Resurgence, Matrices, and Strings

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QFT Toy Model: Compute $\lambda\varphi^4$ Partition Function

- QFT toy model ($d = 0$):

$$Z(m,\lambda) = \int_\Gamma \mathrm{d}\varphi \, \exp\left(-\frac{m}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4\right).$$

- Perturbative solution? $\Gamma = \mathbb{R}$, compute gaussian moments:

$$\frac{1}{n!}\left(\frac{\lambda}{4!}\right)^n \int_{\mathbb{R}} \mathrm{d}\varphi \, \varphi^{4n} \, \mathrm{e}^{-\frac{m}{2}\varphi^2}.$$

- Obtain (all-orders) perturbative solution to partition function:

$$Z(m,\lambda) = \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{+\infty} \left(\frac{2}{3}\right)^n \frac{(4n)!}{2^{6n}\,(2n)!\,n!} \left(\frac{\lambda}{m^2}\right)^n.$$

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QFT Toy Model: Sum Perturbative Expansion?

- Radius of convergence of the perturbative series?

$$Z(m,\lambda) = Z(m,0) \sum_{n=0}^{+\infty} Z_n \times \left(\frac{\lambda}{m^2}\right)^n \quad \Rightarrow \quad R = \lim_{n\to+\infty} \frac{Z_n}{Z_{n+1}} = 0.$$

Perturbative series asymptotic $\Rightarrow$ does not converge for any λ...

- Niels Henrik Abel (1802–1829):

"...Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever..." (1826)

- Félix Édouard Justin Émile Borel (1871–1956):
  - Factorial growth $(-1)^n\, n!$ can be Borel summable:

$$1 - 1 + 2 - 6 + 24 - 120 + \cdots \simeq 0.596347....$$

QFT Toy Model: Borel Transform of Perturbative Series

- Factorial growth of perturbative $Z_n \sim n!$ tackled via Borel transform:

$$\mathcal{B}: \left(\frac{\lambda}{m^2}\right)^{n+1} \mapsto \frac{s^n}{n!}.$$

Can compute Borel transform of perturbative series exactly:

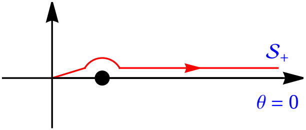
$$\mathcal{B}[Z](s) = \frac{1}{8}\sqrt{\frac{2\pi}{m}}\,{}_2F_1\left(\frac{5}{4}, \frac{7}{4}, 2 \,\middle|\, \frac{2s}{3}\right).$$

- Borel resummation given by inverse Borel transform ($s \to s \in \mathbb{C}$)

$$\mathcal{S}Z(m,\lambda) = \int_0^{+\infty} \mathrm{d}s\, \mathcal{B}[Z](s)\, \mathrm{e}^{-s\frac{m^2}{\lambda}}.$$

...Only defined if $\mathcal{B}[Z](s)$ has no singularities along $\mathbb{R}^+$...
But Borel singularities reflect asymptotic nature of original series ✓

QFT Toy Model: Nature of Borel Singularities?



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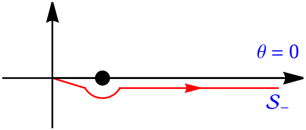

- ${}_2F_1\left(a, b, c \,|z\right)$ hypergeometric function has branch-point at $z = 1$.
- $\exists$ Borel singularity at $s = \frac{3}{2} \equiv A$. What is nature of this singularity?

$$\mathcal{B}[Z](s)\Big|_{s=A} \approx (+2) \times \underbrace{\Psi\left(s - A\right)}_{\text{regular @ } A} \times \frac{\log\left(s - A\right)}{2\pi \mathrm{i}} + \text{regular},$$

with

$$\Psi\left(s\right) = \frac{\mathrm{i}}{8}\sqrt{\frac{\pi}{m}}\, {}_2F_1\left(\frac{5}{4}, \frac{7}{4}, 2 \,\middle|\, -\frac{2s}{3}\right).$$

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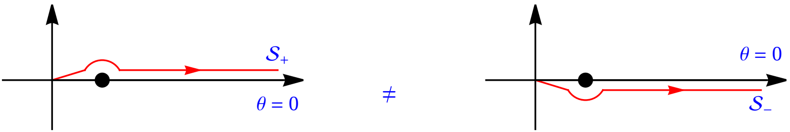
QFT Toy Model: Discontinuity Across Stokes Line



- Can exactly compute discontinuity $\mathrm{Disc} = \mathcal{S}_+ - \mathcal{S}_-$:

$$\mathrm{Disc}\, Z = (+2) \times \underbrace{\mathrm{e}^{-\frac{3m^2}{2\lambda}}}_{\text{exp shift}} \times \underbrace{(-\mathrm{i}) \sqrt{\frac{\pi}{m}} \sum_{n=0}^{+\infty} \left(-\frac{2}{3}\right)^n \frac{(4n)!}{2^{6n}\,(2n)!\, n!} \left(\frac{\lambda}{m^2}\right)^n}_{\text{asymptotic series}}.$$

- A *"new"* asymptotic series/sector resurges at Borel singularity of *"old"* asymptotic series/perturbative sector. What is it??

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QFT Toy Model: Nontrivial Saddles in Partition Function

- All saddles of $\lambda\varphi^4$ partition function?

$$Z(m,\lambda) = \int_\Gamma \mathrm{d}\varphi\, \exp\left(-\frac{m}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4\right).$$

  - Perturbative saddle: $\varphi_\star^{(0)} = 0 \;\Rightarrow\; Z^{(0)}(m,\lambda)$ ✓
  - "Instanton" saddle: $\varphi_\star^{(\pm)} = \pm\sqrt{\frac{6m}{\lambda}} \;\Rightarrow\; Z^{(1)}(m,\lambda) \sim \mathrm{e}^{-\frac{3m^2}{2\lambda}}$ ←

- Instanton solution? $\Gamma = \Gamma_1$. Obtain (all-orders) instanton solution:

$$Z^{(1)}(m,\lambda) = (-\mathrm{i})\sqrt{\frac{\pi}{m}}\,\mathrm{e}^{-\frac{3m^2}{2\lambda}} \sum_{n=0}^{+\infty} \left(-\frac{2}{3}\right)^n \frac{(4n)!}{2^{6n}\,(2n)!\,n!} \left(\frac{\lambda}{m^2}\right)^n.$$

- Without great surprise, $\Psi(s) = \mathcal{B}[Z^{(1)}](s)$.

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QFT Toy Model: True Nature of Borel Singularities

- Rewrite earlier Borel singularity of perturbative sector:

$$\mathcal{B}[Z^{(0)}](s)\Big|_{s=A} \approx \underbrace{(+2)}_{\text{Stokes}} \times \mathcal{B}[Z^{(1)}]\,(s-A) \times \frac{\log(s-A)}{2\pi\mathrm{i}}.$$

- Borel singularity of instanton sector:

$$\mathcal{B}[Z^{(1)}](s)\Big|_{s=-A} \approx \underbrace{(-1)}_{\text{Stokes}} \times \mathcal{B}[Z^{(0)}]\,(s+A) \times \frac{\log(s+A)}{2\pi\mathrm{i}}.$$

- *At the singularity of the (Borel) perturbative sector one finds the resurgence of the (Borel) instanton sector, and vice-versa!* **[Écalle]**
- Information encoded in terms of Stokes data $\mathsf{S}_k \in \mathbb{C}$.

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QFT Toy Model: Resurgence and Transseries

- Resurgence: coefficients $Z_n^{(0)}$ and $Z_{n'}^{(1)}$ relate to each other...

$$Z^{(0)}_{n\gg 1} \simeq -\frac{2}{2\pi\mathrm{i}}\,\frac{(n-1)!}{A^n}\left(Z_0^{(1)} + \frac{Z_1^{(1)} A}{n-1} + \frac{Z_2^{(1)} A^2}{(n-1)\,(n-2)} + \cdots\right).$$

$\Rightarrow$ Obtain nonperturbative data out of perturbative data!

- Complete nonperturbative solution $\Rightarrow$ transseries:

$$\mathcal{Z}(\sigma\,|\,m,\lambda) = Z^{(0)}\underbrace{(m,\lambda)}_{\text{monomials}} + \sigma\,\underbrace{\mathrm{e}^{-\frac{3m^2}{2\lambda}}}_{\text{transmonomial}}\,\widehat{Z}^{(1)}\underbrace{(m,\lambda)}_{\text{monomials}}.$$

- Built-in: crossing Stokes line yields Stokes phenomena:

$$\sigma \mapsto \sigma + \mathsf{S}.$$

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- If the original perturbative series is resurgent, then the above construction holds—alongside *many* resurgent relations between perturbative and nonperturbative sectors—and, in fact, the complete nonperturbative solution is given by a transseries!

Outline

Outline

Analyticity versus Non-Analyticity

- Analytic functions described by power series. But how to describe general non-analytic functions?
- Transseries augment power series with non-analytic terms, the trans-monomials, in order to describe non-analytic functions.
- For example with exponentials,

$$\exp\left(-\frac{1}{\lambda}\right), \quad \exp\left(-\exp\left(\frac{1}{\lambda}\right)\right), \quad \exp\left(-\exp\left(\exp\left(\frac{1}{\lambda}\right)\right)\right), \quad \cdots$$

  or with logarithms,

$$\log(\lambda), \quad \log(\log(\lambda)), \quad \log(\log(\log(\lambda))), \quad \cdots$$

- Here, mainly address (familiar) dependences $\mathrm{e}^{-\frac{1}{\lambda}}$ and $\log\lambda$.

