use $x = \sin^{-1} 0.5$, this way $\mathrm{BGfine}_{\sin x} = 1.81$. For $\sin 1/x$, . . .

"Fine tuning" in a Toy Model

Let us look at two functions, one *well behaved* one and one that requires **fine tuning** in x to bring the **"observable"** (i.e. $f(x)$) into a rather narrow "experimental window" of 0.5 ± 0.005:

$$\sin(x) \qquad \text{and} \qquad \sin(1/x)$$

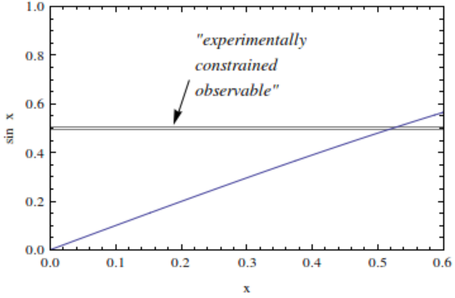
Traditional definition of fine tuning: Barbieri-Guidice NPB 306:63(1988)

$$\text{BGfine}_{O(x)} = \left| \frac{x}{O^2} \frac{\partial O^2(x)}{\partial x} \right|$$

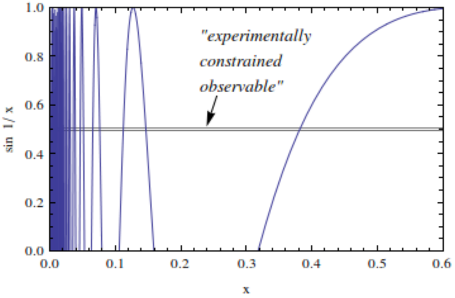
Analytically very easy to calculate, but which value of x to choose?

For $\sin x$, use $x = \sin^{-1} 0.5$, this way $\text{BGfine}_{\sin x} = 1.81$. For $\sin 1/x$, ...

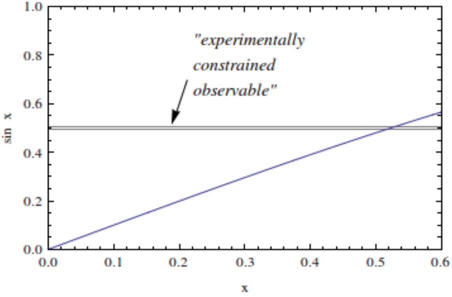
$\sin x$


"experimentally
constrained
observable"
sin x
x
0.0
0.2
0.4
0.6
0.8
1.0
0.1
0.3
0.5

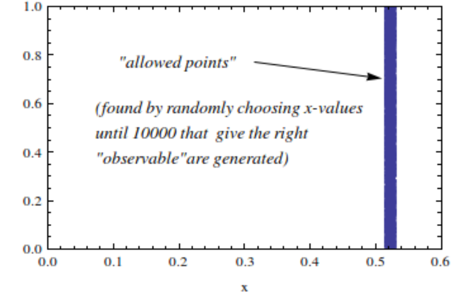
$\sin 1/x$


"experimentally
constrained
observable"
sin 1/ x
x
0.0
0.2
0.4
0.6
0.8
1.0
0.1
0.3
0.5

$\sin x$

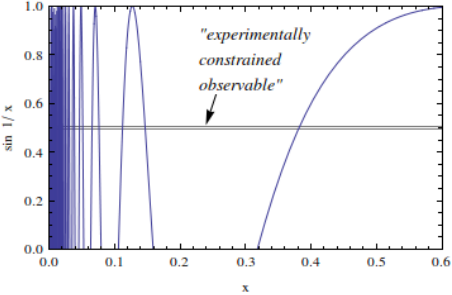


"experimentally
constrained
observable"
sin x
x
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6



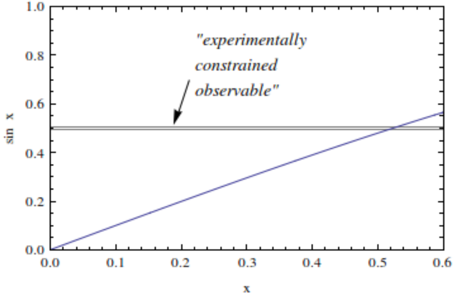
"allowed points"
(found by randomly choosing x-values
until 10000 that give the right
"observable"are generated)
x
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6

