Triality in Little String Theories

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based on work in collaboration with: Brice Bastian, Amer Iqbal and Soo-Jong Rey

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UCB Lyon 1

Little String Theories

Over the last decades string theory has provided insights into **strongly coupled** quantum systems

Specifically: prediction of existence of new interacting conformal field theories in dimensions $D > 4$

e.g.: [Seiberg 1996]

String theory
- extended objects
- gravitation

suitable decoupling of gravity →

quantum field theory
- point-like degrees of freedom
- well defined energy momentum tensor

String theory also predicts the existence of new 'non-local theories', e.g. **little string theories (LSTs)**

String theory
- extended objects
- gravitation
- compactification to $D > 4$

suitable decoupling of gravity →

little string theory
- intrinsic string scale M_{string} remains
  - $\ll M_{\text{string}}$: effective QFT (point-like dofs)
  - $> M_{\text{string}}$: UV-comp. contains stringy dofs
- well defined energy momentum tensor

different approaches: [Witten 1995]
[Aspinwall, Morrison 1997]
[Seiberg 1997]
[Intriligator 1997]
[Hanany, Zaffaroni 1997]
[Brunner, Karch 1997]

Classification of LSTs (ADE type for theories with $\mathcal{N} = (2,0)$ supersymmetry)

[Bhardwaj, Del Zotto, Heckman, Morrison, Rudelius, Vafa 2016]

Rich class of examples realised in 11-dimensional **M-theory** through systems of parallel M5-branes with M2-branes stretched between them

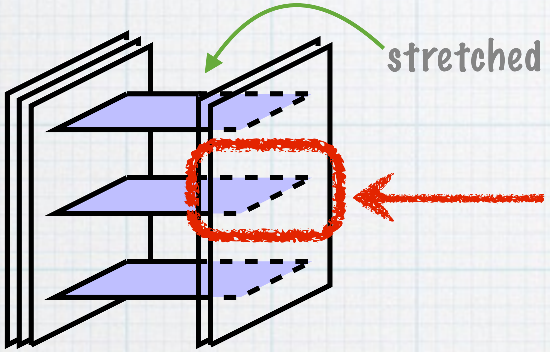
[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
[Haghighat, Kozçaz, Lockhart, Vafa 2013]
[SH, Iqbal 2013]
[Haghighat 2015]
[SH, Iqbal, Rey 2015]
[Haghighat, Murthy, Vafa, Vandoren 2015]

Parallel M5-branes

M-branes (M2- and M5) are extended in objects in 11-dimensional M-theory

They can be arranged in a fashion to preserve (some amount of) supersymmetry: **brane webs**

* String-like objects arise at the intersection of M5- and M2-branes



* many dual realisations allowing to explicitly compute quantities (e.g. partition function)

notably: F-theory compactification on toric, non-compact Calabi-Yau threefolds

[Morrison, Vafa 1996]
[Heckman, Morrison, Vafa 2013]
[Del Zotto, Heckman, Tomasiello, Vafa 2014]
[Heckman 2014]
[Haghighat, Klemm, Lockhart, Vafa 2014]
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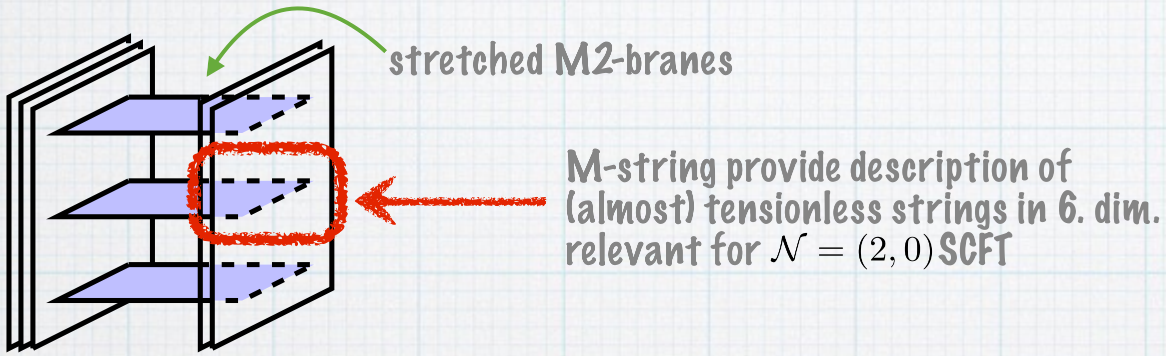

* many dual realisations allowing to explicitly compute quantities (e.g. partition function)

  notably: F-theory compactification on toric, non-compact Calabi-Yau threefolds

* depending on the details of the brane configuration, a large class of different **Little Strings** (or their duals) can be realised and studied very explicitly

* low energy limit associated with non-abelian supersymmetric field theories (mass deformed $\mathcal{N} = 2^*$ theories upon compactification to 4 dimensions)

Class of theories exhibits interesting (and non-expected) **dualities**!   in this talk: **triality**

6-dimensional systems:
- gravity is decoupled
- have an intrinsic string scale
- obtained from type II string theory through the decoupling limit

$$g_{\text{st}} \to 0 \text{ while } \ell_{\text{st}} = \text{fixed}$$

Little String Theories with 16 supercharges (A-series)

* IIb LST of type A_{N-1} with $\mathcal{N} = (2,0)$ supersymmetry
  - -) decoupling limit of N M5-branes with transverse space $\mathbb{S}^1 \times \mathbb{R}^4$
  - -) decoupling limit of a stack of N NS5-branes in type IIA with transverse space $\mathbb{R}^4$
  - -) type IIB string theory on A_{N-1} orbifold background

related by T-duality

* IIa LST of type A_{N-1} with $\mathcal{N} = (1,1)$ supersymmetry
  - -) decoupling limit of a stack of N NS5-branes in type IIB with transverse space $\mathbb{R}^4$
  - -) type IIA string theory on A_{N-1} orbifold background

BPS states from the point of view of M5-branes correspond to M2-branes ending on them

Little Strings

6-dimensional systems:
- gravity is decoupled
- have an intrinsic string scale
- obtained from type II string theory through the decoupling limit

$$g_{\rm st} \to 0 \quad \text{while} \quad \ell_{\rm st} = \text{fixed}$$

Little String Theories with 8 supercharges: **particular class** obtained as

* $\mathbb{Z}_N$ orbifold of IIa LST of type A_{M-1} with $\mathcal{N}=(1,0)$ supersymmetry
  - -) decoupling limit of M M5-branes with transverse space $\mathbb{S}^1 \times \text{ALE}_{A_{N-1}}$
  - -) decoupling limit of a stack of N NS5-branes in type IIB with transverse space $\mathbb{R}^4/\mathbb{Z}_N$

related by T-duality

* $\mathbb{Z}_M$ orbifold of IIb LST of type A_{N-1} with $\mathcal{N}=(1,0)$ supersymmetry
  - -) decoupling limit of N M5-branes with transverse space $\mathbb{S}^1 \times \text{ALE}_{A_{M-1}}$
  - -) decoupling limit of a stack of N NS5-branes in type IIA with transverse space $\mathbb{R}^4/\mathbb{Z}_M$

Explicit computation of BPS partition function using various methods

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
[Haghighat, Kozçaz, Lockhart, Vafa 2013]
[SH, Iqbal 2013]
[SH, Iqbal, Rey 2015]

in this talk:
further dualities

Brane Configurations

The most general configuration of branes in M-theory in 11 dimensions looks like

	0	1	2	3	4	5	6	7	8	9	10
M5-branes	•	•	•	•	•	•					
M2-branes	•	•					•				

Directions 2–5: $\mathbb{R}^4_{||}$

Directions 7–10: $\mathrm{ALE}_{A_{M-1}} \sim \mathbb{R}^4/\mathbb{Z}_M$

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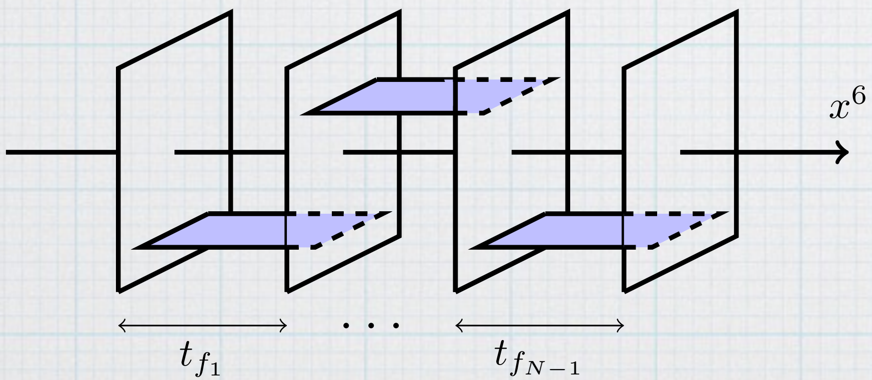
Directions 2–5: $\mathbb{R}^4_{||}$ Directions 7–10: $\text{ALE}_{A_{M-1}} \sim \mathbb{R}^4/\mathbb{Z}_M$

non-compact case: $\mathbb{R}$

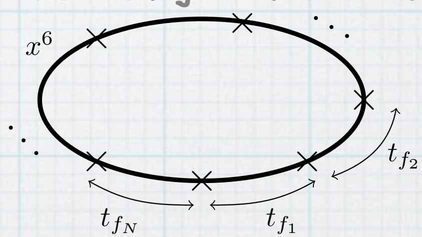
M5-branes distributed along non-comp. (6)-direction

with M2-branes stretched between them

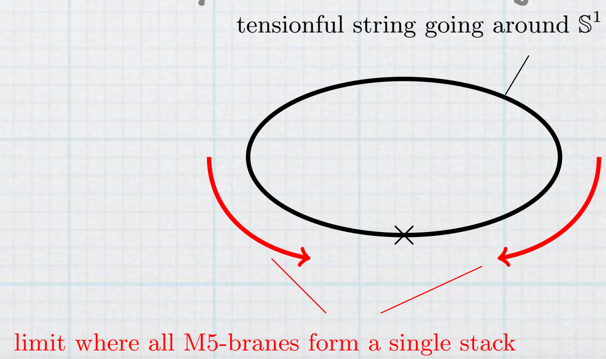


compact case: S^1

M5-branes arranged on a circle $R_6 = \frac{\rho}{2\pi i}$



necessary for little-string interpretation



Brane Configurations

The most general configuration of branes in M-theory in 11 dimensions looks like

	(0)	(1)	2	3	4	5	6	7	8	9	10
M5-branes	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$					
M2-branes	$\bullet$	$\bullet$					$\bullet$				
ϵ_1			$\circ$	$\circ$				$\circ$	$\circ$	$\circ$	$\circ$
ϵ_2					$\circ$	$\circ$		$\circ$	$\circ$	$\circ$	$\circ$

Compactification: Compactify (0,1) to $T^2 \sim S^1 \times S^1$ with radii R_0 and $R_1 =: \frac{\tau}{2\pi i}$

Deformations: there are two types of deformations with respect to the compactified (0,1)-directions introducing complex coordinates $(z_1, z_2) = (x_2 + ix_3, x_4 + ix_5)$ and $(w_1, w_2) = (x_7 + ix_8, x_9 + ix_{10})$

(0)-direct.: $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$: $(z_1, z_2) \to (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2)$ and $(w_1, w_2) \to (e^{-i\pi(\epsilon_1+\epsilon_2)} w_1, e^{-i\pi(\epsilon_1+\epsilon_2)} w_2)$

(1)-direct.: $U(1)_m$: $(w_1, w_2) \to (e^{2\pi i m} w_1, e^{-2\pi i m} w_2)$

gauge theory: Omega-background [Nekrasov 2012]