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Asymptotic Expansions: Perturbative and Nonperturbative

- Perturbative series asymptotic $\Rightarrow$ Coefficients grow as $Z_k^{(0)} \sim k!\ldots$

$$Z^{(0)}(\lambda) \simeq \sum_{k=0}^{+\infty} Z_k^{(0)} \lambda^{k+1}.$$

- One-parameter transseries (generically multi-parameters...):

$$\mathcal{Z}(\sigma \,|\, \lambda) = \sum_{n=0}^{+\infty} \sigma^n \, \underbrace{Z^{(n)}(\lambda)},$$

$$Z^{(n)}(\lambda) \simeq \mathrm{e}^{-n\frac{A}{\lambda}} \sum_{k=1}^{+\infty} Z_k^{(n)} \lambda^{k+\beta_n} \equiv \mathrm{e}^{-n\frac{A}{\lambda}} \Phi_n(\lambda).$$

  - Double "perturbative" expansion, both in λ and $\sigma \, \mathrm{e}^{-\frac{A}{\lambda}}\ldots$

- Transseries: recursively solve for unknowns $Z_k^{(n)}$ ✓
- Resurgence: coefficients $Z_k^{(n)}$, $Z_{k'}^{(n')}$ relate to each other ✓

Resurgent Functions and Borel Singularities

- What class of singularities does one usually find?
- With instanton action A, Borel singularities located at $n\,A$ with $n \in \mathbb{N}$.
- Can compute *all* Borel singularities in complete transseries!

$$\mathcal{B}[\Phi_n](s)\Big|_{s=kA} \approx \mathsf{S}_k \times \mathcal{B}[\Phi_{n+k}](s-kA)\,\frac{\log(s-kA)}{2\pi\mathrm{i}}, \qquad k \in \mathbb{Z}^\times.$$

- *At the kth singularity of the (Borel) n-instanton sector one finds the resurgence of the (Borel) $(n+k)$-instanton sector!* **[Écalle]**
- Information encoded in terms of Stokes data $\mathsf{S}_k \in \mathbb{C}$.

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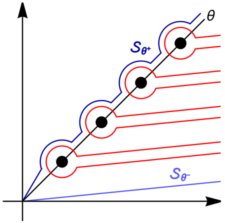
Discontinuity upon Crossing a Stokes Line



$$\Rightarrow \qquad \mathrm{Disc}_{\,\theta} Z = \sum_n \mathsf{S}_n \, \mathrm{e}^{-n\frac{A}{\lambda}} \, \Phi_n.$$

- All sectors Φ_n must be included in full solution, as perturbative series not enough! $\Rightarrow$ *Transseries* and *Resurgence* ✓

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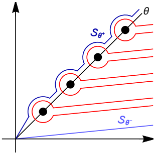
Resummation: Stokes Phenomena



- On transseries, Stokes transitions occur upon crossing Stokes lines: can turn on/off infinite set of multi-instanton corrections:

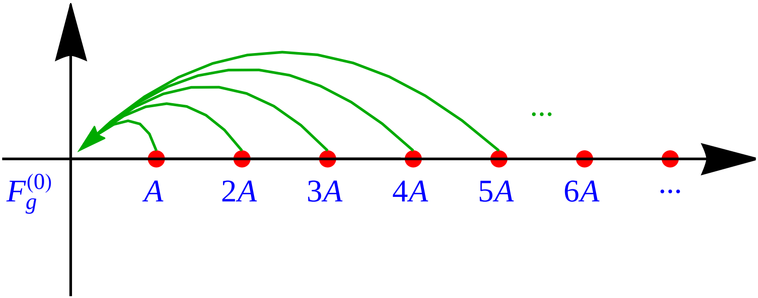
$$\mathcal{S}_+ \mathcal{Z}(\sigma) = \mathcal{S}_- \mathcal{Z}(\sigma + S_1).$$

- Exponentially suppressed contributions in some region of the complex plane may become exponentially enhanced contributions elsewhere ⇒ Stokes phenomena leads to dominance of instantons!

- Transseries resummation allows to reach arbitrary coupling (strong coupling!) and further venture (incorporating Stokes phenomena) into the complex plane. **[Mariño, Couso-RS-Vaz, Couso-Mariño-RS, Codesido-Mariño-RS]**

Outline

Asymptotics of (One-Parameter) Perturbative Series



- From perturbative expansion and Cauchy dispersion relation

$$F_g^{(0)} \simeq \frac{S_1}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{A^{g-\beta}} \left(F_1^{(1)} + \frac{A}{g-\beta-1} F_2^{(1)} + \cdots \right) + \\ + \frac{S_1^2}{2\pi\mathrm{i}} \frac{\Gamma(g-2\beta)}{(2A)^{g-2\beta}} \left(F_1^{(2)} + \frac{2A}{g-2\beta-1} F_2^{(2)} + \cdots \right) + \cdots$$

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Using Resurgent Asymptotics: Main Idea

- Predicting the large-order behavior:

$$\Phi^{(0)}_{g\gg 1} \longleftrightarrow \left[\Phi^{(1)}_h\right] + \left[\Phi^{(2)}_h\right] + \left[\Phi^{(3)}_h\right] + \cdots .$$

- Decoding the nonperturbative content:

$$\Phi^{(0)}_{g\gg 1} \longleftrightarrow \left[\Phi^{(1)}_h\right] + \left[\Phi^{(2)}_h\right] + \left[\Phi^{(3)}_h\right] + \cdots .$$

- The structural need for the Stokes coefficients:

$$\Phi^{(0)}_{g\gg 1} \longleftrightarrow \boldsymbol{S_1}\left[\Phi^{(1)}_h\right] + \boldsymbol{S_2}\left[\Phi^{(2)}_h\right] + \boldsymbol{S_3}\left[\Phi^{(3)}_h\right] + \cdots .$$

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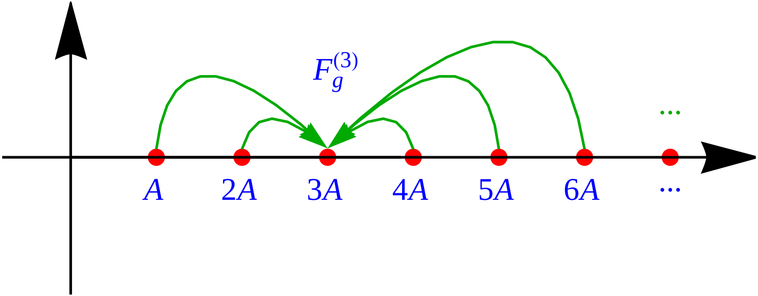
Asymptotics of (One-Parameter) Multi-Instanton Series

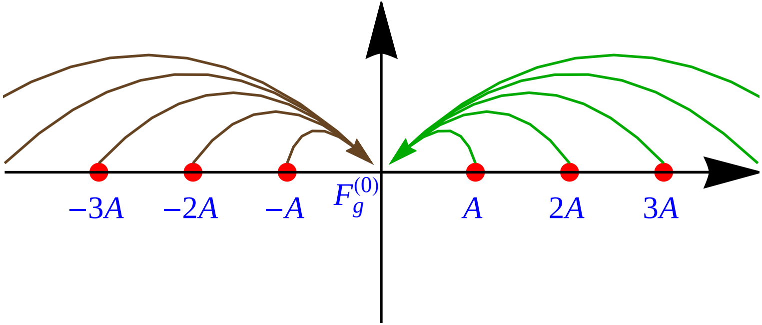


- Stokes discontinuities $\mathrm{Disc}_{\,0}\,\Phi_n$ and $\mathrm{Disc}_{\,\pi}\,\Phi_n$, imply [Aniceto-RS-Vonk]

$$F_g^{(n)} \simeq \frac{S_1}{2\pi\mathrm{i}}\,(n+1)\,\frac{\Gamma\left(g-\beta-1\right)}{(A)^{g-\beta-1}}\,F_1^{(n+1)} + \frac{S_{-1}}{2\pi\mathrm{i}}\,(n-1)\,\frac{\Gamma\left(g+\beta-1\right)}{(-A)^{g+\beta-1}}\,F_1^{(n-1)} + \cdots.$$

- All Stokes factors now needed $\Rightarrow$ Hard (analytical) computation...