$\sin 1/x$

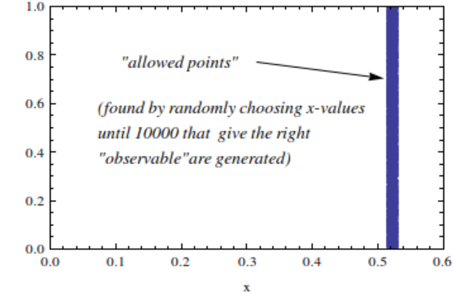


"experimentally
constrained
observable"
sin 1/ x
x
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6

$\sin x$

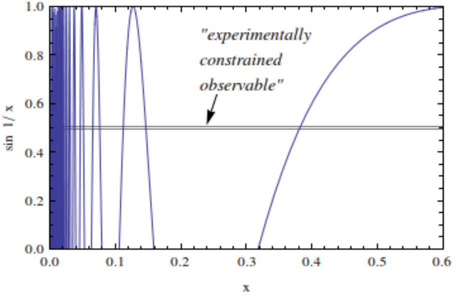


"experimentally
constrained
observable"
sin x
x
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.8
1.0

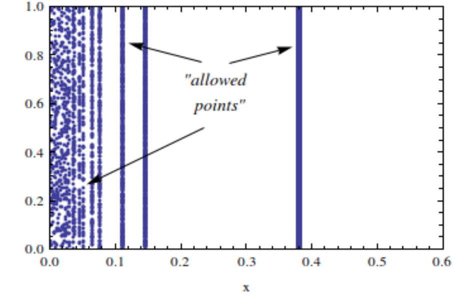


"allowed points"
(found by randomly choosing x-values
until 10000 that give the right
"observable"are generated)
x

$\sin 1/x$



"experimentally
constrained
observable"
sin 1/ x
x



"allowed
points"
x

Average of $\text{BGfine}_{O(x)} = \left| \frac{x}{O^2} \frac{\partial O^2(x)}{\partial x} \right|$ over the generated "valid" points:

$$\text{BGfine}_{\sin x} = 1.81 \ (= \text{naïve result}), \quad \text{BGfine}_{\sin 1/x} = 93.58.$$

Average of $\text{BGfine}_{O(x)} = \left| \frac{x}{O^2} \frac{\partial O^2(x)}{\partial x} \right|$ over the generated "valid" points:

$\text{BGfine}_{\sin x} = 1.81$ (= naïve result), $\text{BGfine}_{\sin 1/x} = 93.58$.

Our proposal: Much easier to calculate (does not depend on $O \rightarrow$ multiple observables!) and at least as instructive:

Box-counting dimension of valid points

No fine tuning: dim $\approx$ # parameters $\leftrightarrow$ Extreme fine tuning: dim ≈ 0

Average of $\text{BGfine}_{O(x)} = \left|\frac{x}{O^2}\frac{\partial O^2(x)}{\partial x}\right|$ over the generated "valid" points:

$\text{BGfine}_{\sin x} = 1.81$ (= naïve result), $\text{BGfine}_{\sin 1/x} = 93.58$.

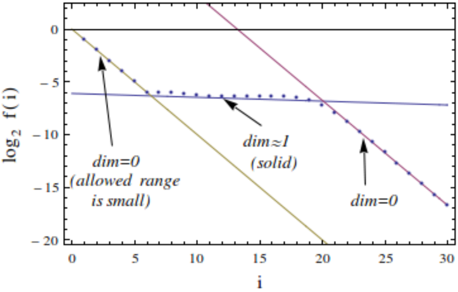
Our proposal: Much easier to calculate (does not depend on $O \to$ multiple observables!) and at least as instructive:

Box-counting dimension of valid points

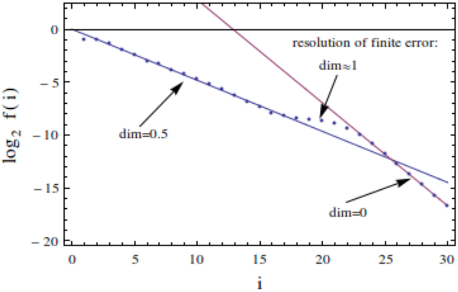
No fine tuning: dim $\approx$ # parameters $\leftrightarrow$ Extreme fine tuning: dim ≈ 0

What are the **dimensionalities** of the allowed ranges for x?