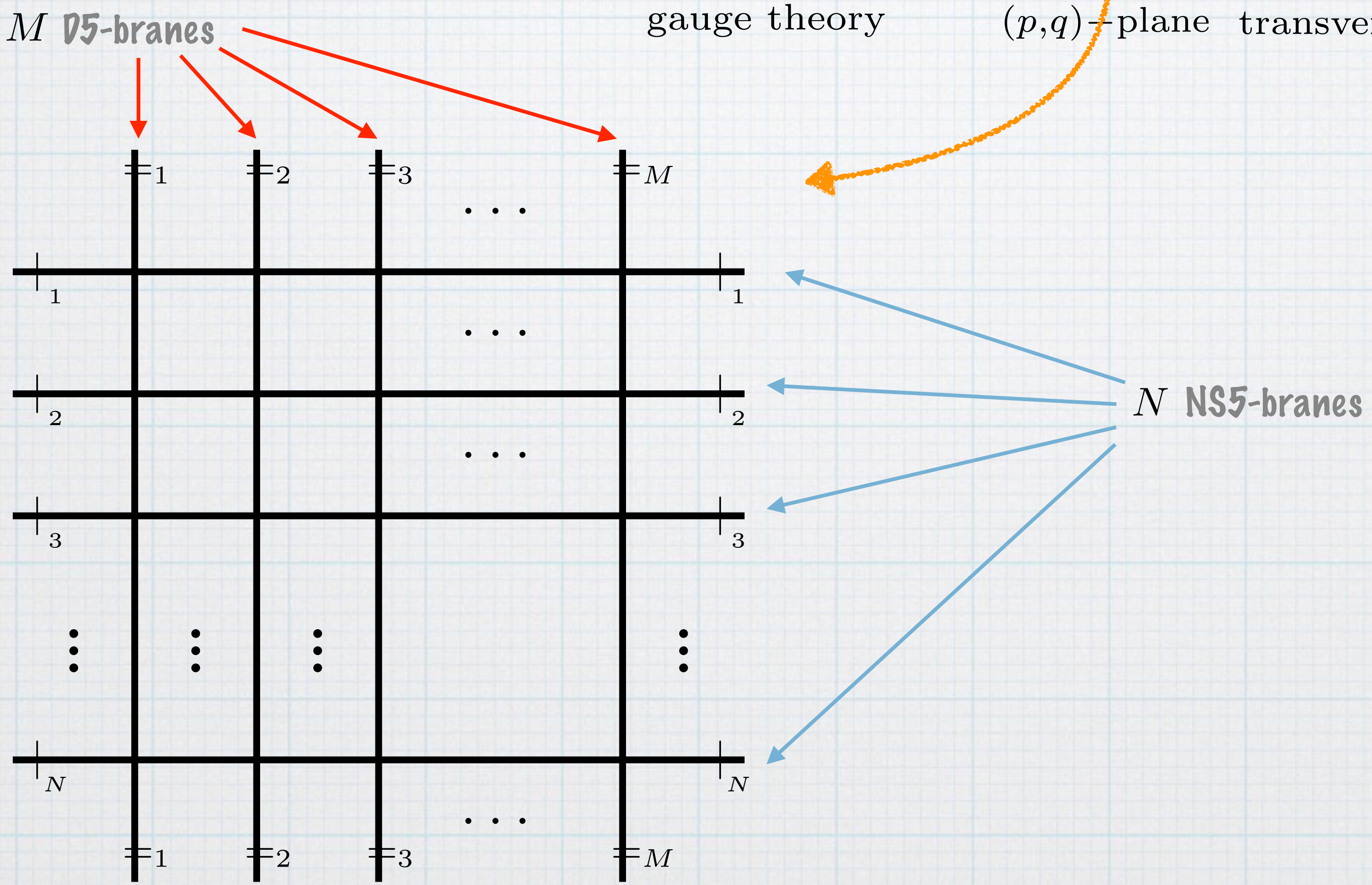
mass-deformation

Dual Setups to Brane Configurations

For vanishing mass deformation ($m = 0$) the M-brane configuration is dual to D5-NS5-branes in IIB

	0	1	2	3	4	5	6	7	8	9
D5 branes	●	●	●	●	●	●	–			
NS5 branes	●	●	●	●	●	–	●			

Dimensions 0–4: gauge theory; 5–6: (p,q)-plane; 7–9: transverse $\mathbb{R}^3$



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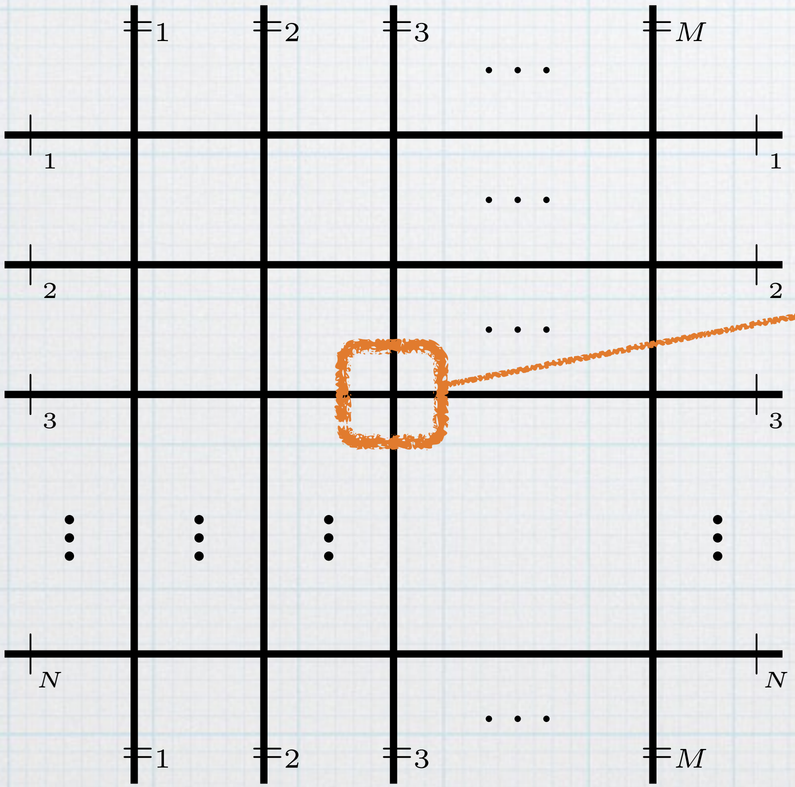
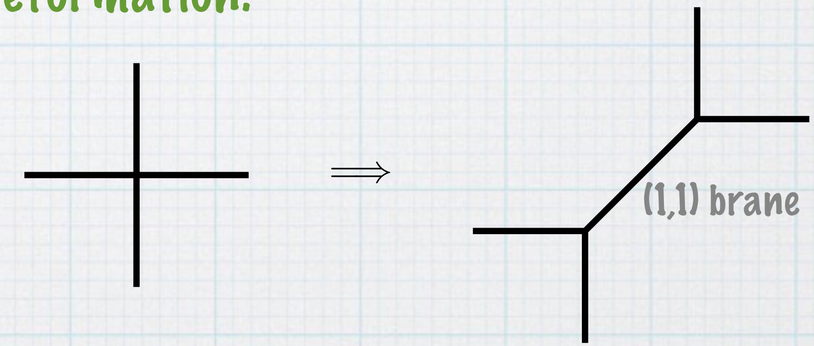
D5 branes	•	•	•	•	•	•	–			
NS5 branes	•	•	•	•	•	–	•			
	gauge theory					(p,q)-plane		transverse $\mathbb{R}^3$		



Deformation:



uplift the deformed type II configuration to M-th.
on an elliptically fibered Calabi-Yau threefold $X_{N,M}$

[Leung, Vafa 1997]

topic diagram of $X_{N,M}$ same as deformed brane web

Dual Construction of LSTs: Toric Calabi-Yau 3folds

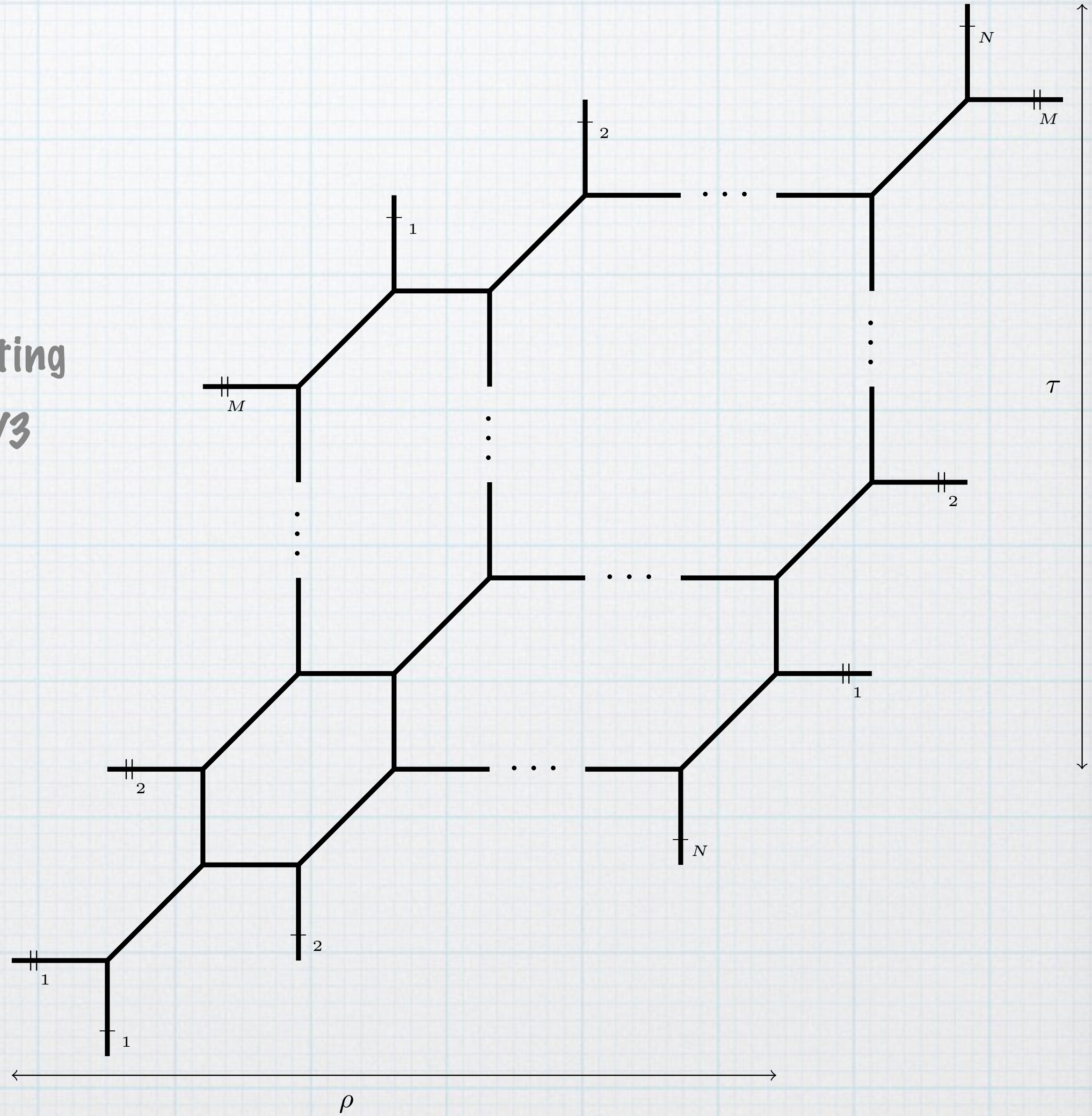
Specific, 2-parameter series of toric, double elliptically fibered Calabi-Yau threefolds $X_{N,M}$

Toric Web Diagram:

* (N, M) web on a torus
* double elliptic fibration structure with parameters (ρ, τ)
* $3NM$ different parameters representing the area of various curves C of the CY3

$$d = \int_C \omega$$

Kähler form



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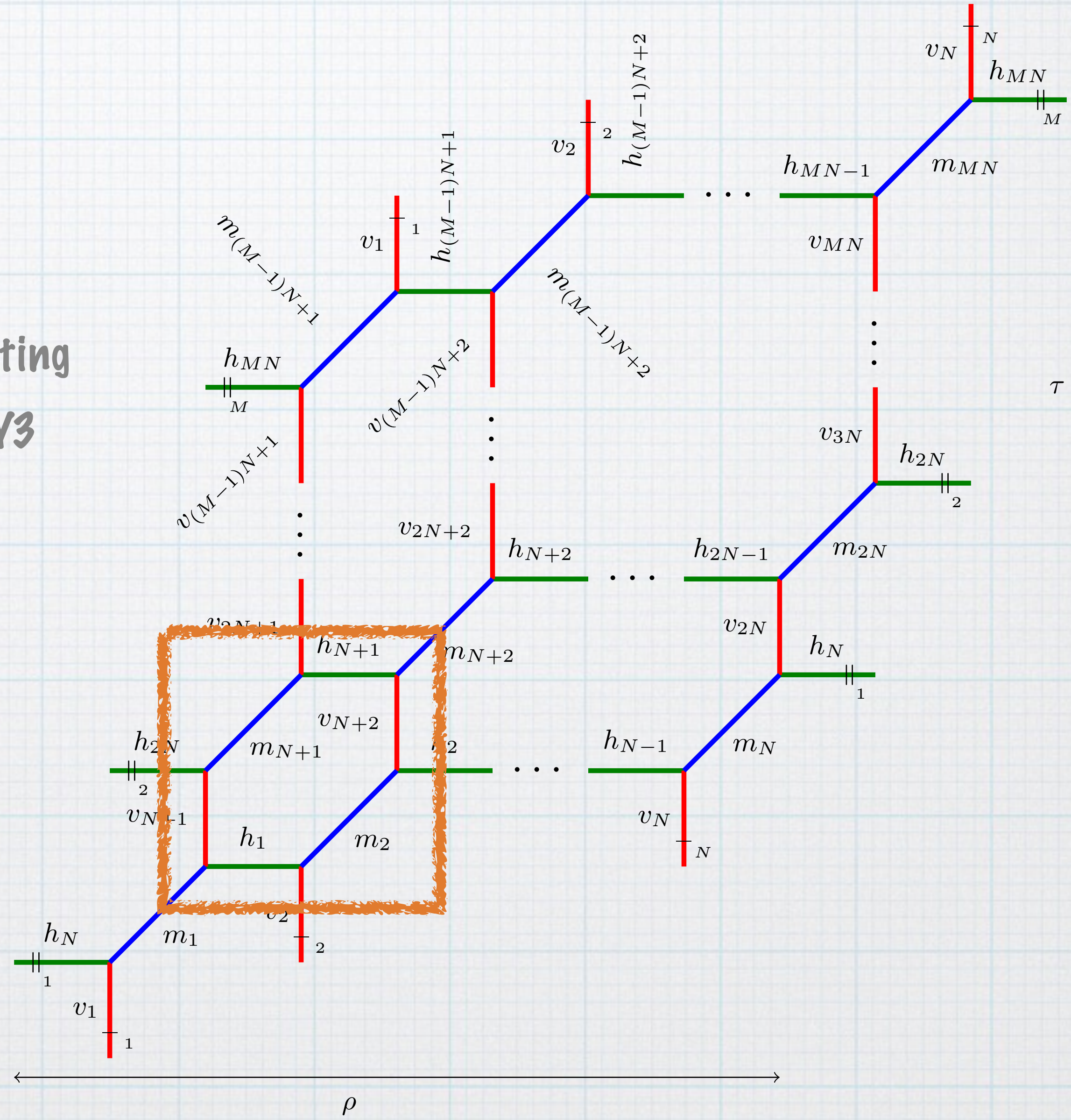
Kähler form

-) NM horizontal lines $h_{1,\ldots,NM}$

-) NM vertical lines $v_{1,\ldots,NM}$

-) NM diagonal lines $m_{1,\ldots,NM}$

* only $NM+2$ independent parameters due to consistency conditions



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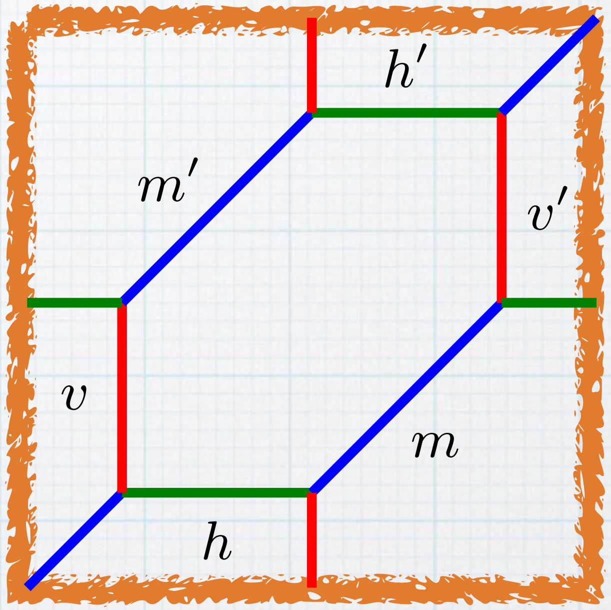

$$h + m = h' + m'$$
$$v + m' = m + v'$$

different possible choices for set of independent parameters

BPS States and Topological String

Free Energy: Counts number of BPS configurations, i.e. M2-branes wrapping holomorphic curves on the CY3 $X_{N,M}$. Captured by topological free energy $F_{N,M} = \ln \mathcal{Z}_{N,M}$ of $X_{N,M}$

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
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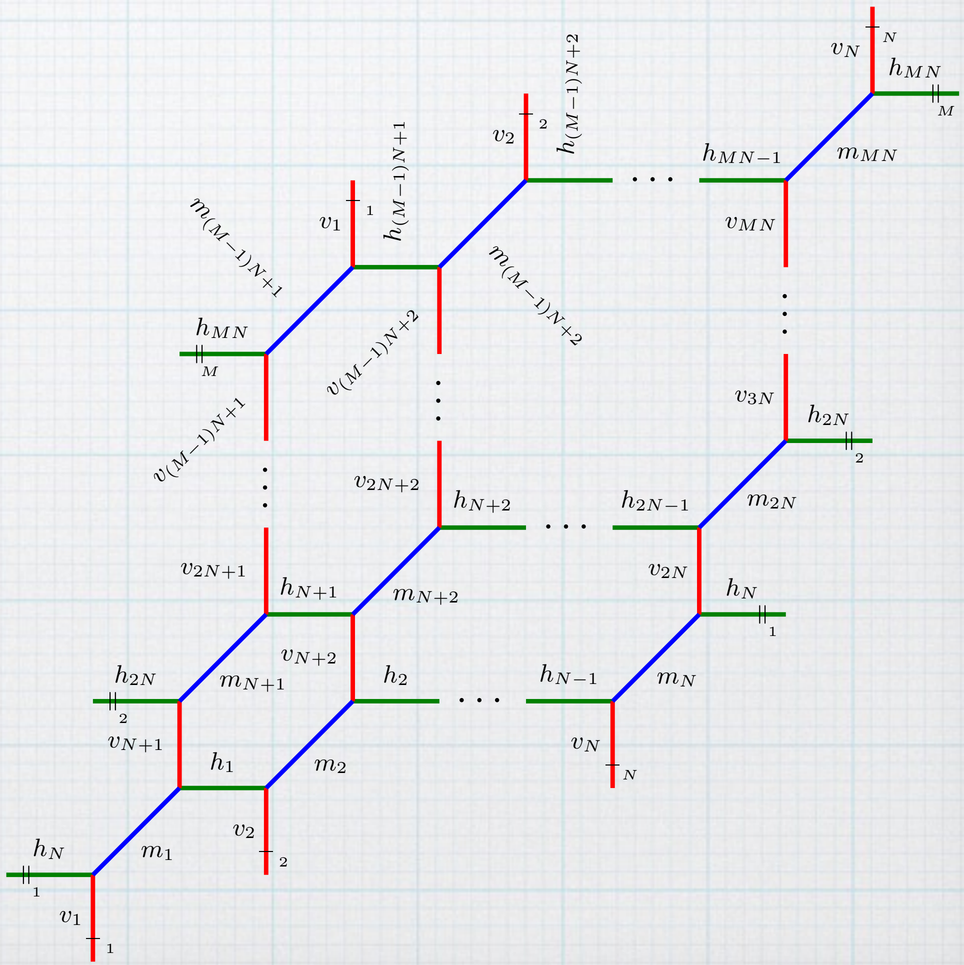
Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the **refined topological vertex**



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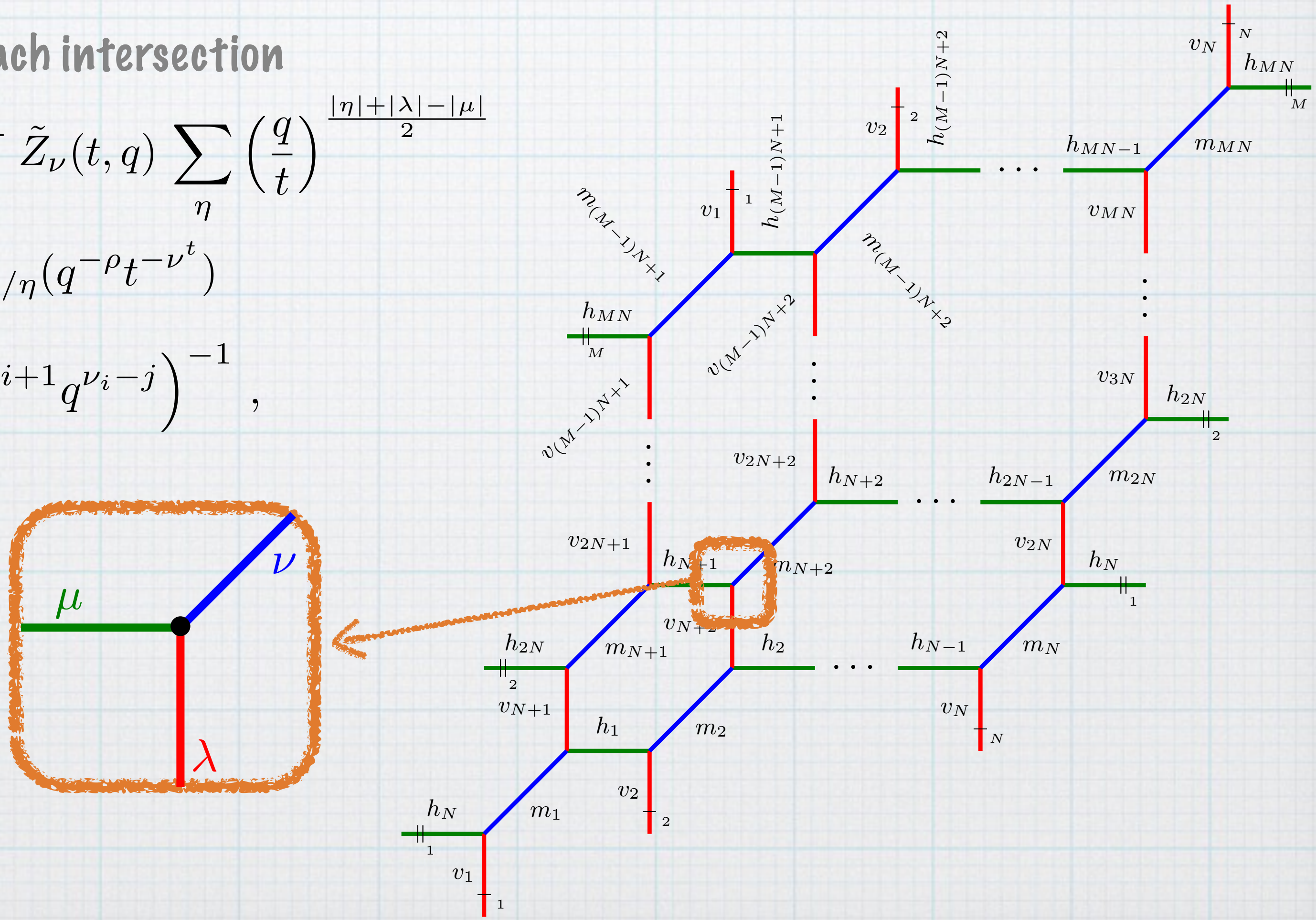
Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the **refined topological vertex**

-) assign trivalent vertex to each intersection

$$C_{\lambda\mu\nu} = q^{\frac{||\mu||^2}{2}} t^{-\frac{||\mu^t||^2}{2}} q^{\frac{||\nu||^2}{2}} \tilde{Z}_\nu(t,q) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}}$$

$$\times\, s_{\lambda^t/\eta}(t^{-\rho} q^{-\nu})\, s_{\mu/\eta}(q^{-\rho} t^{-\nu^t})$$

$$\tilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu} \left(1 - t^{\nu^t_j - i + 1} q^{\nu_i - j}\right)^{-1},$$



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Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the refined topological vertex

-) assign trivalent vertex to each intersection

$$C_{\lambda\mu\nu} = q^{\frac{||\mu||^2}{2}} t^{-\frac{||\mu^t||^2}{2}} q^{\frac{||\nu||^2}{2}} \tilde{Z}_\nu(t,q) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}}$$

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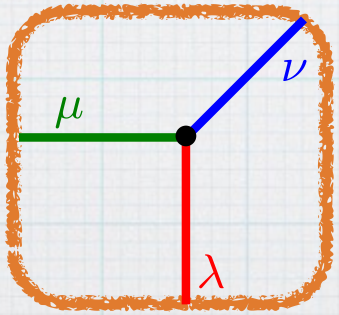
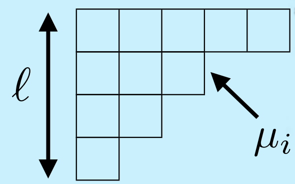