Asymptotics of *Resonant* Perturbative Series

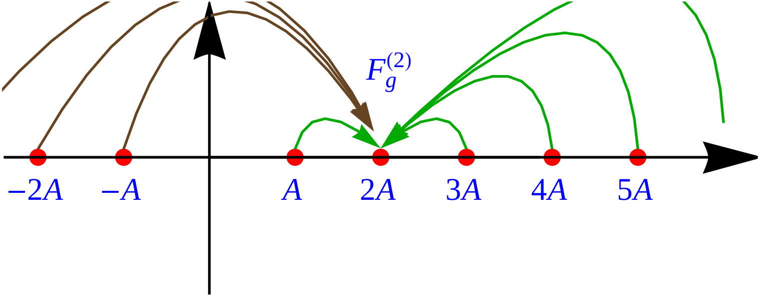


- Two-parameter transseries with instanton actions A and $-A$...
- Already at perturbative level leading asymptotics clearly distinct

[Aniceto-RS-Vonk]

$$F_g^{(0|0)} \simeq \frac{S_1^{(0)}}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{A^{g-\beta}} F_1^{(1|0)} + \frac{\widetilde{S}_{-1}^{(0)}}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{(-A)^{g-\beta}} F_1^{(0|1)} + \cdots$$

Asymptotics of *Resonant* Multi-Instanton Series



- String theoretic examples always resonant: open *and* closed strings ✓

Asymptotics of Two-Parameter Multi-Instanton Series

- New, distinct, visible, features in both perturbative and multi-instantonic asymptotics... Resonance!
- Precisely these asymptotics found in many examples!
  - Chern–Simons on $\mathbb{S}^3$ (milder) [Pasquetti-RS]
  - Chern–Simons on lens spaces (milder) [Aniceto-Russo-RS]
  - ABJM gauge theory on $\mathbb{S}^3$ (milder) [Aniceto-Russo-RS]
  - Painlevé I equation (2d gravity) [Garoufalidis-Its-Kapaev-Mariño, Aniceto-RS-Vonk]
  - Painlevé II equation (2d supergravity) [Mariño, RS-Vaz]
  - One-cut quartic matrix model [Mariño, Aniceto-RS-Vonk]
  - Two-cut quartic matrix model [RS-Vaz]
  - Topological strings in the conifold (milder) [Pasquetti-RS]
  - ABJM on $\mathbb{S}^3$ as string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ [Drukker-Mariño-Putrov]
  - Topological strings in local $\mathbb{P}^2$ [Couso-Edelstein-RS-Vonk]

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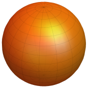
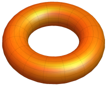
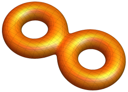
Examples: Matrix Models and Topological Strings

- String theory generically defined *perturbatively*:

$$F = \log Z \simeq \sum_{g=0}^{+\infty} F_g(t)\, g_s^{2g-2} =$$

$$= \frac{1}{g_s^2}$$  $$+$$  $$+ \; g_s^2$$  $$+ \cdots.$$

- Obtain nonperturbative definition/construction of string theory?
- Obtain semiclassical interpretation of this nonperturbative answer?
- Beyond perturbative $\sim g_s^{\#} \Rightarrow$ nonperturbative $\sim \exp\left(-\frac{\#}{g_s}\right)$...

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Double-Scaled Matrix Model and Painlevé I Equation

- Painlevé I = "specific heat" of 2d gravity/minimal strings with $d = 0$:

  [Douglas-Shenker, Brézin-Kazakov, Gross-Migdal]

$$u^2(z) - \frac{1}{6} u''(z) = z.$$

- Free energy and partition function follow as:

$$F''(z) = -u(z) \quad \Rightarrow \quad Z = \exp F.$$

- String-theoretic genus expansion ($g_{\mathsf{s}} = z^{-5/4}$):

$$\begin{aligned} F &= \log Z \simeq \sum_{g=0}^{+\infty} g_{\mathsf{s}}^{2g-2} F_g = \\ &\simeq -\frac{4}{15} \frac{1}{g_{\mathsf{s}}^2} + \frac{1}{60} \log g_{\mathsf{s}} + \frac{7}{5760} g_{\mathsf{s}}^2 + \frac{245}{331776} g_{\mathsf{s}}^4 + \cdots. \end{aligned}$$

- Perturbative series is asymptotic! $\Rightarrow$ Coefficients grow as $F_g \sim (2g)! \ldots$

Painlevé I Equation: Perturbative and Nonperturbative?

- String theory aside, consider Painlevé I equation:

$$u^2(z) - \frac{1}{6}u''(z) = z.$$

- Its perturbative solution (now at large $z \equiv$ small g_s)

$$u(z) \simeq \sqrt{z} \sum_{g=0}^{+\infty} \frac{u_g}{z^{\frac{5}{2}g}},$$

yields recursion equation; leading to *asymptotic* expansion

$$u(z) \simeq \sqrt{z} \left(1 - \frac{1}{48} z^{-\frac{5}{2}} - \frac{49}{4608} z^{-5} + \cdots \right).$$

- Second order differential equation ⇒ Yields *two* instanton actions

$$A = \pm \frac{8\sqrt{3}}{5}.$$

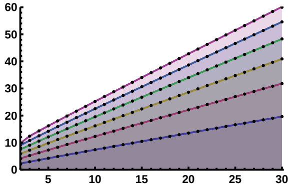
Resonant Two-Parameter Transseries Solution

- General two-parameter transseries solution:

$$u(g_{\mathsf{s}},\sigma_1,\sigma_2)=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sigma_1^n\sigma_2^m\,\mathrm{e}^{-(n-m)\frac{A}{g_{\mathsf{s}}}}\left(\sum_{k=0}^{\min(n,m)}\log^k(g_{\mathsf{s}})\cdot\Phi_{(n|m)}^{[k]}(g_{\mathsf{s}})\right).$$

  Trans-monomials ordering: $\mathrm{e}^{-\frac{A}{g_{\mathsf{s}}}}\ll g_{\mathsf{s}}\ll\log(g_{\mathsf{s}})\ll\mathrm{e}^{+\frac{A}{g_{\mathsf{s}}}}$.

- Checked nonperturbative sectors via resurgent large-order analysis.



- Resurgence allows extremely accurate tests: at genus $g=30$, including six instantons, results correct up to 60 decimal places!

  [Aniceto-RS-Vonk]

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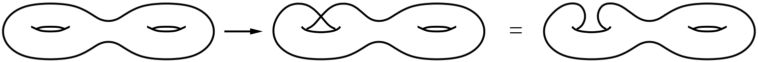
Topological Strings on Calabi–Yau Geometries

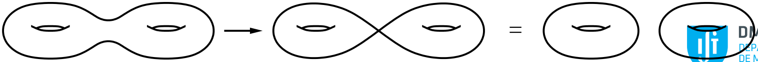
- Consider topological string B-model on local Calabi–Yau, mirror to some toric threefold. Non-trivial information about CY geometry encoded in Riemann surface ⇔ mirror curve of the geometry.
- Compute the string free energy using the holomorphic anomaly!

$$F \simeq \sum_{g=0}^{+\infty} F_g(z, \bar{z})\, g_s^{2g-2}.$$

- Holomorphic anomaly equations are recursive: [Bershadsky-Cecotti-Ooguri-Vafa]



(a)



(b)

(Nonperturbative) Holomorphic Anomaly Equations

- Consider example of B-model on mirror of local $\mathbb{P}^2$ $\Rightarrow$ Solved perturbatively via holomorphic anomaly equations...

  [Bershadsky-Cecotti-Ooguri-Vafa, Grimm-Klemm-Mariño-Weiss, Haghighat-Klemm-Rauch]

- Rewrite holomorphic anomaly equations for partition function Z $\Rightarrow$ Naturally solved with transseries *ansatz*...
- Instanton action is holomorphic: $\partial_{\bar{z}} A = 0$ $\Rightarrow$ Can *still* compute A as appropriate combinations of periods in the geometry. [Drukker-Mariño-Putrov]
- Nonperturbative holomorphic anomaly equations ✓ [Couso-Edelstein-RS-Vonk]

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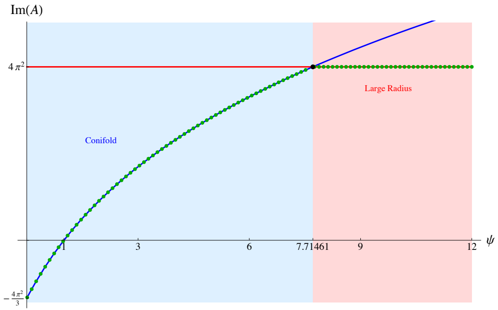
Local $\mathbb{P}^2$: Checks of Conifold Instanton Actions

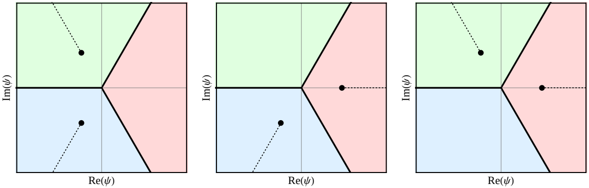


- Near conifold $z = -\frac{1}{27}$ use coordinate $\psi^{-3} = -27z \Rightarrow 3$ conifold points at cubic roots of unity $\Rightarrow$ Instanton actions $A_i(\psi) = \frac{2\pi \mathrm{i}}{\sqrt{3}}\, t_{\mathsf{c},i}(\psi)$, with $t_{\mathsf{c},1}(\psi) = t_{\mathsf{c}}(\psi)$, $t_{\mathsf{c},2}(\psi) = t_{\mathsf{c}}(\mathrm{e}^{-2\pi\mathrm{i}/3}\,\psi)$, $t_{\mathsf{c},3}(\psi) = t_{\mathsf{c}}(\mathrm{e}^{+2\pi\mathrm{i}/3}\,\psi)$.

$$\left(t_{\mathsf{c}} = \frac{2\pi}{\sqrt{3}} \left(\frac{3\psi}{\Gamma\left(\frac{2}{3}\right)^3}\; {}_3F_2\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{2}{3}, \frac{4}{3} \,\middle|\, \psi^3 \right) - \frac{\frac{9}{2}\psi^2}{\Gamma\left(\frac{1}{3}\right)^3}\; {}_3F_2\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{4}{3}, \frac{5}{3} \,\middle|\, \psi^3 \right) - 1 \right) \right)$$

Local $\mathbb{P}^2$: Structure of Instanton Actions

- Branch points and cuts of actions A_1, A_2, A_3, in complex ψ plane. Each wedge $2\pi/3$ in correspondence with one full complex z plane.