$\sin x$



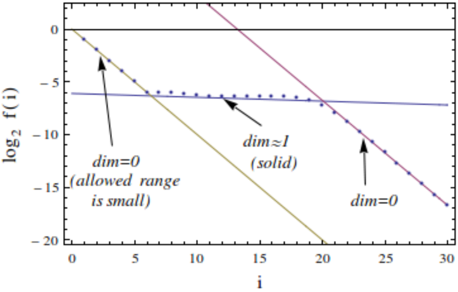
$\sin 1/x$



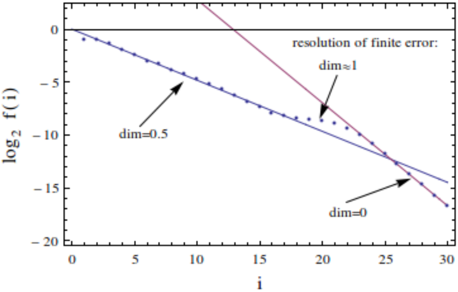
For sin(x) we observe a constant fraction of filled boxes over several orders of magnitude in box size.

What are the **dimensionalities** of the allowed ranges for x?

$\sin x$



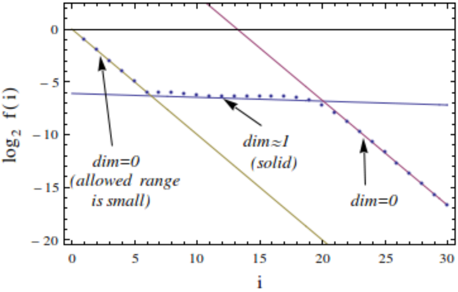
$\sin 1/x$



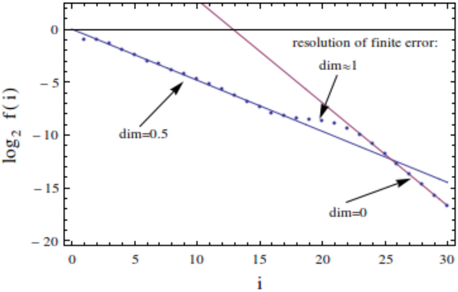
For $\sin(x)$ we observe a constant fraction of filled boxes over several orders of magnitude in box size. ⇒ No fine tuning

What are the **dimensionalities** of the allowed ranges for x?

$\sin x$



$\sin 1/x$

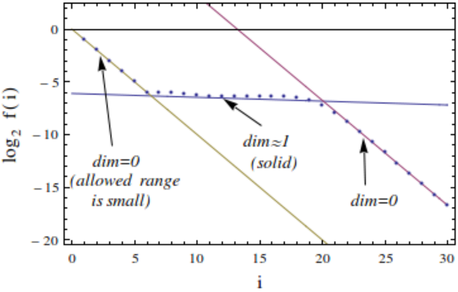


For $\sin(x)$ we observe a constant fraction of filled boxes over several orders of magnitude in box size. $\Rightarrow$ **No fine tuning**

For $\sin(1/x)$ we observe a falling fraction of filled boxes for increasing numbers of boxes corresponding to dim ≈ 0.5.

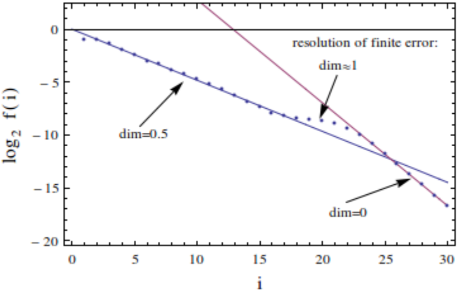
What are the **dimensionalities** of the allowed ranges for x?