Notation:

$q = e^{2\pi i \epsilon_1}$ and $t = e^{-2\pi i \epsilon_2}$

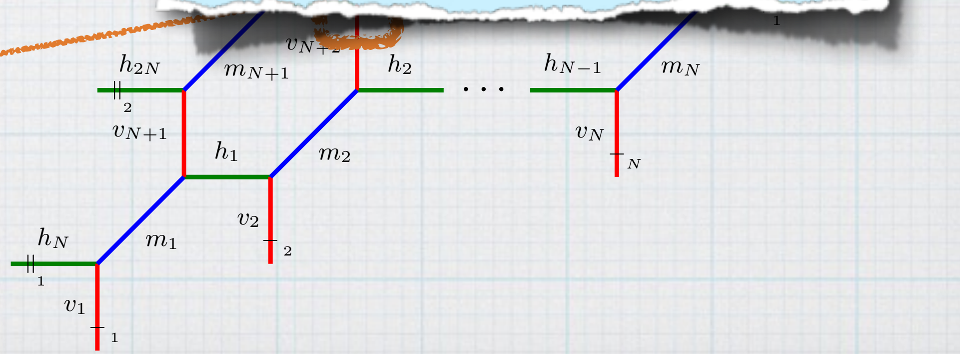
μ, ν, λ integer partitions

$$|\mu| = \sum_{i=1}^{\ell} \mu_i$$



$$||\mu||^2 = \sum_{i=1}^{\ell} \mu_i^2$$

$s_{\mu/\eta}$ skew Schur function



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-) assign trivalent vertex to each intersection

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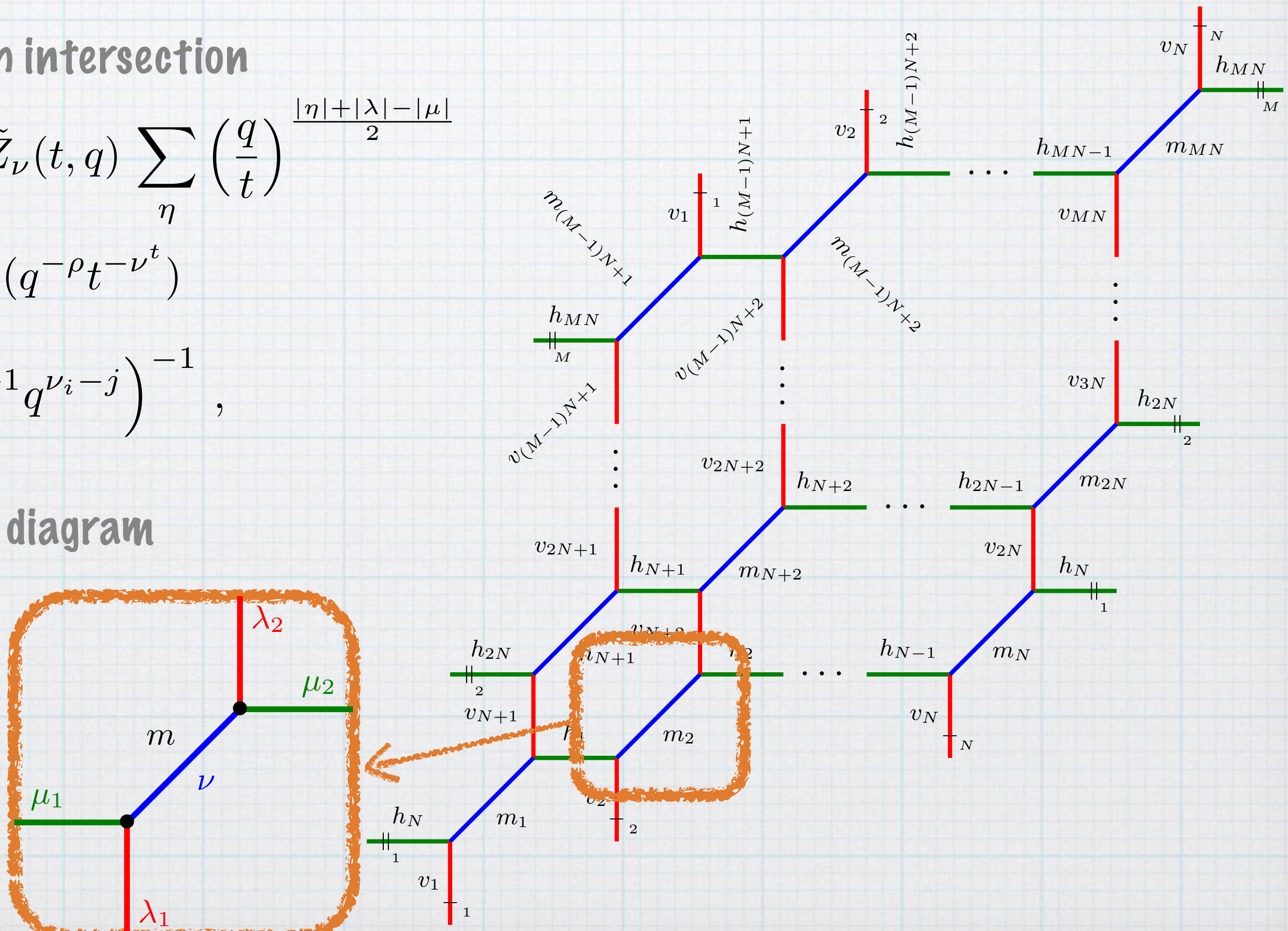
$$\times\, s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu})\, s_{\mu/\eta}(q^{-\rho}t^{-\nu^t})$$

$$\tilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu} \left(1 - t^{\nu_j^t - i + 1} q^{\nu_i - j}\right)^{-1},$$

-) glue vertices according to web diagram

$$\sum_\nu (-e^{2\pi i m})^{|\nu|} C_{\mu_1\lambda_1\nu} C_{\mu_2^t\lambda_2^t\nu^t}$$



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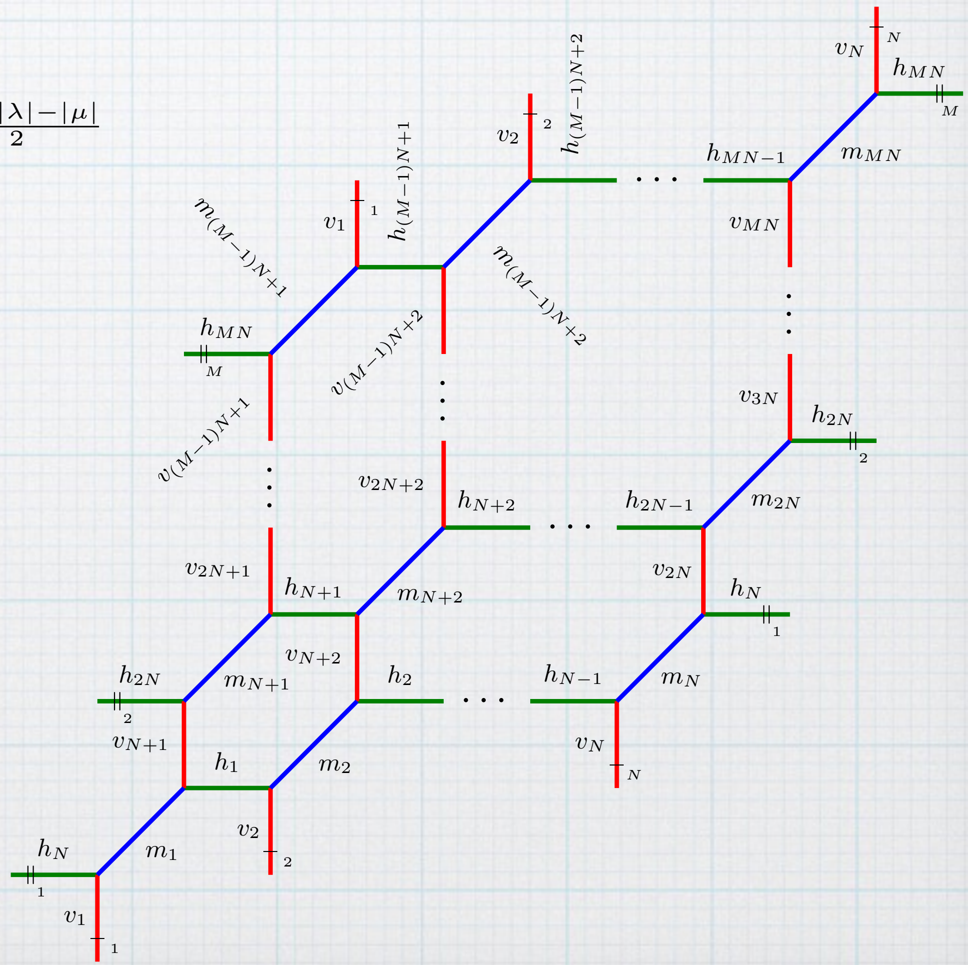
-) glue vertices according to web diagram

$$\sum_\nu (-e^{2\pi i m})^{|\nu|} C_{\mu_1\lambda_1\nu} C_{\mu_2^t\lambda_2^t\nu^t}$$

-) choose **preferred direction**

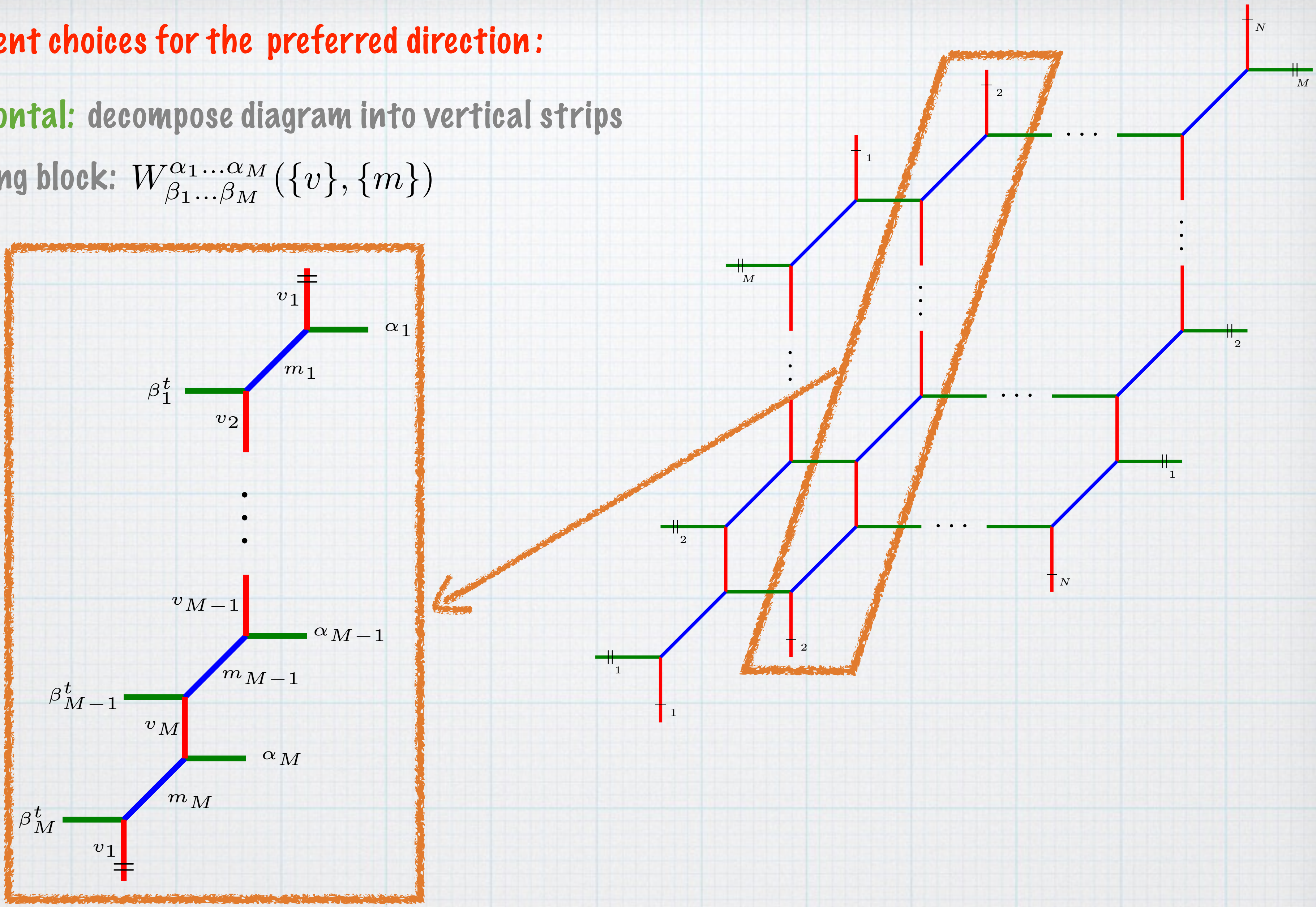
must be common to all vertices of diagram