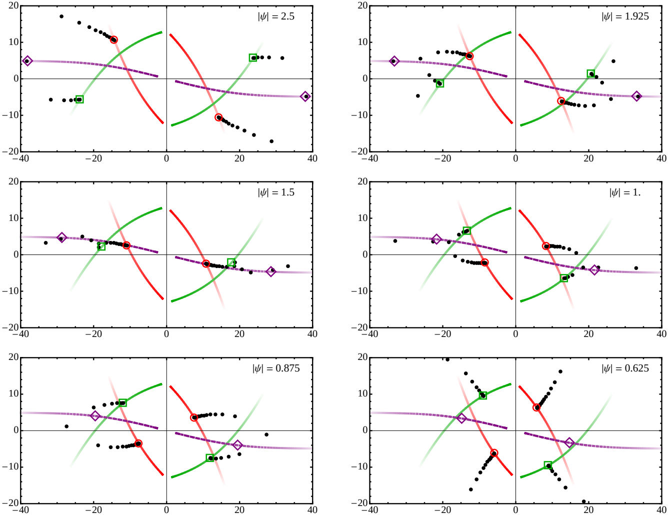


- Large-radius point $z = 0$, $\psi^{-1} = 0$ use mirror map to write Kähler parameter as

$$T(\psi) = -\frac{1}{2\pi \mathrm{i}} \frac{\sqrt{3}}{2\pi} G^{22}_{33} \left(\begin{array}{ccc|c} \frac{1}{3} & \frac{2}{3} & 1 & -\frac{1}{\psi^3} \\ 0 & 0 & 0 & \end{array} \right).$$

$\Rightarrow$ In region of moduli space associated to large-radius point, dominant instanton action is $A_{\mathsf{K}}(\psi) = 4\pi^2 \mathrm{i}\, T(\psi)$.

Local $\mathbb{P}^2$: Conifold versus Kähler Actions on Borel Plane

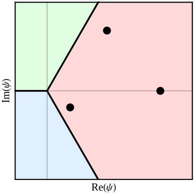


Local $\mathbb{P}^2$: Checks of Conifold One-Instanton Sector I

- Conifold one-instanton: $F_g^{(\mathbf{e}_1)} = \frac{\mathrm{i}\pi}{S_{1,1}}\, \mathrm{e}^{\frac{1}{2}(\partial_z A_1)^2 \left(\bar{z}-\bar{z}_{1,\mathsf{hol}}\right)}\, \mathsf{Pol}\left(\bar{z}; 3g\right)$.
- Testable at large-order using the sequence:

$$\frac{S_{1,1}}{\mathrm{i}\pi} F_h^{(\mathbf{e}_1)} = \lim_{g\to\infty} \frac{A_1^{2g-1-h}}{\Gamma(2g-1-h)} \left(F_g^{(\mathbf{0})} - \sum_{h'=0}^{h-1} \frac{\Gamma\left(2g-1-h'\right)}{A_1^{2g-1-h'}} \frac{S_{1,1}}{\mathrm{i}\pi} F_{h'}^{(\mathbf{e}_1)} \right).$$

- Fig: for $h = 0, 1, 2, 3$, at three different points in ψ-moduli space:



- Fig: x changes value of propagator around its holomorphic value.
- Fig: numerical blue, green dots correspond to real, imaginary.

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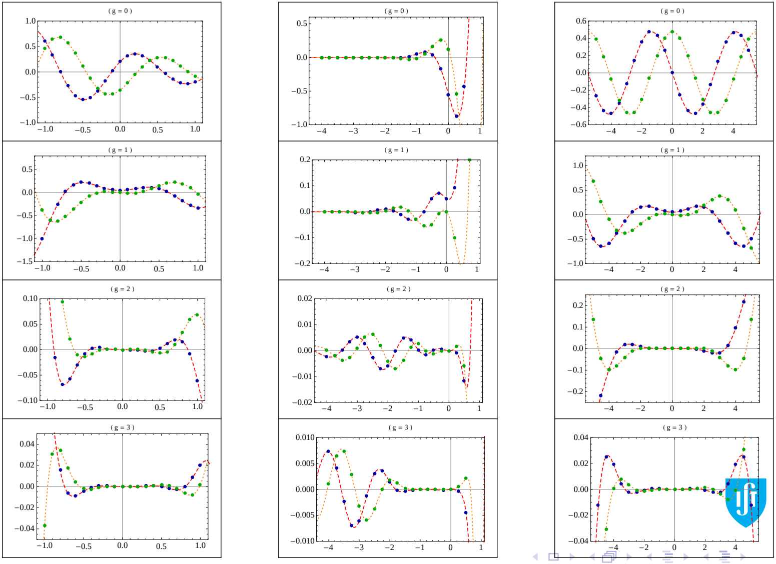
Local $\mathbb{P}^2$: Checks of Conifold One-Instanton Sector II



Local $\mathbb{P}^2$: Checks of Conifold Two-Instanton Sectors I

- At two-instanton level find a "pure" contribution

$$F_h^{(2\mathbf{e}_1)} = \mathrm{e}^{(\partial_z A_1)^2 (\bar{z}-\bar{z}_{1,\mathsf{hol}})}\, \mathsf{Pol}\,(\bar{z}; 3h) + \mathrm{e}^{2(\partial_z A_1)^2 (\bar{z}-\bar{z}_{1,\mathsf{hol}})}\, \mathsf{Pol}\,(\bar{z}; 3h),$$

alongside a "mixed" contribution

$$F_h^{(\mathbf{e}_{1,1})} = \mathrm{e}^{(\partial_z A_1)^2 (\bar{z}-\bar{z}_{1,\mathsf{hol}})}\, \mathsf{Pol}\left(\bar{z}; \frac{5h}{2}\right) + \mathsf{Pol}\left(\bar{z}; \frac{3h}{2} - 1\right).$$

- These are testable at large-order using the sequence:

$$F_g^{(\mathbf{e}_1)} \simeq \sum_{h=0}^{+\infty} \left\{ \frac{\Gamma(g+1-h)}{(+A_1)^{g+1-h}} \frac{S_{1,1}}{\mathrm{i}\pi} F_h^{(2\mathbf{e}_1)} + \frac{\Gamma(g+1-h)}{(-A_1)^{g+1-h}} \frac{\widetilde{S}_{-1,1}}{2\pi\mathrm{i}} F_h^{(\mathbf{e}_{1,1})} \right\} + \cdots.$$

- Top Fig: real and imaginary parts of $\frac{S_{1,1}^2}{(\mathrm{i}\pi)^2} F_0^{(2\mathbf{e}_1)}$.
- Bottom Fig: real and imaginary parts of $\frac{S_{1,1}}{\mathrm{i}\pi} \frac{\widetilde{S}_{-1,1}}{2\pi\mathrm{i}} F_0^{(\mathbf{e}_{1,1})}$.

(numerical tests at fixed $\psi = 2\,\mathrm{e}^{-\mathrm{i}\pi/36}$ and varying $\bar{z} = 10^{-8}\,(1+\mathrm{i}x)$)

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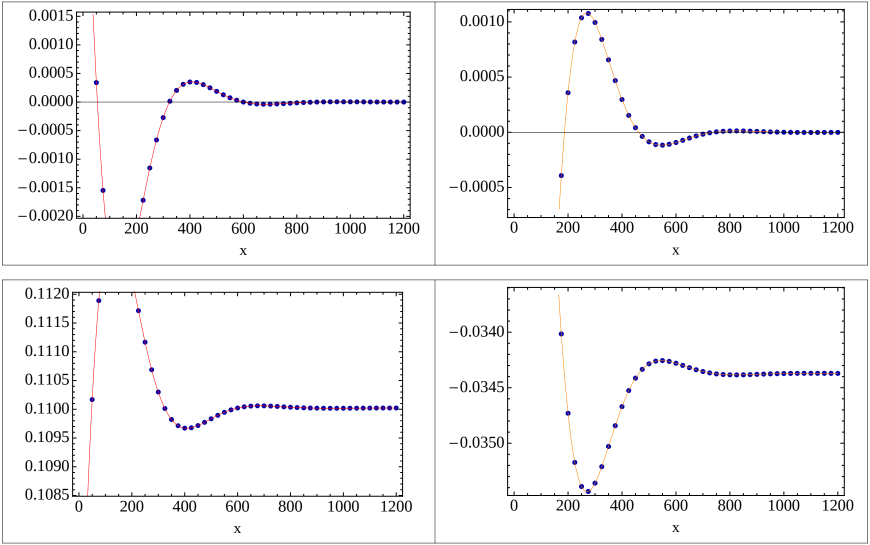
Local $\mathbb{P}^2$: Checks of Conifold Two-Instanton Sectors II



Outline

Matrix Models and 't Hooft Large N Limit

- What is the resurgent, nonperturbative nature of the large N limit?...
- Hermitian one-matrix model with polynomial potential $V(z)$,

$$Z = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int \mathrm{d}M \, \exp\left(-\frac{1}{g_{\mathsf{s}}} \mathrm{Tr} V(M)\right).$$