$\sin x$



For $\sin(x)$ we observe a constant fraction of filled boxes over several orders of magnitude in box size. ⇒ **No fine tuning**

$\sin 1/x$

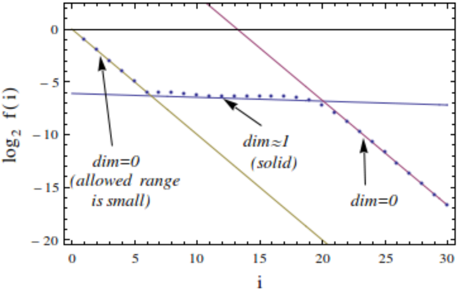


For $\sin(1/x)$ we observe a falling fraction of filled boxes for increasing numbers of boxes corresponding to $\dim \approx 0.5$.

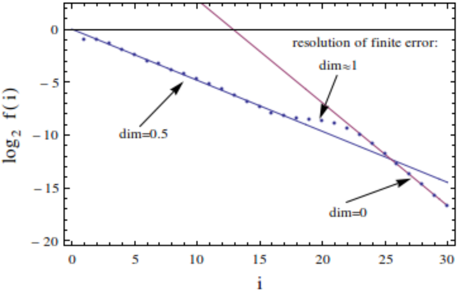
⇒ **Moderate fine tuning**

What are the **dimensionalities** of the allowed ranges for x?

$\sin x$



$\sin 1/x$



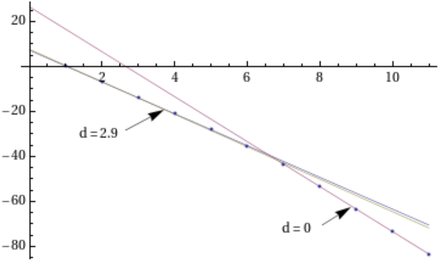
For $\sin(x)$ we observe a constant fraction of filled boxes over several orders of magnitude in box size. $\Rightarrow$ **No fine tuning**

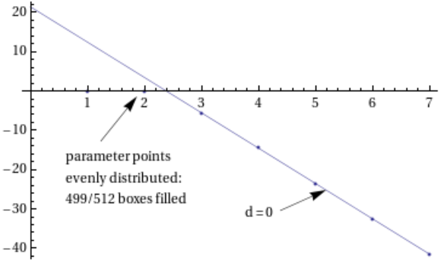
For $\sin(1/x)$ we observe a falling fraction of filled boxes for increasing numbers of boxes corresponding to $\dim \approx 0.5$.
$\Rightarrow$ **Moderate fine tuning**

Physics

Effective dim. of the parameter space of SM4 and LHT

Distrib. of valid points in **SM4** ... ... and in **LHT**

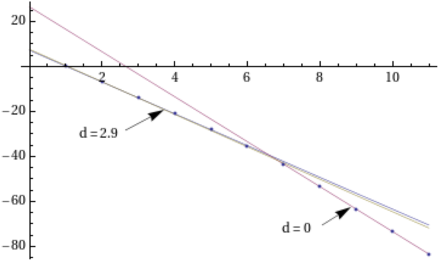




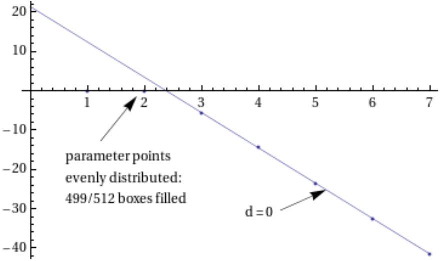
In **SM4**, the valid points in parameter space lie on a rather connected structure in 10-dim. space with an **effective dimension** of ~ 3.

Effective dim. of the parameter space of SM4 and LHT

Distrib. of valid points in **SM4** ...



... and in **LHT**

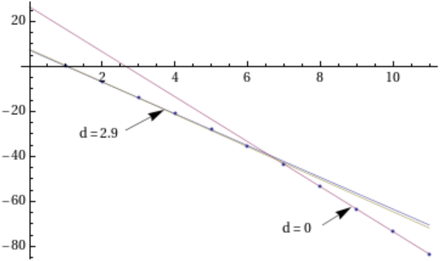


In **SM4**, the valid points in parameter space lie on a rather connected structure in 10-dim. space with an **effective dimension** of ~ 3.