3 different choices for the preferred direction:

1) horizontal: decompose diagram into vertical strips

building block: $W^{\alpha_1 \dots \alpha_M}_{\beta_1 \dots \beta_M}(\{v\},\{m\})$



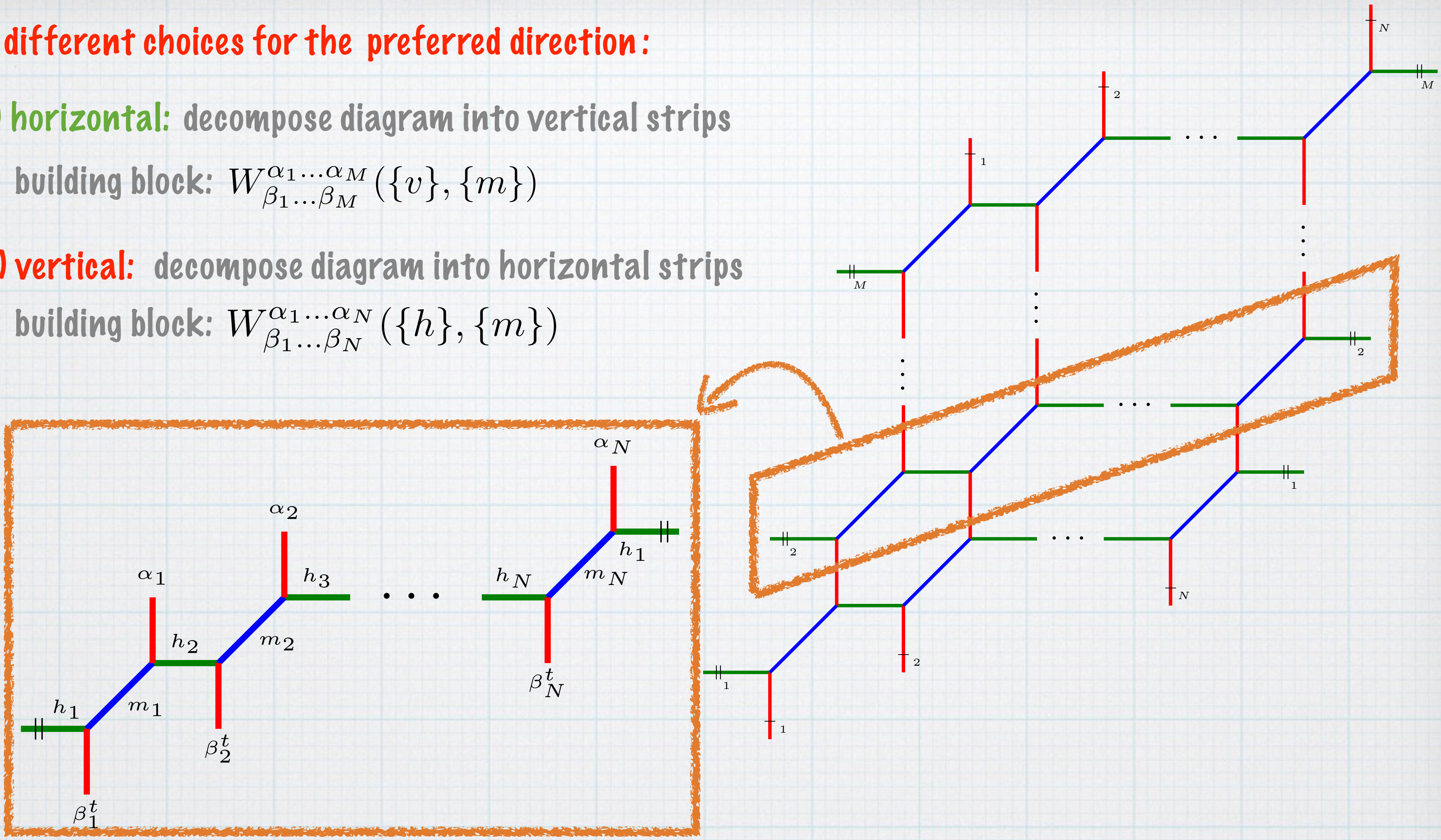
3 different choices for the preferred direction:

1) horizontal: decompose diagram into vertical strips

building block: $W^{\alpha_1 \ldots \alpha_M}_{\beta_1 \ldots \beta_M}(\{v\},\{m\})$

2) vertical: decompose diagram into horizontal strips

building block: $W^{\alpha_1 \ldots \alpha_N}_{\beta_1 \ldots \beta_N}(\{h\},\{m\})$



3 different choices for the preferred direction:

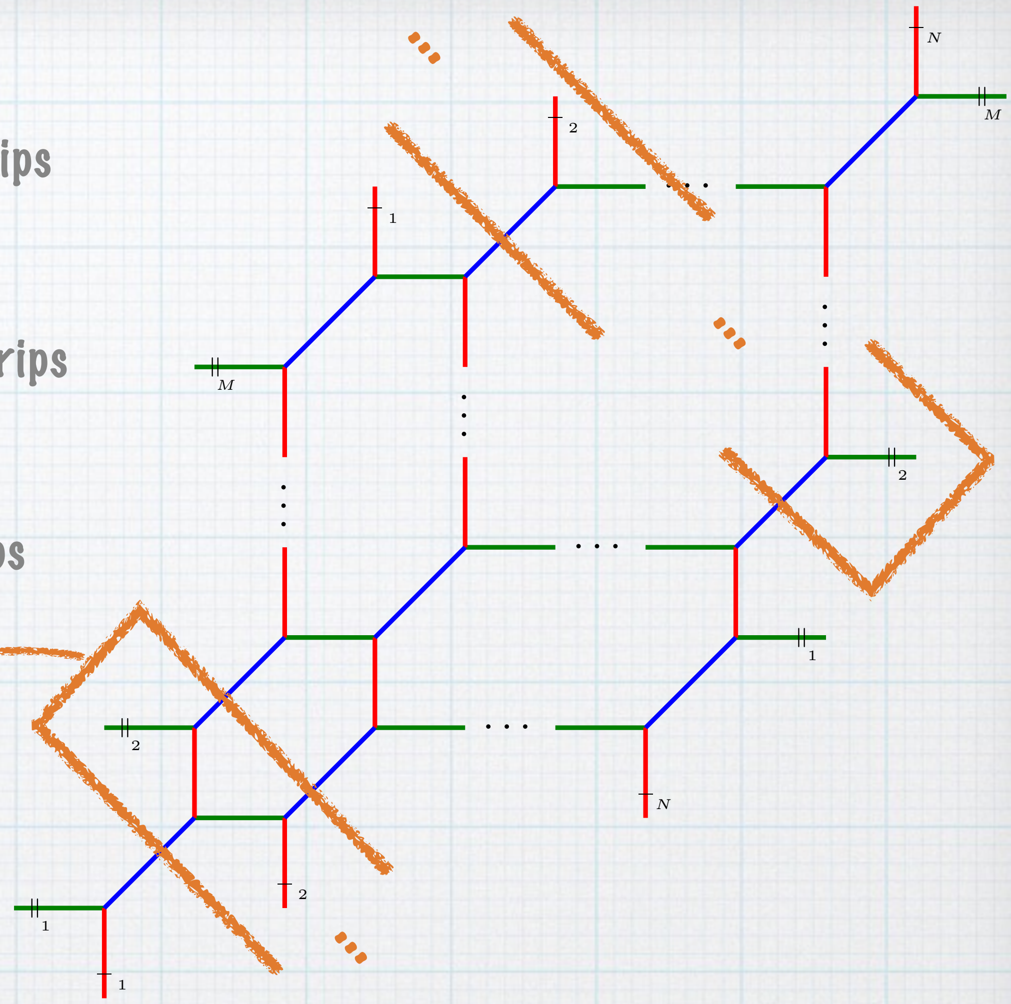
1) horizontal: decompose diagram into vertical strips

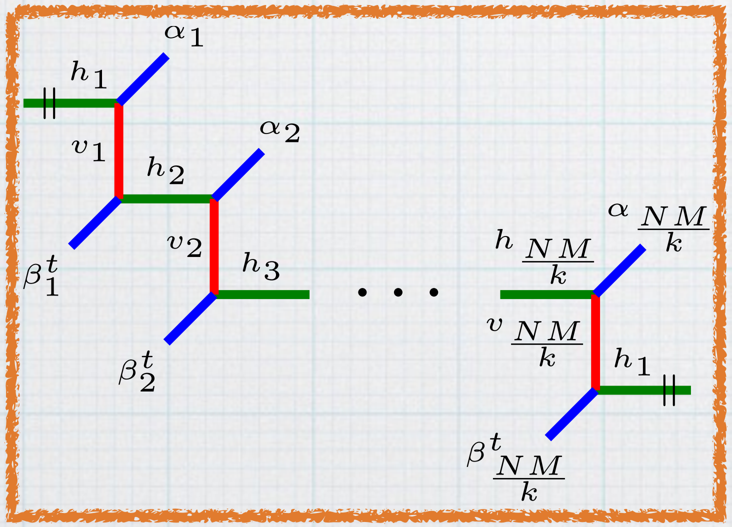
building block: $W^{\alpha_1 \ldots \alpha_M}_{\beta_1 \ldots \beta_M}(\{v\},\{m\})$

2) vertical: decompose diagram into horizontal strips

building block: $W^{\alpha_1 \ldots \alpha_N}_{\beta_1 \ldots \beta_N}(\{h\},\{m\})$

3) diagonal: decompose diagram into diagonal strips





where $k = \gcd(N, M)$

3 different choices for the preferred direction:

1) horizontal: decompose diagram into vertical strips

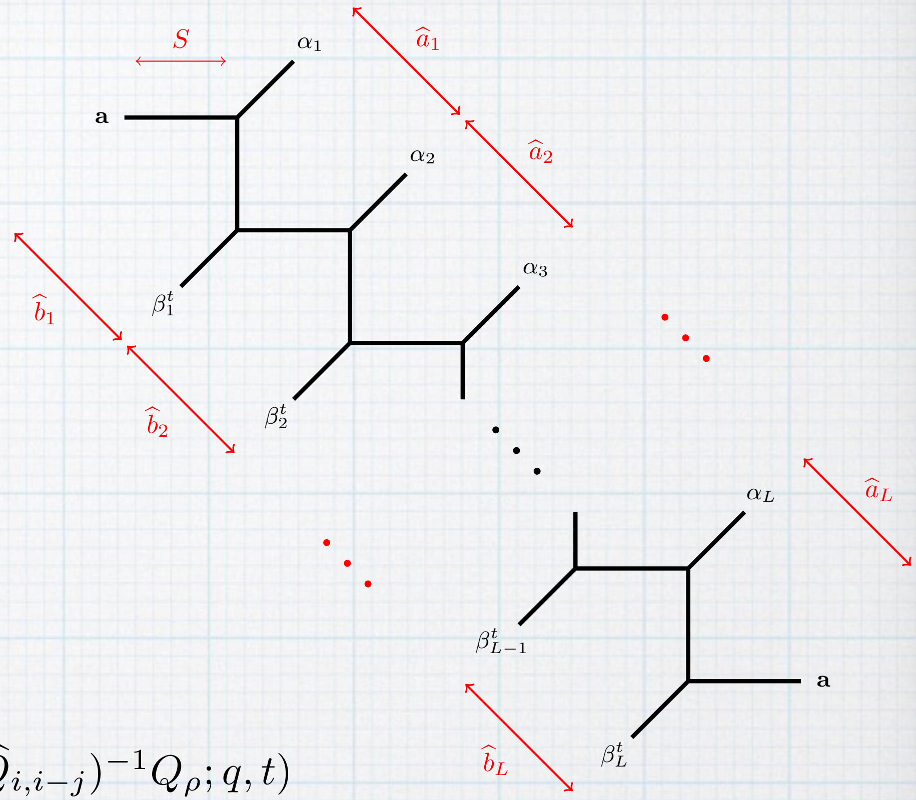
building block: $W^{\alpha_1\ldots\alpha_M}_{\beta_1\ldots\beta_M}(\{v\},\{m\})$