- Consider limit $N \to +\infty$ while $t = g_{\mathsf{s}} N$ fixed ['t Hooft]. Free energy $F = \log Z$ has *asymptotic* genus expansion

$$F \simeq \sum_{g=0}^{+\infty} F_g(t)\, g_{\mathsf{s}}^{2g-2}.$$

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Borel–Padé–Écalle Finite N Transseries Resummation

- Transseries resummation allows to reach finite N (strong coupling!) and further venture (incorporating Stokes phenomena) into the complex plane! [Mariño, Couso-RS-Vaz]
- Asymptotic series "building" the transseries,

$$\mathcal{F}^{(n)}(N,t) \simeq \sum_{g=0}^{+\infty} N^{-g-\beta_n^{\mathcal{F}}}\, t^{g+\beta_n^{\mathcal{F}}}\, \mathcal{F}_g^{(n)}(t),$$

need to be Borel resummed (in practice Borel–Padé),

$$\mathcal{S}_\theta \mathcal{F}^{(n)}(N,t) = \int_0^{\mathrm{e}^{i\theta}\infty} \mathrm{d}s\, \mathcal{B}[\mathcal{F}^{(n)}](s,t)\, \mathrm{e}^{-sN}.$$

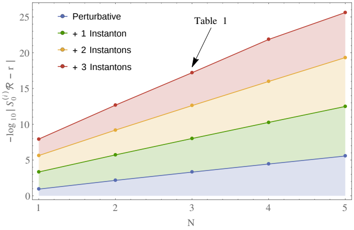
- Assemble back into transseries to obtain Borel–Padé–Écalle resummation

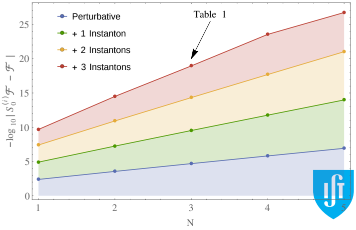
$$\mathcal{S}\mathcal{F}(N,t) = \sum_{n=0}^{+\infty} \sigma^n\, \mathrm{e}^{-nN\frac{A(t)}{t}}\, \mathcal{S}_\theta \mathcal{F}^{(n)}(N,t).$$

Resummation of the Transseries and Instanton Corrections

Sector	$S_0\mathcal{R}^{(n)}$	$S_0\mathcal{F}^{(n)}$
Perturbative	$2.615\,796\,570\,569\,705\,50\ldots$	$-1.973\,899\,279\,493\,161\,74\ldots$
1-Instanton	$0.000\,487\,953\,495\,567\,22\ldots$	$-0.000\,020\,359\,080\,917\,15\ldots$
2-Instanton	$0.000\,000\,009\,807\,788\,15\ldots$	$-0.000\,000\,000\,300\,789\,88\ldots$
3-Instanton	$0.000\,000\,000\,000\,245\,38\ldots$	$-0.000\,000\,000\,000\,004\,71\ldots$
Total	$2.616\,284\,533\,873\,306\,27\ldots$	$-1.973\,919\,638\,874\,873\,50\ldots$
Exact	$2.616\,284\,533\,873\,306\,26\ldots$	$-1.973\,919\,638\,874\,873\,50\ldots$

Decimal-place precision of multi-instanton corrections at $t = 6$:





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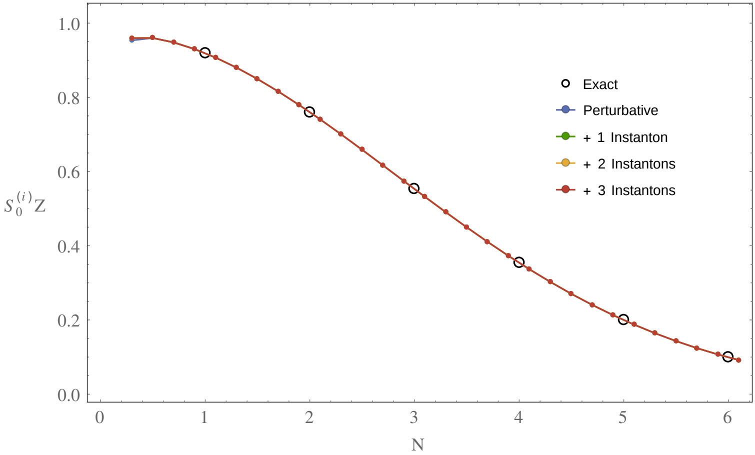
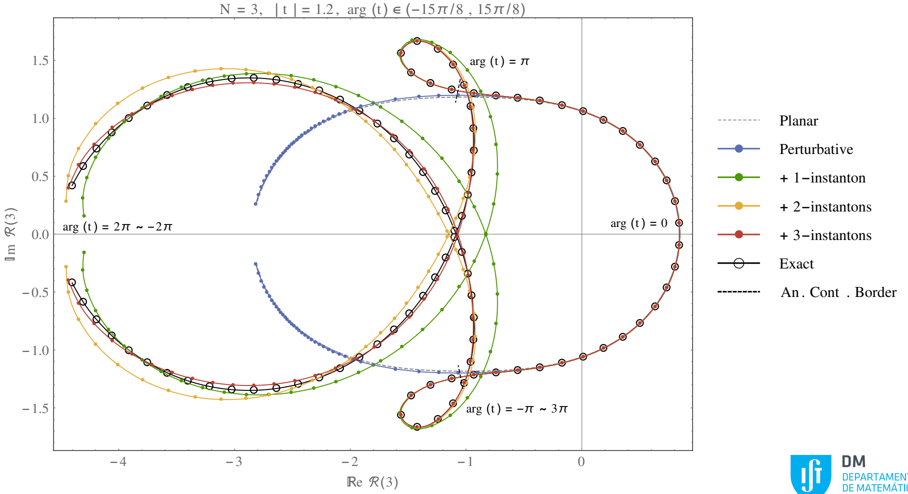
Partition Function: Interpolation at Non-Integer N

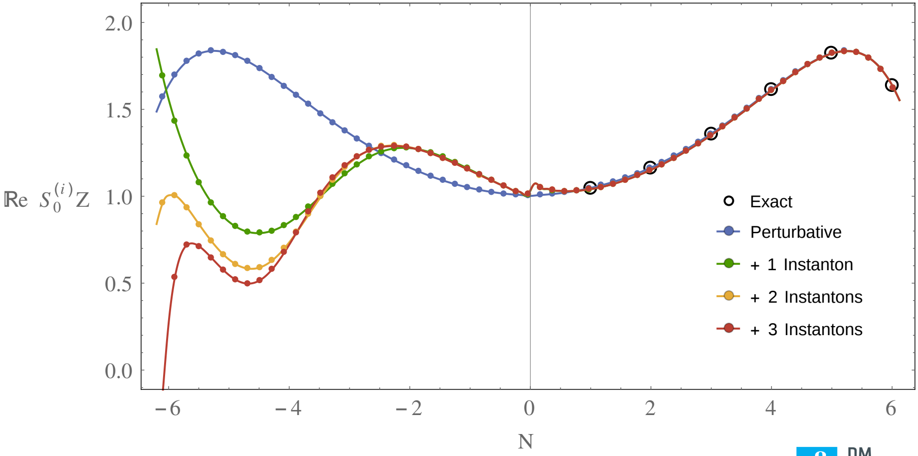


(interpolation at continuous $N \in (0,6)$ for $t = 1$)

Instantons *Dominate* Over the $1/N^2$ Perturbative Series!

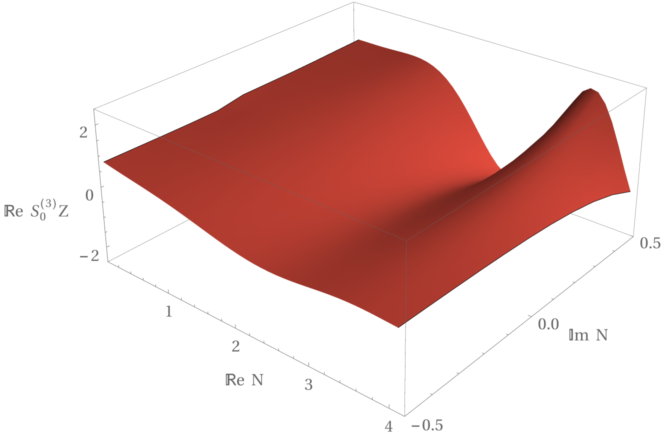


Partition Function at *Negative* Rank: $N \in \mathbb{R}$



(fixed $t = \frac{1}{2}\, \mathrm{e}^{\frac{5\pi \mathrm{i}}{6}}$ — accuracy no longer reliable for $N \lesssim -4$)

Partition Function for *Complex* Rank: $N \in \mathbb{C}$



- Fixed $t = 10\,\mathrm{e}^{\frac{99\pi i}{100}}$.
- What are the analytical properties of the partition function?...