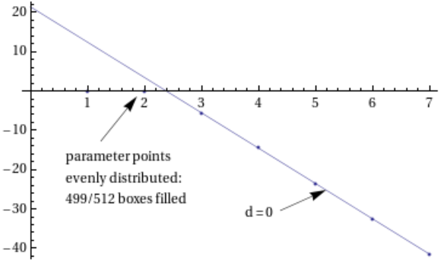
In **LHT**, the valid points are single points **distributed evenly** over the parameter space

Effective dim. of the parameter space of SM4 and LHT

Distrib. of valid points in **SM4** ...



... and in **LHT**



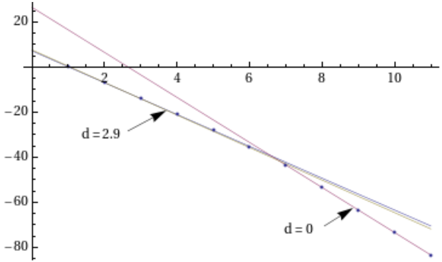
In **SM4**, the valid points in parameter space lie on a rather connected structure in 10-dim. space with an **effective dimension** of ~ 3.

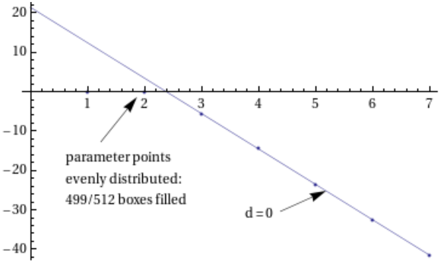
In **LHT**, the valid points are single points **distributed evenly** over the parameter space, the exp. constraints are fulfilled by **fine tuning** the mixing parameters and the mirror fermion masses.

Feldmann/Promberger/SR EPJC 72:1867 (2012)

Effective dim. of the parameter space of SM4 and LHT

Distrib. of valid points in **SM4** ... ... and in **LHT**





In **SM4**, the valid points in parameter space lie on a rather connected structure in 10-dim. space with an **effective dimension** of ~ 3.

In **LHT**, the valid points are single points **distributed evenly** over the parameter space, the exp. constraints are fulfilled by **fine tuning** the mixing parameters and the mirror fermion masses.

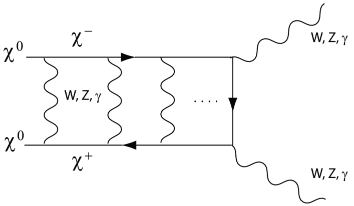
Feldmann/Promberger/SR EPJC 72:1867 (2012)

Sommerfeld enhancement in Dark Matter annihilation

Sommerfeld enhancement in Dark Matter annihilation

WIMP miracle: A weakly interacting massive particle of mass $\mathcal{O}(1\,TeV)$ (e.g. MSSM neutralino), thermally produced in the early universe and then frozen out could give the correct relic density $\Omega_{DM}h^2{}_{\mathrm{Planck}} = 0.1187(17)$.

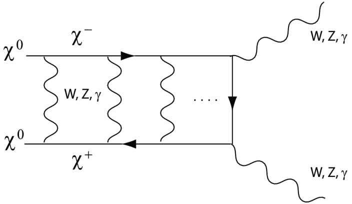
If there are other states close (in mass) to this **dark matter candidate**, resonant annihilation can change the relic density contribution significantly.



Sommerfeld enhancement in Dark Matter annihilation

WIMP miracle: A weakly interacting massive particle of mass $\mathcal{O}(1\,TeV)$ (e.g. MSSM neutralino), thermally produced in the early universe and then frozen out could give the correct relic density $\Omega_{DM}h^2{}_{\mathrm{Planck}} = 0.1187(17)$.

If there are other states close (in mass) to this **dark matter candidate**, resonant annihilation can change the relic density contribution significantly.

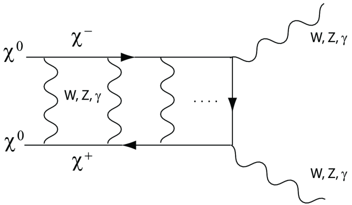


This is called **Sommerfeld Enhancement**.

Sommerfeld enhancement in Dark Matter annihilation

WIMP miracle: A weakly interacting massive particle of mass $\mathcal{O}(1\,TeV)$ (e.g. MSSM neutralino), thermally produced in the early universe and then frozen out could give the correct relic density $\Omega_{DM} h^2{}_{\mathrm{Planck}} = 0.1187(17)$.