2) vertical: decompose diagram into horizontal strips

building block: $W^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_N}(\{h\},\{m\})$

3) diagonal: decompose diagram into diagonal strips

building block: $W^{\alpha_1\ldots\alpha_{\frac{NM}{k}}}_{\beta_1\ldots\beta_{\frac{NM}{k}}}(\{h\},\{v\})$



generic form of the building block

$$W^{\alpha_1\ldots\alpha_L}_{\beta_1\ldots\beta_L}=W_L(\emptyset)\cdot\hat{Z}\cdot\prod_{i,j=1}^{L}\frac{\mathcal{J}_{\alpha_i\beta_j}(\widehat{Q}_{i,i-j};q,t)\mathcal{J}_{\beta_j\alpha_i}((\widehat{Q}_{i,i-j})^{-1}Q_\rho;q,t)}{\mathcal{J}_{\alpha_i\alpha_j}(\overline{Q}_{i,i-j}\sqrt{q/t};q,t)\mathcal{J}_{\beta_j\beta_i}(\dot{Q}_{i,j-i}\sqrt{t/q};q,t)}$$

with

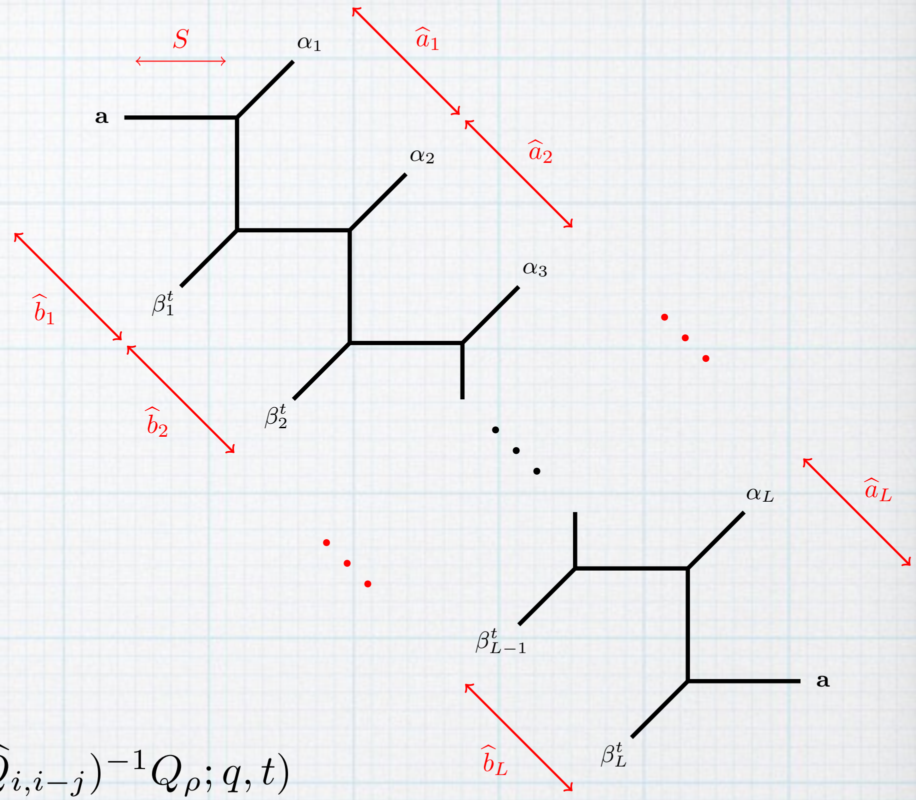
$$W_L(\emptyset)=\prod_{i,j=1}^{L}\prod_{k,r,s=1}^{\infty}\frac{(1-\widehat{Q}_{i,j}\,Q_\rho^{k-1}q^{r-\frac{1}{2}}t^{s-\frac{1}{2}})(1-\widehat{Q}_{i,j}^{-1}Q_\rho^{k}q^{s-\frac{1}{2}}t^{r-\frac{1}{2}})}{(1-\overline{Q}_{i,j}Q_\rho^{k-1}q^{r}t^{s-1})(1-\dot{Q}_{i,j}Q_\rho^{k-1}q^{s-1}t^{r})},$$

$$\hat{Z}=\prod_{i=1}^{L}t^{\frac{||\alpha_k||^2}{2}}q^{\frac{||\alpha_k^t||^2}{2}}\tilde{Z}_{\alpha_k}(q,t)\tilde{Z}_{\alpha_k^t}(t,q)\,,\qquad \tilde{Z}_\nu(t,q)=\prod_{(i,j)\in\nu}\left(1-t^{\nu_j^t-i+1}q^{\nu_i-j}\right)^{-1}$$

$$\mathcal{J}_{\mu\nu}(x;t,q)=\prod_{k=1}^{\infty}J_{\mu\nu}(Q_\rho^{k-1}x;t,q)\,,$$

$$J_{\mu\nu}(x;t,q)=\prod_{(i,j)\in\mu}\left(1-x\,t^{\nu_j^t-i+\frac{1}{2}}q^{\mu_i-j+\frac{1}{2}}\right)\times\prod_{(i,j)\in\nu}\left(1-x\,t^{-\mu_j^t+i-\frac{1}{2}}q^{-\nu_i+j-\frac{1}{2}}\right)$$

3 different choices for the preferred direction:



1) horizontal: decompose diagram into vertical strips

building block: $W^{\alpha_1\dots\alpha_M}_{\beta_1\dots\beta_M}(\{v\},\{m\})$

2) vertical: decompose diagram into horizontal strips

building block: $W^{\alpha_1\dots\alpha_N}_{\beta_1\dots\beta_N}(\{h\},\{m\})$

3) diagonal: decompose diagram into diagonal strips

building block: $W^{\alpha_1\dots\alpha_{\frac{NM}{k}}}_{\beta_1\dots\beta_{\frac{NM}{k}}}(\{h\},\{v\})$

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Notation:

$$\widehat{Q}_{i,j} = Q_S\prod_{r=1}^{i}(Q_{a_r}Q_{b_r}^{-1})\prod_{k=1}^{j-1}Q_{a_{i-k}}\,,$$

$$\overline{Q}_{i,j} = \begin{cases}1 & \text{if } j=L\\ \prod_{k=1}^{j}Q_{a_{i-k}} & \text{if } j\neq L\end{cases}$$

$$\dot{Q}_{i,j} = \prod_{k=1}^{j}Q_{b_{i+k}}$$

and $Q_S = e^{-S}$

$Q_{a_i} = e^{-\widehat{a}_i}$

$Q_{b_i} = e^{-\widehat{b}_i}$

Topological Partition Function

The full partition function is obtained by gluing together the building blocks $W^{\alpha_1 \ldots \alpha_M}_{\beta_1 \ldots \beta_M}$

$$\mathcal{Z}_{N,M} = \sum_{\alpha} \Big(\prod_{i=1,j=1}^{M,N} e^{-u_{ij}\,|\alpha^i_j|} \Big) \prod_{j=1}^{N} W^{\alpha^1_j \cdots \alpha^M_j}_{\alpha^1_{j+1} \cdots \alpha^M_{j+1}}$$

parameters used to glue the strips together

Different choices of preferred direction afford different (but equivalent) expansions:

$$\mathcal{Z}_{N,M}(\{h\},\{v\},\{m\},\epsilon_{1,2}) = Z_p(\{v\},\{m\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{h}}\, Z_{\vec{k}}(\{v\},\{m\}) = Z^{(N,M)}_{\text{hor}}$$

$$= Z_p(\{h\},\{m\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{v}}\, Z_{\vec{k}}(\{h\},\{m\}) = Z^{(N,M)}_{\text{vert}}$$

$$= Z_p(\{h\},\{v\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{m}}\, Z_{\vec{k}}(\{h\},\{v\}) = Z^{(N,M)}_{\text{diag}}$$

common normalisation factor (perturbative partition function)

Compare different series expansions with instanton partition functions of quiver gauge theories.

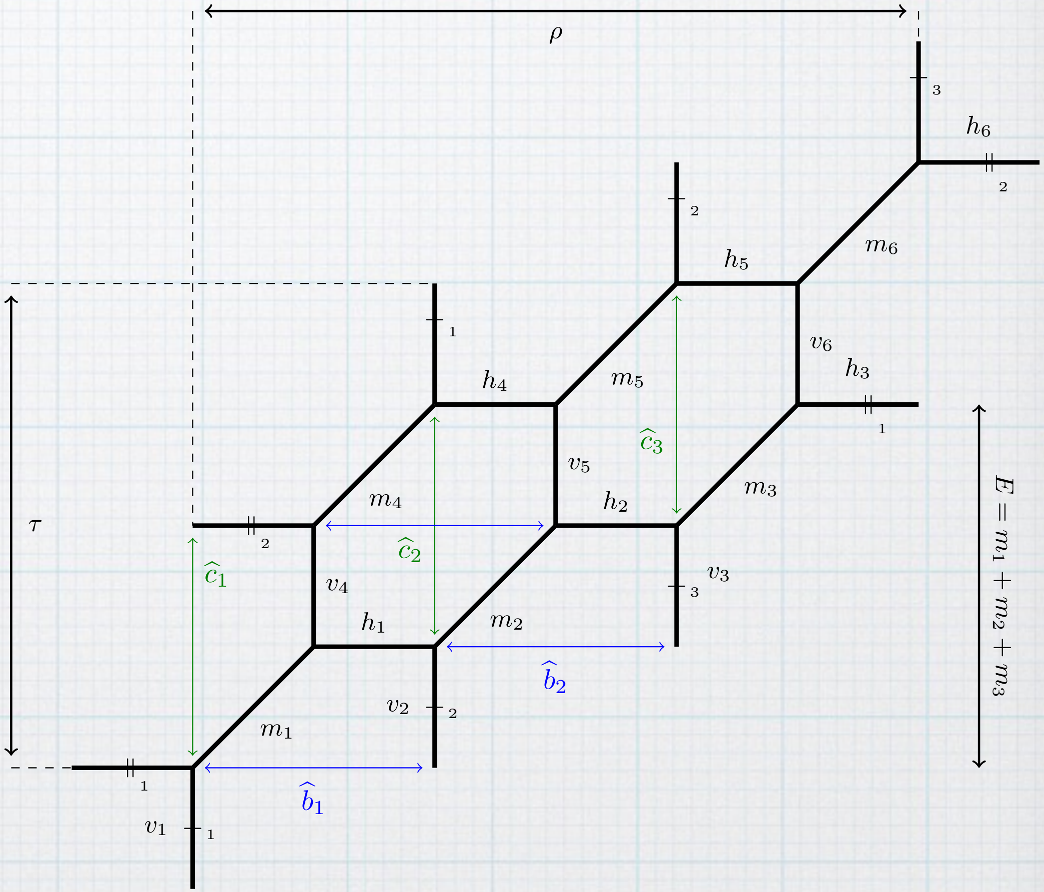
Need to choose **independent Kähler** parameters of $X_{N,M}$

Bases of independent Kähler parameters

For each of the expansion we can choose a suitable set of $NM+2$ independent Kähler parameters:

Example: $(N,M)=(3,2)$

1) horizontal: $(\rho,\widehat{b}_1,\widehat{b}_2;\widehat{c}_1,\widehat{c}_2,\widehat{c}_3;\tau,E)$



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$$\widehat{b}_1\longrightarrow\infty$$