Outline

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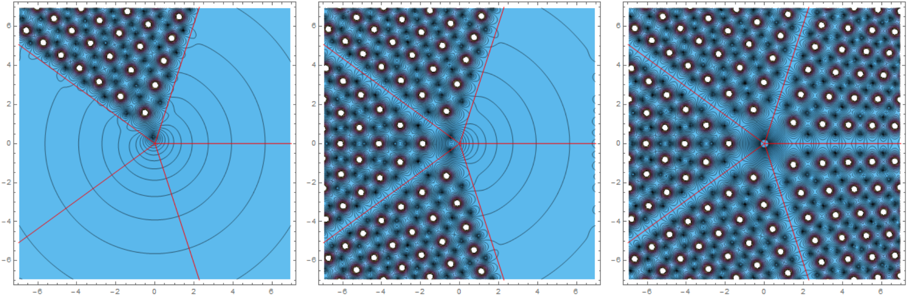
All Painlevé Solutions from Transseries?



$$u^2(z) - \frac{1}{6} u''(z) = z.$$

- Tritronquée solution: perturbative series alone.
- Tronquée solution: one-parameter transseries.
- General solution: full two-parameter transseries.

Resummation of Resurgent Transseries

- Resummation methods help describing different phases of our systems.
- Painlevé solutions generically have double poles,

$$u(z)\Big|_{z=z_0} \approx \frac{1}{(z-z_0)^2} + \cdots,$$

translating to simple zeroes of partition function $Z(z) \approx (z-z_0) + \cdots$.

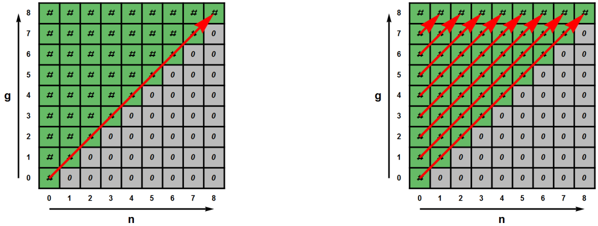
- Can reorganize (one-parameter) transseries double-sum

$$u\left(z,\sigma\right) = \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-nAz^{\frac{5}{4}}} \sum_{g=0}^{+\infty} \frac{u_g^{(n)}}{z^{\frac{5}{4}(g+\beta_n)}},$$

summing first over all instanton numbers?

- Starting order for each sector $\beta_n \propto n \Rightarrow$ *Linear*.

Linear Analytic Transseries Summation



- Linear analytic transseries summation: first sum leading terms for all sectors in the u-transseries.
- Introducing variable $\tau = \frac{\sigma}{12\sqrt{z}}\, \mathrm{e}^{-Az^{\frac{5}{4}}}$ this yields:

$$
\begin{aligned}
u(z,\tau) &= \sum_{g=0}^{+\infty} \frac{\mathbb{U}_g(\tau)}{z^{\frac{5}{4}g}} \simeq \mathbb{U}_0(\tau) + z^{-\frac{5}{4}}\mathbb{U}_1(\tau) + \cdots, \\
\mathbb{U}_g(\tau) &= \sum_{n=0}^{+\infty} \tau^n u_g^{(n)}.
\end{aligned}
$$

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Nonperturbative Arrays of Poles/Zeroes

- Each $\mathbb{U}_g(\tau)$ results in a convergent series! (transasymptotics [Costin])
- Exact result at leading order is

$$\mathbb{U}_0(\tau) = \frac{1 + 10\tau + \tau^2}{(1-\tau)^2}.$$

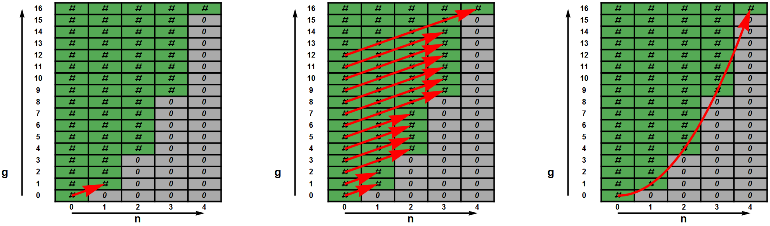
- *First* array of (tronquée) poles located at

$$\tau = 1.$$

(*array* = still need to invert Lambert-W function...)

- Sub-leading results iteratively yield following arrays of poles...

Quadratic Analytic Transseries Summation



- Linear analytic transseries summation: $\mathbb{Z}_0(\tau) = 1 - \tau$.
- For the partition function, starting order $\beta_n \propto n^2 \Rightarrow$ *Quadratic*.
- Quadratic analytic transseries summation: first sum leading terms for all sectors in the Z-transseries (much more efficient).
- At leading order already get **all arrays** of poles (zeroes of Z)!

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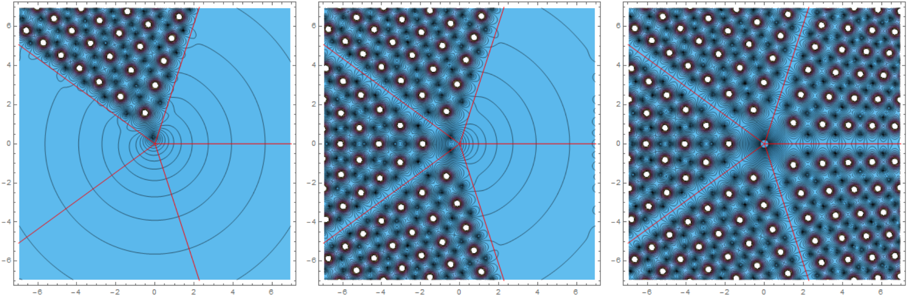
All Painlevé Solutions from Transseries ✓



- Tritronquée solution: perturbative series alone.
- Tronquée solution: one-parameter transseries [Aniceto-RS-Vonk].
- General solution: full two-parameter transseries [Aniceto-RS-Vonk].

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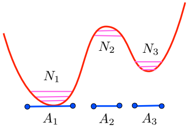
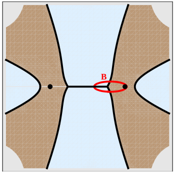
The Quartic Matrix Model and Its Instantons

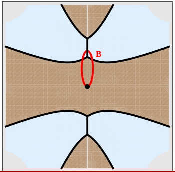


- Potential $V(z) = \frac{1}{2}z^2 - \frac{\lambda}{24}z^4$ generically three-cut solution.



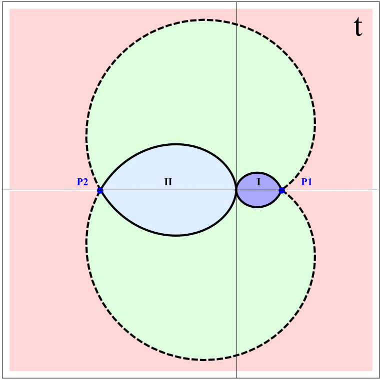
- One-cut $y^2 = \left(1 - \frac{\lambda}{6}\left(z^2 + 2\alpha^2\right)\right)^2 \left(z^2 - 4\alpha^2\right)$.
  [Mariño, Aniceto-RS-Vonk]

- Two-cut $\mathbb{Z}_2$-symmetric
  $y^2 = \frac{1}{36}\lambda^2 z^2 \left(z^2 - a^2\right)\left(z^2 - b^2\right)$. [RS-Vaz]



- Instantons from B-cycles
  [David, Seiberg-Shih, Mariño-RS-Weiss, RS-Vaz].

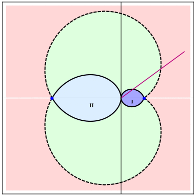
Quartic Phase Diagram at Complex 't Hooft Coupling



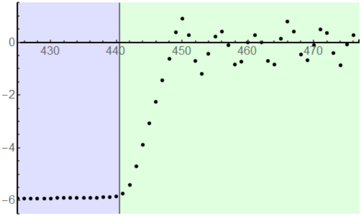
- Anti-Stokes lines (phase boundaries): $\mathbb{R}\mathrm{e}\left(\frac{A(t)}{g_s}\right) = 0$.
- Anti-Stokes phase [Bonnet-David-Eynard, Mariño-Pasquetti-Putrov, Aniceto-RS-Vonk].
- Trivalent-tree phase [David, Bertola, Aniceto-RS-Vonk].

Moving into the Anti-Stokes Phase

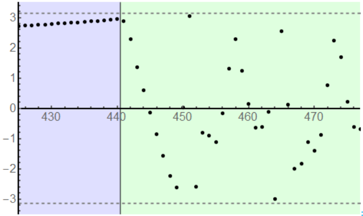
- Numerical (recursive) calculation of recursion coefficients r_n...
- Fix $N = 1000$, $\arg t = \frac{\pi}{12}$; vary $|t|$ from one-cut through anti-Stokes regions...



- Numerical results:



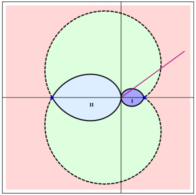
(r_n−r): log of absolute value



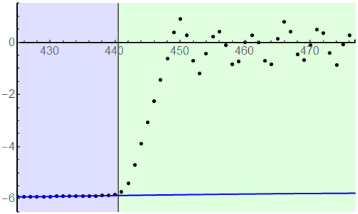
(r_n−r): phase

Perturbative Results in the Anti-Stokes Phase

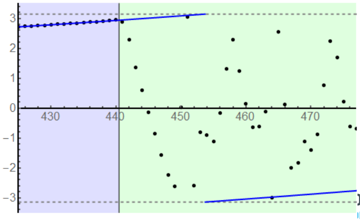
- Numerical (recursive) calculation of recursion coefficients r_n...
- Fix $N = 1000$, $\arg t = \frac{\pi}{12}$; vary $|t|$ from one-cut through anti-Stokes regions...