If there are other states close (in mass) to this **dark matter candidate**, resonant annihilation can change the relic density contribution significantly.



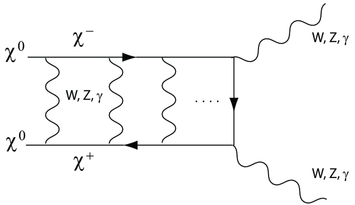
This is called **Sommerfeld Enhancement**.

Sommerfeld (1931);

Sommerfeld enhancement in Dark Matter annihilation

WIMP miracle: A weakly interacting massive particle of mass $\mathcal{O}(1\,TeV)$ (e.g. MSSM neutralino), thermally produced in the early universe and then frozen out could give the correct relic density $\Omega_{DM}h^2{}_{\rm Planck} = 0.1187(17)$.

If there are other states close (in mass) to this **dark matter candidate**, resonant annihilation can change the relic density contribution significantly.



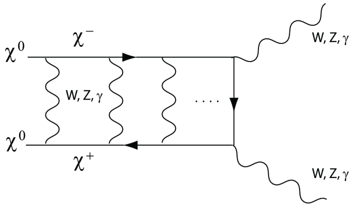
This is called **Sommerfeld Enhancement**.

Sommerfeld (1931); Hisano et al. (2004), . . .

Sommerfeld enhancement in Dark Matter annihilation

WIMP miracle: A weakly interacting massive particle of mass $\mathcal{O}(1\,TeV)$ (e.g. MSSM neutralino), thermally produced in the early universe and then frozen out could give the correct relic density $\Omega_{DM}h^2{}_{\text{Planck}} = 0.1187(17)$.

If there are other states close (in mass) to this **dark matter candidate**, resonant annihilation can change the relic density contribution significantly.



This is called **Sommerfeld Enhancement**.

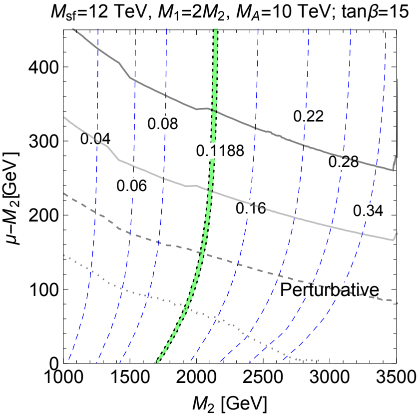
Sommerfeld (1931); Hisano et al. (2004), ...

In the **MSSM**: parameters $M_1, M_2, M_3, \mu, M_A, M_{sf}, \tan_\beta, \ldots$

Allowed (correct RD) **contours** in parameter space are **shifted** by SE!

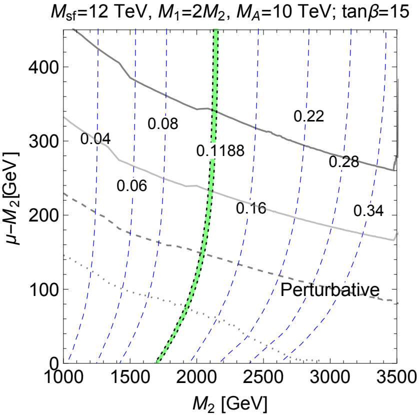
In the **MSSM**: parameters $M_1, M_2, M_3, \mu, M_A, M_{sf}, \tan_\beta, \ldots$
Allowed (correct RD) **contours** in parameter space are **shifted** by SE!

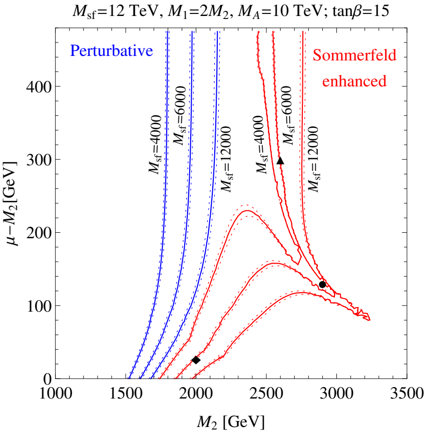
In the **MSSM**: parameters $M_1, M_2, M_3, \mu, M_A, M_{sf}, \tan_\beta, \ldots$
Allowed (correct RD) **contours** in parameter space are **shifted** by SE!