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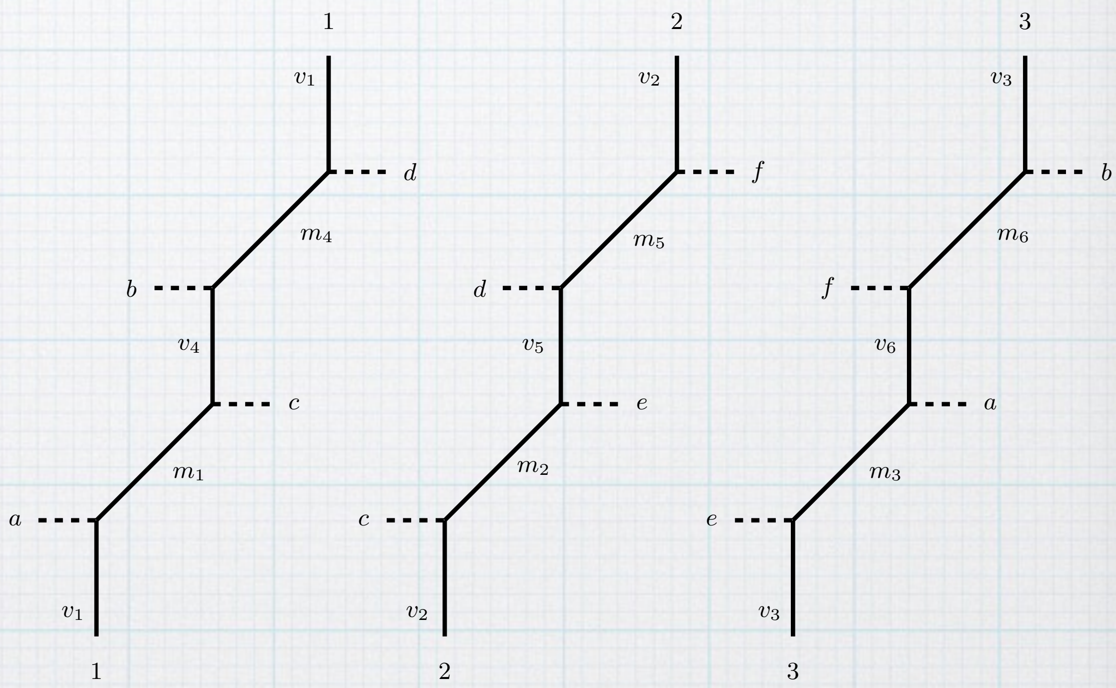
gauge theory: $U(2)\times U(2)\times U(2)$



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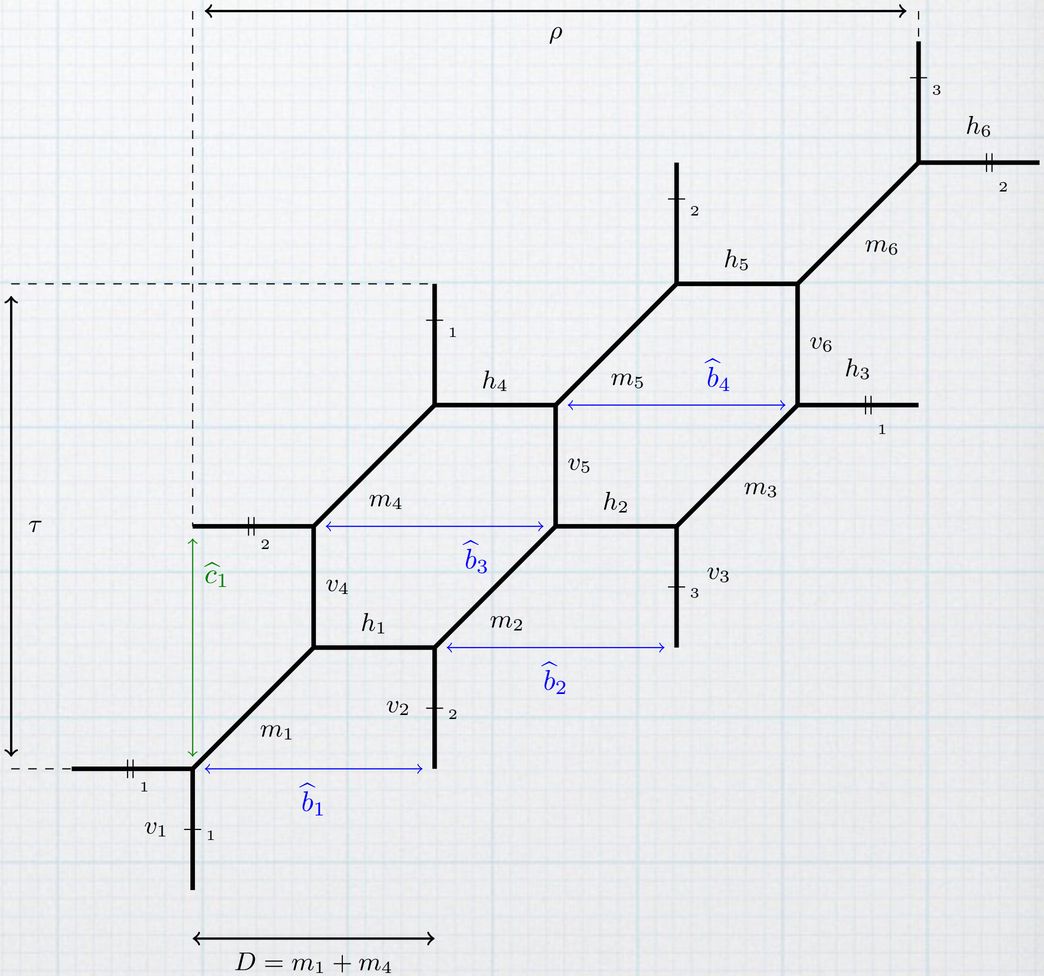
2) vertical: $(\tau,\widehat{c}_1;\widehat{b}_1,\widehat{b}_2,\widehat{b}_3,\widehat{b}_4;\rho,D)$



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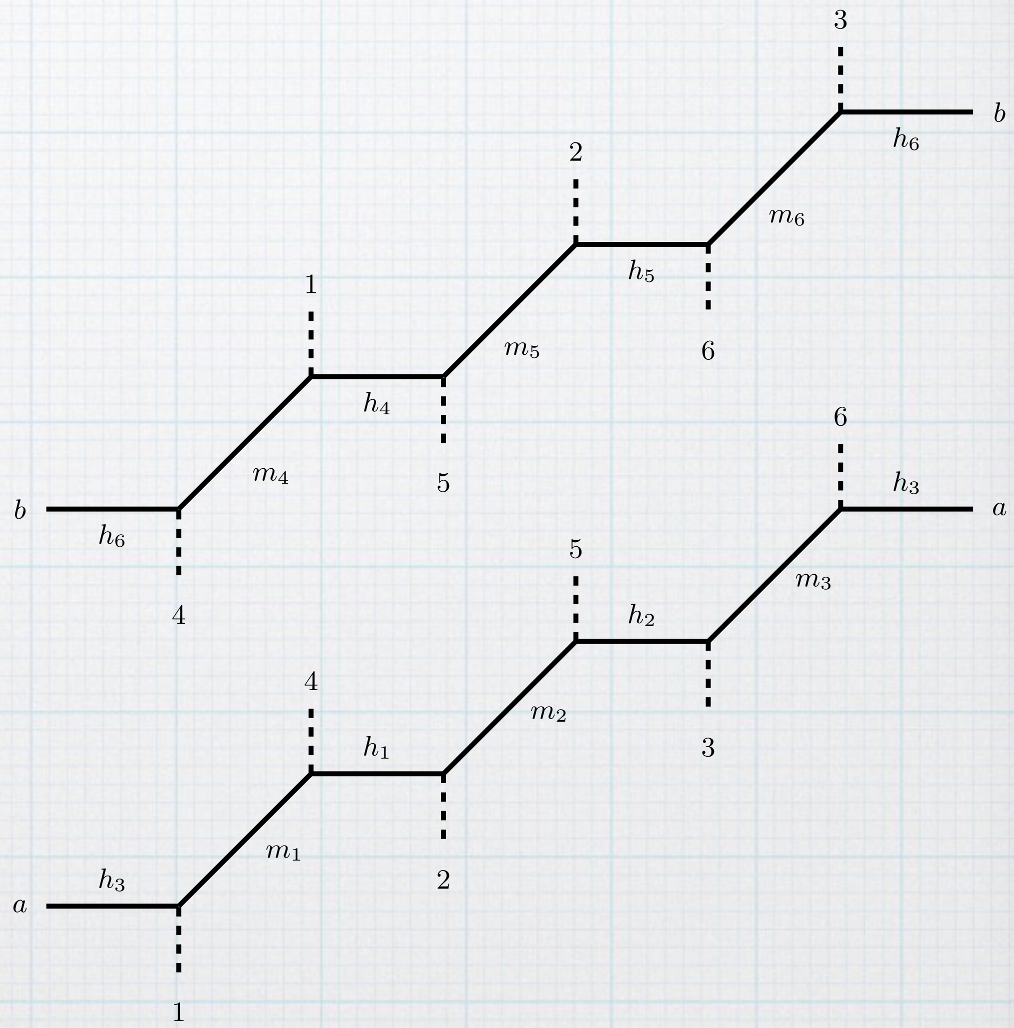

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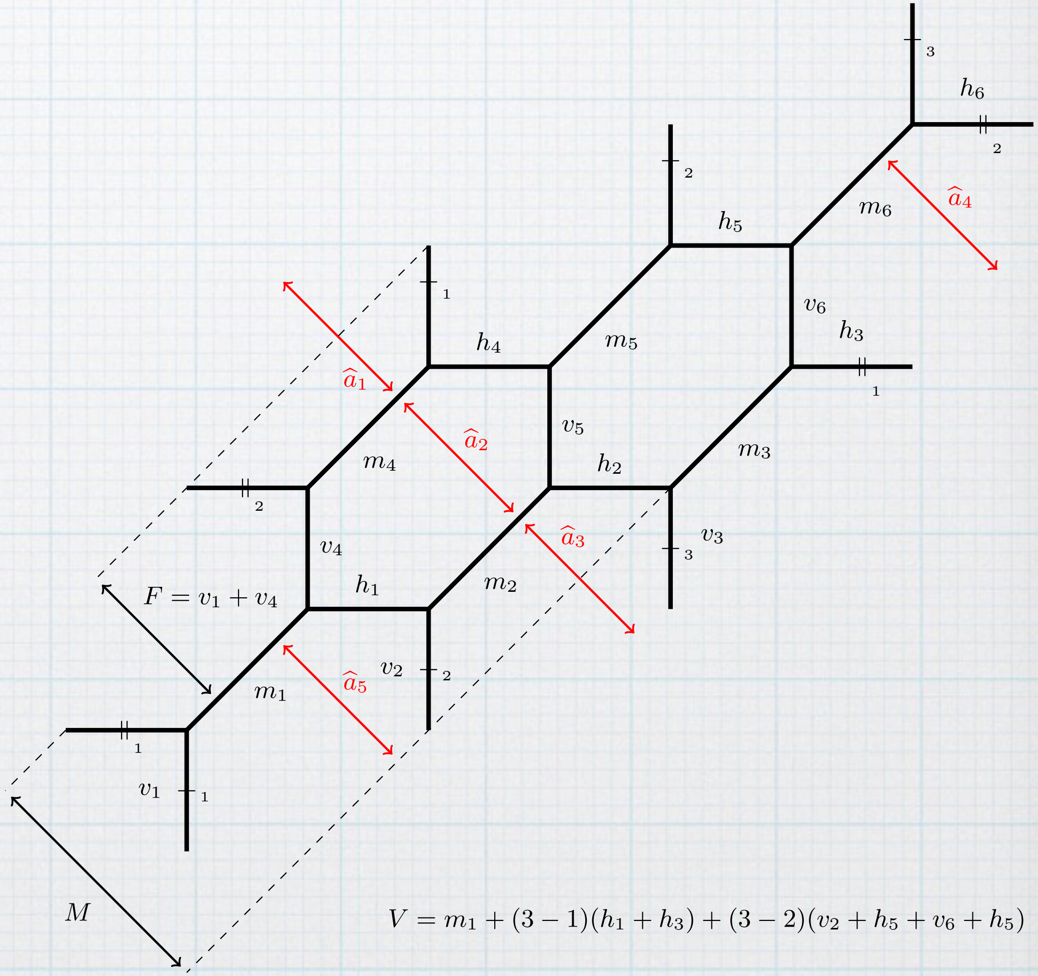
3) diagonal: $(V;\widehat{a}_1,\widehat{a}_2,\widehat{a}_3,\widehat{a}_4,\widehat{a}_5;M,F)$