- Comparison with the perturbative transseries:



(r_n−r) vs (R_{pert}−r): log of absolute value



(r_n−r) vs (R_{pert}−r): phase

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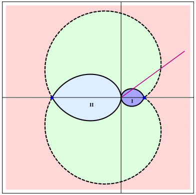
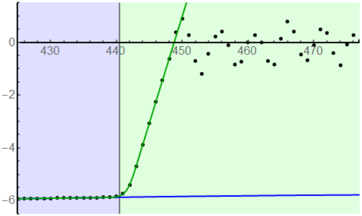
One-Instanton Results in the Anti-Stokes Phase

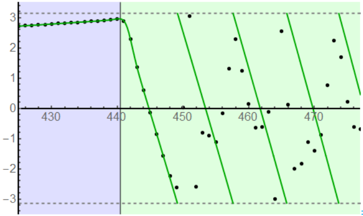
- Numerical (recursive) calculation of recursion coefficients r_n...
- Fix $N = 1000$, $\arg t = \frac{\pi}{12}$; vary $|t|$ from one-cut through anti-Stokes regions...



- Comparison with perturbative plus one-instanton:



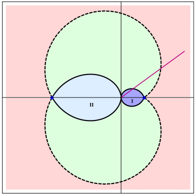
(r_n−r) vs ($R_{1\text{−inst}}$−r): log of absolute value



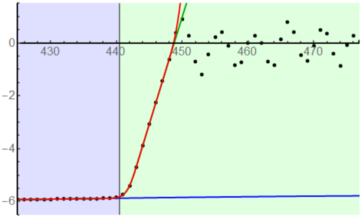
(r_n−r) vs ($R_{1\text{−inst}}$−r): phase

Three-Instanton Results in the Anti-Stokes Phase

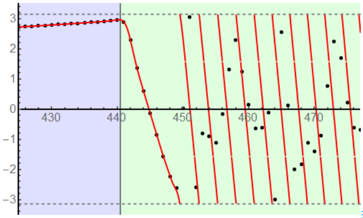
- Numerical (recursive) calculation of recursion coefficients r_n...
- Fix $N = 1000$, $\arg t = \frac{\pi}{12}$; vary $|t|$ from one-cut through anti-Stokes regions...



- Comparison up to three-instanton contributions:



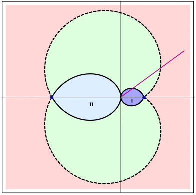
$(r_n - r)$ vs $(R_{3\text{-inst}} - r)$: log of absolute value



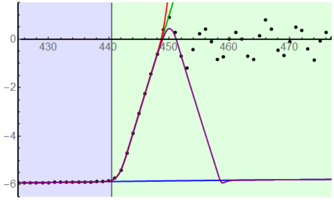
$(r_n - r)$ vs $(R_{3\text{-inst}} - r)$: phase

Linear Analytic Transseries Summation

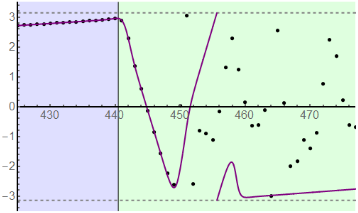
- Numerical (recursive) calculation of recursion coefficients r_n...
- Fix $N = 1000$, $\arg t = \frac{\pi}{12}$; vary $|t|$ from one-cut through anti-Stokes regions...



- Linear analytic transseries summation (*leading* in g_s):



$(r_n - r)$ vs $(R_{\text{lin.ATS},1} - r)$: log of absolute value



$(r_n - r)$ vs $(R_{\text{lin.ATS},1} - r)$: phase

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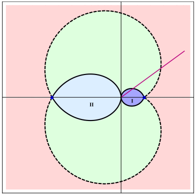
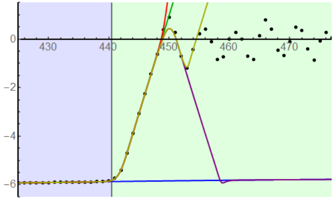
Linear Analytic Transseries Summation

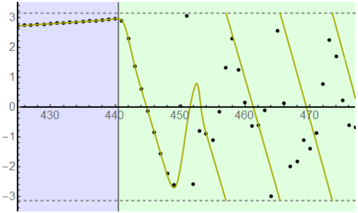
- Numerical (recursive) calculation of recursion coefficients r_n...
- Fix $N = 1000$, $\arg t = \frac{\pi}{12}$; vary $|t|$ from one-cut through anti-Stokes regions...



- Linear analytic transseries summation (*next-to-leading* in g_s):



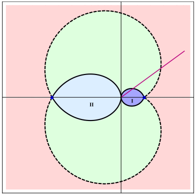
$(r_n - r)$ vs $(R_{\text{lin.ATS},2} - r)$: log of absolute value



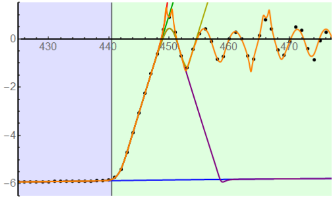
$(r_n - r)$ vs $(R_{\text{lin.ATS},2} - r)$: phase

Quadratic Analytic Transseries Summation

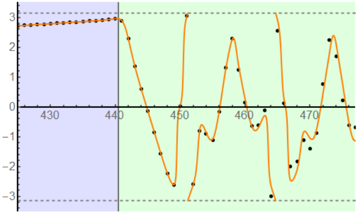
- Numerical (recursive) calculation of recursion coefficients r_n...
- Fix $N = 1000$, $\arg t = \frac{\pi}{12}$; vary $|t|$ from one-cut through anti-Stokes regions...



- **Quadratic** analytic transseries summation (leading in g_s):



$(r_n - r)$ vs $(R_{\text{quad.ATS},1} - r)$: log of absolute value



$(r_n - r)$ vs $(R_{\text{quad.ATS},1} - r)$: phase

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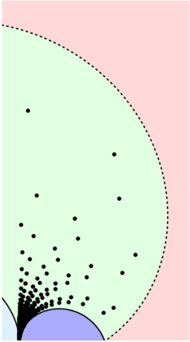
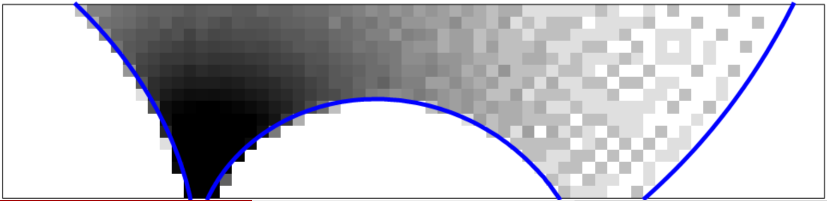
Fisher/Lee–Yang Zeroes of Partition Function

- Comparison of Fisher/Lee–Yang Zeroes for the partition function: quadratic analytic transseries summation ($N = 10$ eigenvalues) *versus* numerical matrix-integral calculation ($N = 100$ eigenvalues)... **[Aniceto-RS-Vonk]**





Outline

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Semiclassical Decodings via Transseries?

- Nonperturbative definition: not always available in string theory...
- Perturbative series: *intuitive*, but *incomplete* across parameter space...
- Resurgence ⇒ perturbative series ⇒ nonperturbative transseries...
- Semiclassical decoding via transseries: **[Mariño]**

  "...A nonperturbative function in quantum theory can be semiclassically decoded if it can be written as the Borel–Écalle resummation of a transseries..."

Local $\mathbb{P}^2$ and Mirror Quantization

- For our example of B-model on mirror of local $\mathbb{P}^2$ $\Rightarrow$ There is a nonperturbative definition in the literature! [Grassi-Hatsuda-Mariño, Kashaev-Mariño, Mariño-Zakany, Codesido-Grassi-Mariño ... Gu-Sulejmanpasic]
  - Actually extends for toric Calabi–Yau geometries...
- Mirror curve of the geometry is given by

$$W_{\mathbb{P}^2}(x,p;\boldsymbol{z}) = \mathrm{e}^{x} + \mathrm{e}^{p} + \mathrm{e}^{-x-p} + \boldsymbol{z}.$$

- Nonperturbative definition based upon quantization of mirror curve:

$$\hat{\mathcal{O}}_{\mathbb{P}^2}(\hat{x},\hat{p}) = \mathrm{e}^{\hat{p}} + \mathrm{e}^{\hat{x}} + \mathrm{e}^{-\hat{p}-\hat{x}},$$

with commutation relations $[\hat{x},\hat{p}] = \mathrm{i}\hbar$.