Beneke/Bharucha/Hryczuk/Ruiz-Femenia/SR (2016)

In the **MSSM**: parameters $M_1, M_2, M_3, \mu, M_A, M_{sf}, \tan_\beta, \ldots$
Allowed (correct RD) **contours** in parameter space are **shifted** by SE!



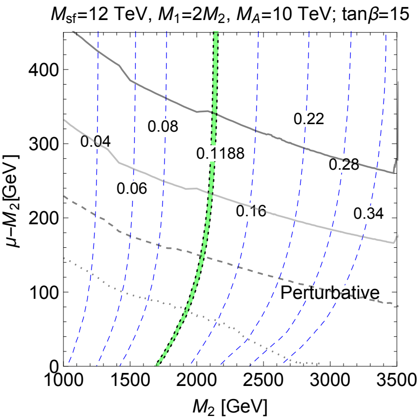


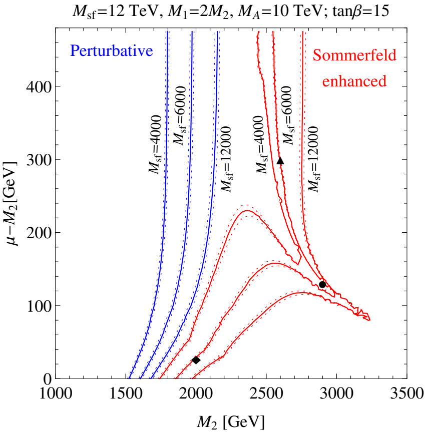
Beneke/Bharucha/Hryczuk/Ruiz-Femenia/SR (2016)

But, there are many **more parameters** in the MSSM!

In the **MSSM**: parameters $M_1, M_2, M_3, \mu, M_A, M_{sf}, \tan_\beta, \ldots$

Allowed (correct RD) **contours** in parameter space are **shifted** by SE!



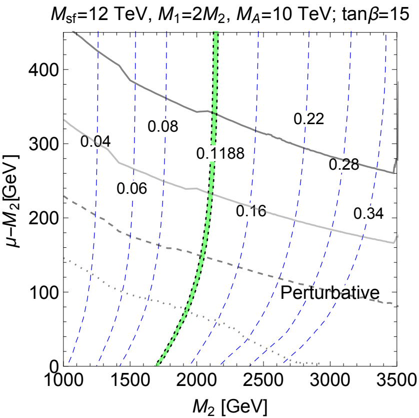


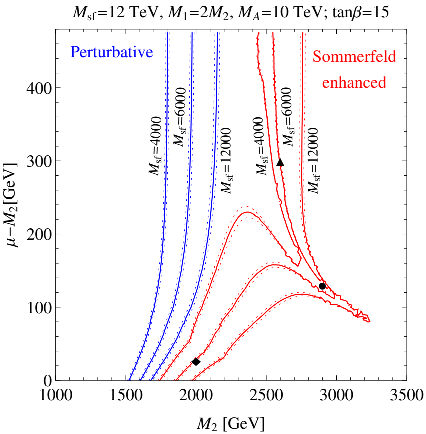
Beneke/Bharucha/Hryczuk/Ruiz-Femenia/SR (2016)

But, there are many **more parameters** in the MSSM!

Dimensionality of the allowed points in **parameter space**?

In the **MSSM**: parameters $M_1, M_2, M_3, \mu, M_A, M_{sf}, \tan_\beta, \ldots$
Allowed (correct RD) **contours** in parameter space are **shifted** by SE!

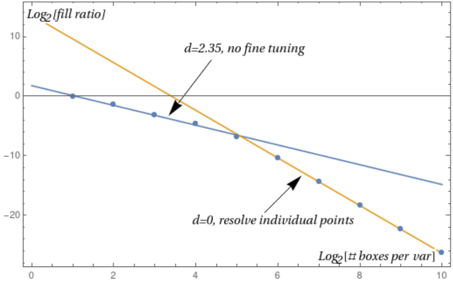




Beneke/Bharucha/Hryczuk/Ruiz-Femenia/SR (2016)

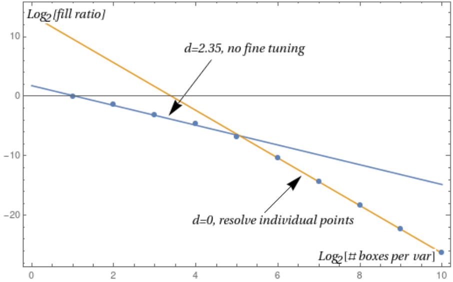
But, there are many **more parameters** in the MSSM!
Dimensionality of the allowed points in **parameter space**?

Dimensionality of the allowed points in **parameter space**:



Dimensionality of valid points ≈ 2.4

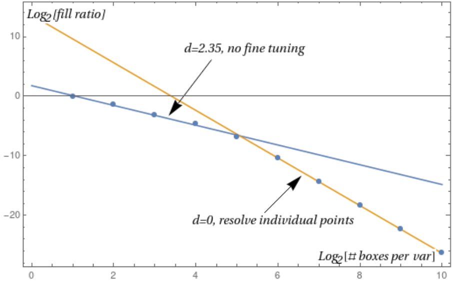
Dimensionality of the allowed points in **parameter space**:



Dimensionality of valid points ≈ 2.4

(as expected/necessary: falling to 0 when # data points ≈ # boxes)

Dimensionality of the allowed points in **parameter space**:

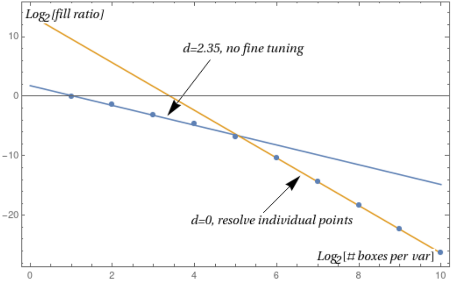


Dimensionality of valid points ≈ 2.4

(as expected/necessary: falling to 0 when # data points $\approx$ # boxes)

No fine tuning necessary to reproduce correct RD with MSSM neutralino as DM candidate!

Dimensionality of the allowed points in **parameter space**:



Dimensionality of valid points ≈ 2.4

(as expected/necessary: falling to 0

when # data points ≈ # boxes)

No fine tuning necessary to reproduce correct RD
with MSSM neutralino as DM candidate!

Conclusions

Conclusions

- In the study of models of **New Physics**, we want to know how **natural** it is to fulfil observational constraints $\Rightarrow$ **fine tuning**?

Conclusions

- In the study of models of **New Physics**, we want to know how **natural** it is to fulfil observational constraints $\Rightarrow$ **fine tuning**?
- What is the structure of the allowed areas in parameter space?

Conclusions

- In the study of models of **New Physics**, we want to know how **natural** it is to fulfil observational constraints ⇒ **fine tuning**?
- What is the structure of the allowed areas in parameter space?
- **Fractal Dimensions** can be calculated very easily (box counting algorithm) for a given set of points

Conclusions

- In the study of models of **New Physics**, we want to know how **natural** it is to fulfil observational constraints $\Rightarrow$ **fine tuning**?
- What is the structure of the allowed areas in parameter space?
- **Fractal Dimensions** can be calculated very easily (box counting algorithm) for a given set of points
- The **Fractal Dimension** of the allowed region(s) of parameter space is a good measure for **fine tuning**

Conclusions

- In the study of models of **New Physics**, we want to know how **natural** it is to fulfil observational constraints $\Rightarrow$ **fine tuning**?
- What is the structure of the allowed areas in parameter space?
- **Fractal Dimensions** can be calculated very easily (box counting algorithm) for a given set of points
- The **Fractal Dimension** of the allowed region(s) of parameter space is a good measure for **fine tuning**
- Consistent with standard definition of fine tuning.

Conclusions

- In the study of models of **New Physics**, we want to know how **natural** it is to fulfil observational constraints $\Rightarrow$ **fine tuning**?
- What is the structure of the allowed areas in parameter space?
- **Fractal Dimensions** can be calculated very easily (box counting algorithm) for a given set of points
- The **Fractal Dimension** of the allowed region(s) of parameter space is a good measure for **fine tuning**
- Consistent with standard definition of fine tuning.

شكرا بزاف

Thank you!