Local $\mathbb{P}^2$ and Nonperturbative Completions

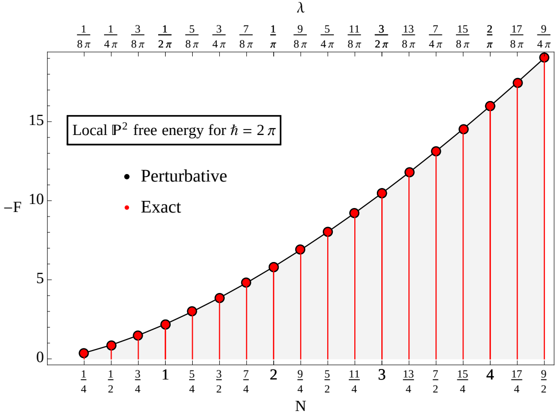
- Inverse operator $\hat{\rho} = \hat{\mathcal{O}}^{-1}$ acting on $L^2(\mathbb{R})$ is trace class and positive definite $\Rightarrow$ Analytic spectral determinant:

$$\mathcal{Z}(\kappa, \hbar) = \det\left(1 + \kappa \hat{\rho}\right).$$

- Canonical partition function follows from spectral-trace expansion:

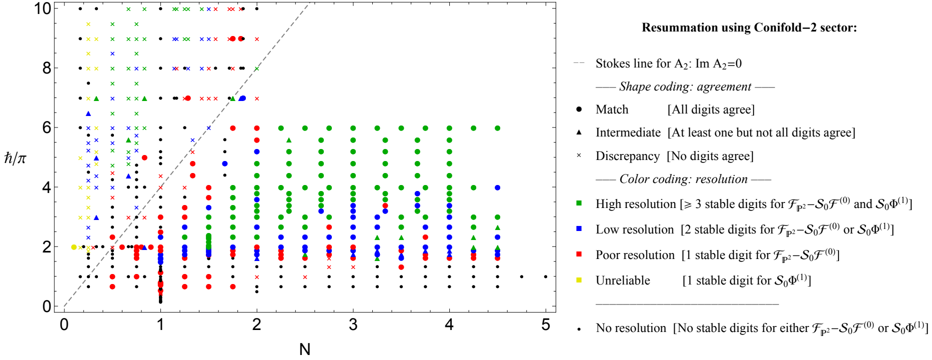
$$\det\left(1 + \kappa \hat{\rho}\right) = 1 + \sum_{N=1}^{+\infty} Z(N, \hbar)\, \kappa^N.$$

Local $\mathbb{P}^2$: Resummation of the String Free Energy



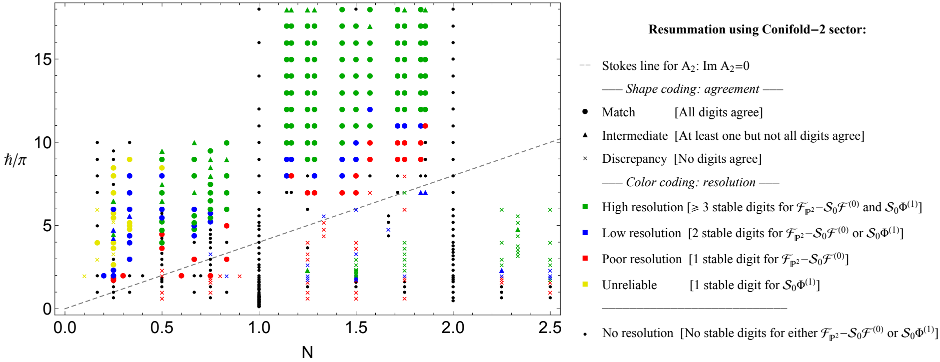
- Matching perturbative results... [Couso-Mariño-RS]
- Notation is $\hbar = \frac{4\pi^2}{g_s}$ and $\lambda = \frac{N}{\hbar}$ (with $\lambda = \frac{\sqrt{3}}{12\pi^2}\, t_c$).

Local $\mathbb{P}^2$: Resummations and Stokes Phenomena I



- Matching nonperturbative one-instanton results before Stokes line...
- Notation is $\hbar = \dfrac{4\pi^2}{g_s}$ and $\lambda = \dfrac{N}{\hbar}$ (with $\lambda = \dfrac{\sqrt{3}}{12\pi^2}\, t_c$).

Local $\mathbb{P}^2$: Resummations and Stokes Phenomena II



- Crossing Stokes line and still find matching results! [Couso-Mariño-RS]
- Stokes jump is: $2\pi\mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}N} \to 2\pi\mathrm{i}\,\mathrm{e}^{2\pi\mathrm{i}N} - 2\pi\mathrm{i}$.

Local $\mathbb{P}^1 \times \mathbb{P}^1$: Different Nonperturbative Completions?

- For the case of strings on local $\mathbb{P}^1 \times \mathbb{P}^1$ there are actually two distinct nonperturbative definitions in the literature!
  - Quantization of mirror curve: $\hat{\mathcal{O}}_{\mathbb{P}^1\times\mathbb{P}^1}(\hat{x},\hat{p}) = \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + \mathrm{e}^{\hat{x}} + m\,\mathrm{e}^{-\hat{x}}$, partition function out of spectral trace expansion of Fredholm determinant $\det\left(1+\kappa\,\rho\right) = 1 + \sum_{N=1}^{+\infty} Z(N,\hbar)\,\kappa^N$ (with $\rho = \hat{\mathcal{O}}^{-1}$).
  - Chern–Simons partition function on lens space $L(2,1) \simeq \mathbb{S}^2/\mathbb{Z}_2$ (localized to 2-cuts matrix model). [Gopakumar-Vafa, Aganagic-Klemm-Mariño-Vafa]

- In what / how much do these definitions differ?
- Can the same transseries match both results?
- How would the corresponding semi-classical decodings differ? [Mariño]

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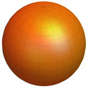
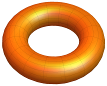
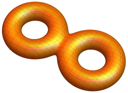
Semiclassics of *Resonant* Transseries? 2D Gravity Example

- Back to Painlevé I describing 2d gravity

$$u^2(z) - \frac{1}{6} u''(z) = z,$$

with asymptotic (perturbative) free energy $F''(z) = -u(z)$,

$F \simeq \frac{1}{g_{\mathsf{s}}^2}$  $+$  $+\ g_{\mathsf{s}}^2$  $+ \cdots .$

- Two-parameter transseries solution is resonant:

$$u(g_{\mathsf{s}}, \sigma_1, \sigma_2) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m \, \mathrm{e}^{-(n-m)\frac{A}{g_{\mathsf{s}}}} \left(\sum_{k=0}^{\min(n,m)} \log^k(g_{\mathsf{s}}) \cdot \Phi_{(n|m)}^{[k]}(g_{\mathsf{s}}) \right).$$

- When $n = m \Rightarrow$ all diagonal *nonperturbative* sectors of same weight!

Analytic Structure of Painlevé *Resonant* Solutions

- Transseries is resonant: $\boldsymbol{A} = (A, -A)$. Cleaner interpretation if mod-out transseries by generator of resonance...
- Explicitly this implies free energy is:

$$\begin{aligned} F(g_{\mathsf{s}}, \boldsymbol{\sigma}) &= \sum_{m=0}^{+\infty} (\sigma_1 \sigma_2)^m \, \Phi_{(m|m)}(g_{\mathsf{s}}) + \\ &+ \sum_{n=1}^{+\infty} \sigma_1^n \, \mathrm{e}^{-n\frac{A}{g_{\mathsf{s}}}} \sum_{m=0}^{+\infty} (\sigma_1 \sigma_2)^m \, \Phi_{(m+n|m)}(g_{\mathsf{s}}) + \\ &+ \sum_{n=1}^{+\infty} \sigma_2^n \, \mathrm{e}^{n\frac{A}{g_{\mathsf{s}}}} \sum_{m=0}^{+\infty} (\sigma_1 \sigma_2)^m \, \Phi_{(m|m+n)}(g_{\mathsf{s}}). \end{aligned}$$

- From explicit data for transseries diagonal components find:

$$\sum_{m=0}^{+\infty} (\sigma_1 \sigma_2)^m \, \Phi_{(m|m)}(g_{\mathsf{s}}) = \sum_{g=0}^{+\infty} \sum_{h=0}^{+\infty} (\sigma_1 \sigma_2)^h \, F_g^{(h|h)} g_{\mathsf{s}}^{2g+h-2}.$$

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Semiclassical Decoding of Painlevé *Resonant* Solutions

- Use 't Hooft-like parameter $s = g_{\mathsf{s}}\sigma_1\sigma_2$ and 't Hooft-like limit $g_{\mathsf{s}} \to 0$, $\sigma_1\sigma_2 \to +\infty$ with fixed s, to find closed string theory

$$\sum_{m=0}^{+\infty} (\sigma_1\sigma_2)^m \, \Phi_{(m|m)}(g_{\mathsf{s}}) = \underbrace{\sum_{g=0}^{+\infty} F_g(s)\, g_{\mathsf{s}}^{2g-2}}_{F_g(s)=\sum_{h=0}^{+\infty} F_g^{(h|h)}\, s^h} \quad .$$

- Modulus dependence $F_g \equiv F_g(s)$... Topological strings inside minimal Painlevé strings?... Yes ✓ [Gamayun-Iorgov-Lisovyy, Bonelli-Lisovyy-Maruyoshi-Sciarappa-Tanzini]
  - $\sim$ Nekrasov–Okounkov (dual) partition function from $4d$ $\mathcal{N} = 2$ SU(2) $N_{\mathsf{f}} = 1$ gauge theory at conformal point $\equiv$ Argyres–Douglas theory H_0...
  - What is (direct) resurgence of this theory?...

Outline

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Summary and Open Questions

- Wrap-up:
  - Whole framework is working very well with matrix models ✓
  - It is also working very well with topological strings ✓
  - Known to be working very well in quantum mechanics ✓

    [Álvarez, Voros, Zinn-Justin ... Dunne–Ünsal]
  - Very recently shown to be working very well with ...superconductors!...

    [Mariño–Reis]

- Questions:
  - Can this be extended to QFT and gauge theories?
  - Can this be extended to *full* string theory?