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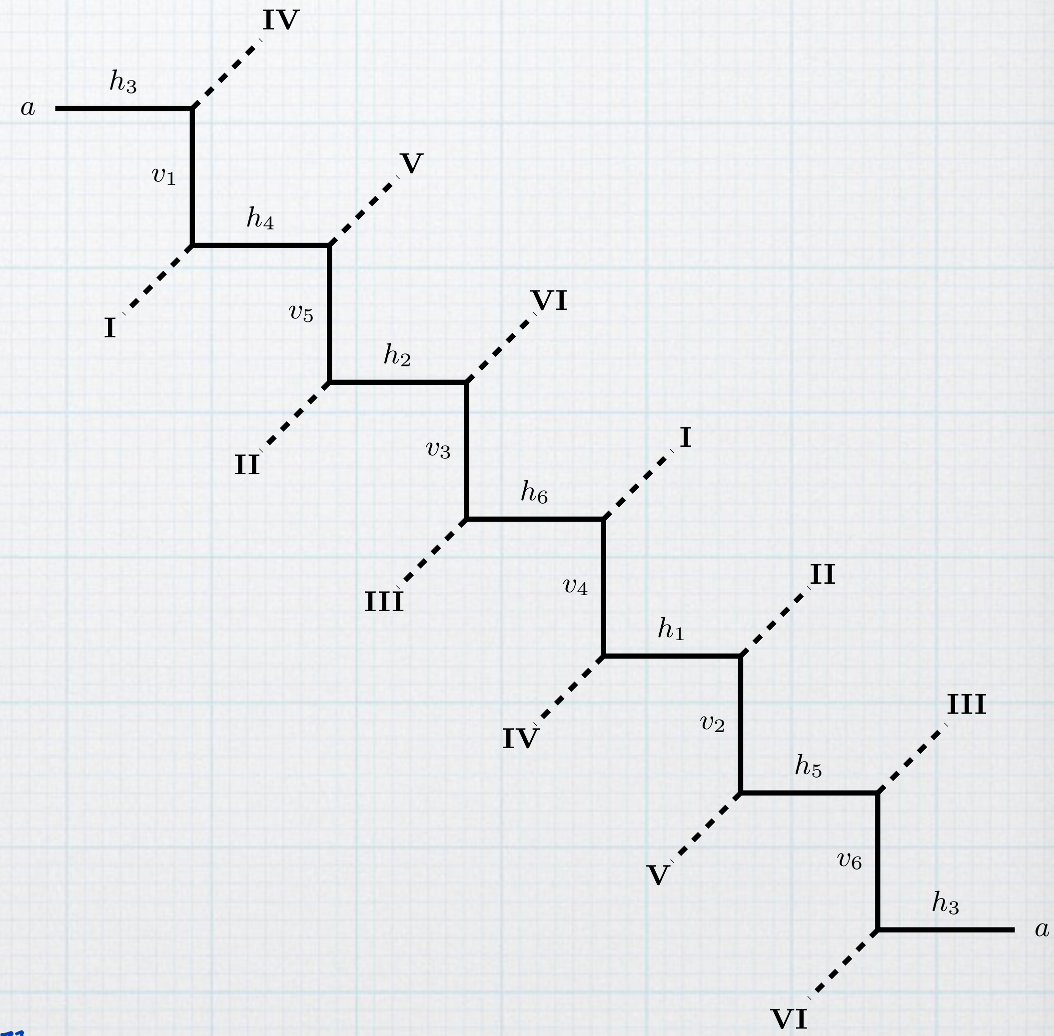
series expansion: $V\longrightarrow\infty$

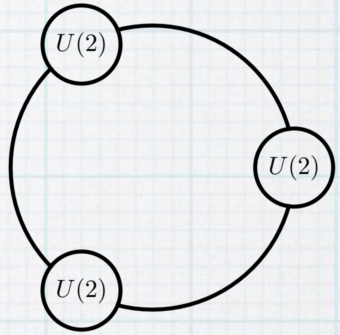
gauge theory: $U(6)$



Similar sets of independent Kähler parameters proposed for generic (N,M) [Bastian, SH, Iqbal, Rey 2017]

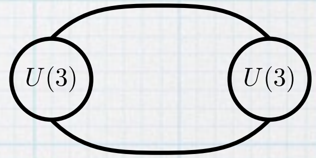
5d Quiver Gauge Theory Interpretation

1) horizontal: $(\rho, \widehat{b}_1, \widehat{b}_2; \widehat{c}_1, \widehat{c}_2, \widehat{c}_3; \tau, E)$: $U(2) \times U(2) \times U(2)$ quiver gauge theory



* $Z^{(3,2)}_{\text{hor}}$ series expansion in $e^{2\pi i(\rho - \widehat{b}_1 - \widehat{b}_2)}$, $e^{2\pi i \widehat{b}_1}$ and $e^{2\pi i \widehat{b}_2}$ related to the **instanton parameters**
* $\widehat{c}_{1,2,3}$ interpreted as simple, positive **roots** of three copies of $\mathfrak{a}_1$
* τ interpreted as (common) **imaginary root** extending $\mathfrak{a}_1$ to $\widehat{\mathfrak{a}}_1$

2) vertical: $(\tau, \widehat{c}_1; \widehat{b}_1, \widehat{b}_2, \widehat{b}_3, \widehat{b}_4; \rho, D)$: $U(3) \times U(3)$ quiver gauge theory



* $Z^{(3,2)}_{\text{vert}}$ series expansion in $e^{2\pi i(\tau - \widehat{c}_1)}$ and $e^{2\pi i \widehat{c}_1}$ related to the **instanton parameters**
* $\widehat{b}_{1,2,3,4}$ interpreted as simple, positive **roots** of two copies of $\mathfrak{a}_2$
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3) diagonal: $(V; \widehat{a}_1, \widehat{a}_2, \widehat{a}_3, \widehat{a}_4, \widehat{a}_5; M, F)$ gauge theory with gauge group $U(6)$

* $Z^{(3,2)}_{\text{diag}}$ can be written as a series expansion in $e^{2\pi i V}$ related to the **instanton parameters**
* $\widehat{a}_{1,2,3,4,5}$ interpreted as simple, positive **roots** of $\mathfrak{a}_5$
* F interpreted as **imaginary root** extending $\mathfrak{a}_5$ to $\widehat{\mathfrak{a}}_5$

Horizontal and vertical gauge theory interpretation well known in the literature

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
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[SH, Iqbal 2013]

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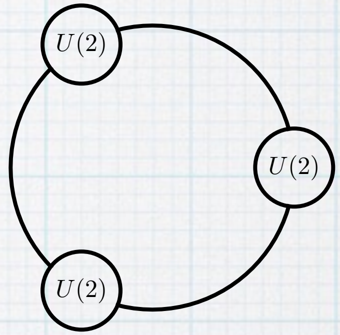

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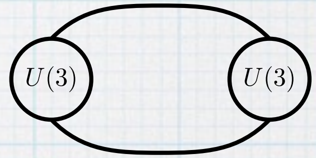

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Horizontal and vertical gauge theory interpretation well known in the literature

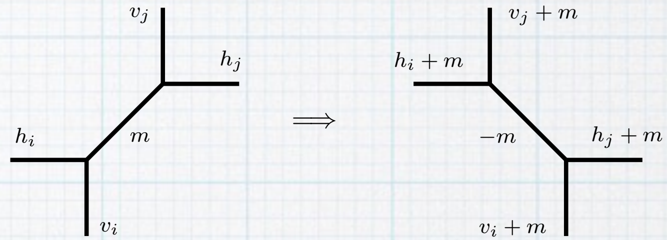
Diagonal expansion leads to novel gauge theory associated with $X_{N,M}$

Triality!

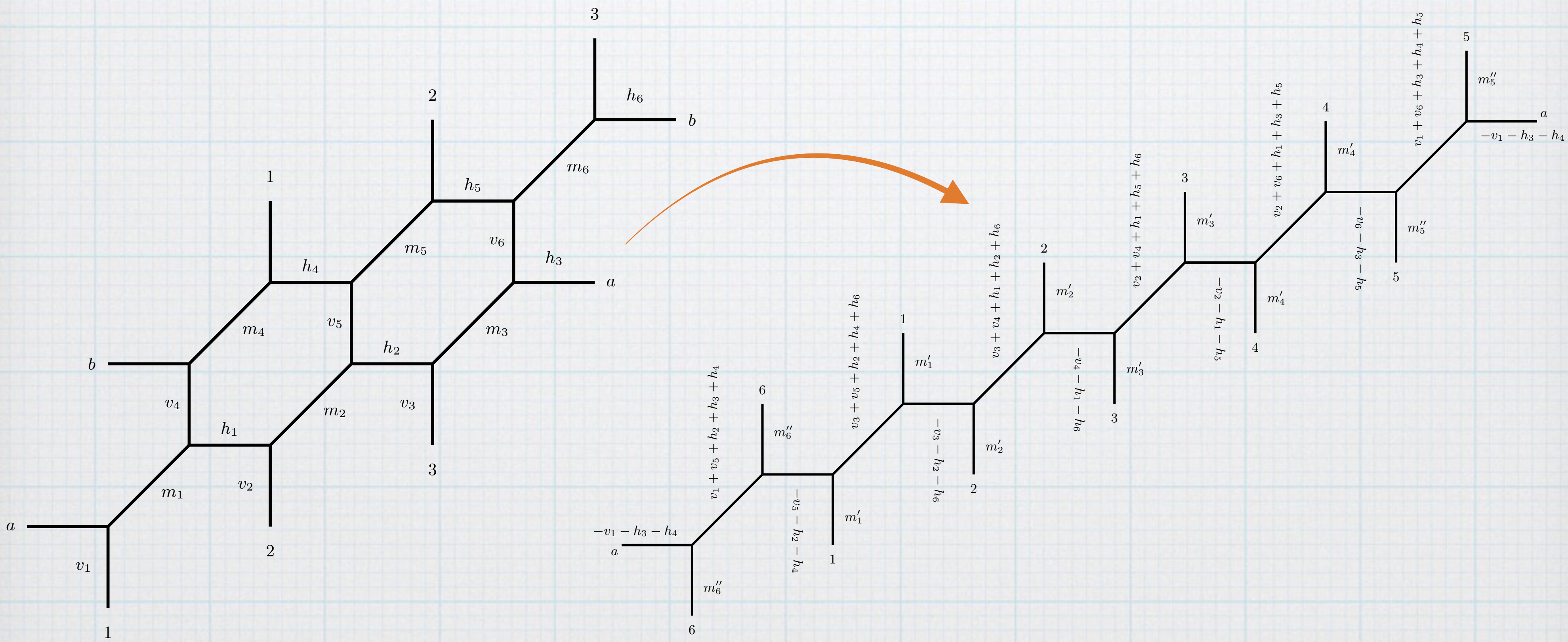
[Bastian, SH, Iqbal, Rey 2017]

Flop Transitions and Duality

Flop transition for any two curves in the diagram:



Example: Series of flop and SL(2,Z) transformations for $X_{3,2} \sim X_{6,1}$ [SH, Iqbal, Rey 2016]



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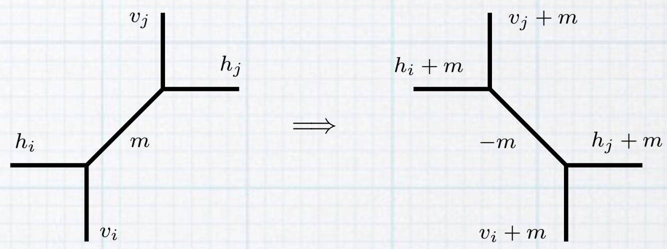

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Duality leaves partiton function invariant

$$\mathcal{Z}_{3,2}(\{h\},\{v\},\{m\},\epsilon_{1,2}) = \mathcal{Z}_{6,1}(\{h'\},\{v'\},\{m'\},\epsilon_{1,2})$$

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Kähler parameters implied by duality transformation

Vertical expansion of $\mathcal{Z}_{6,1}$ gives rise to a gauge theory with gauge group $U(6)$ and part. fct. $\mathcal{Z}^{(6,1)}_{\text{vert}}$

Symmetry transformations do not flop any curve whose area is proportional to V

related to coupling constant of $\mathcal{Z}^{(3,2)}_{\text{diag}}$

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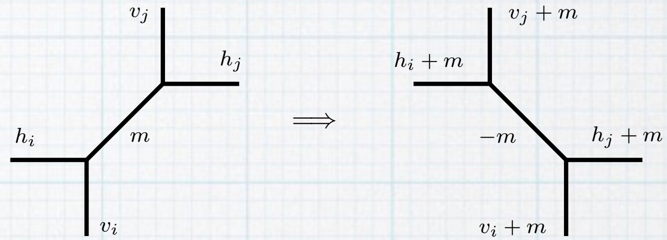

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$\Longrightarrow$ partition functions $\mathcal{Z}_{\text{diag}}^{(3,2)}$ and $\mathcal{Z}_{\text{vert}}^{(6,1)}$ have same asymptotic expansion

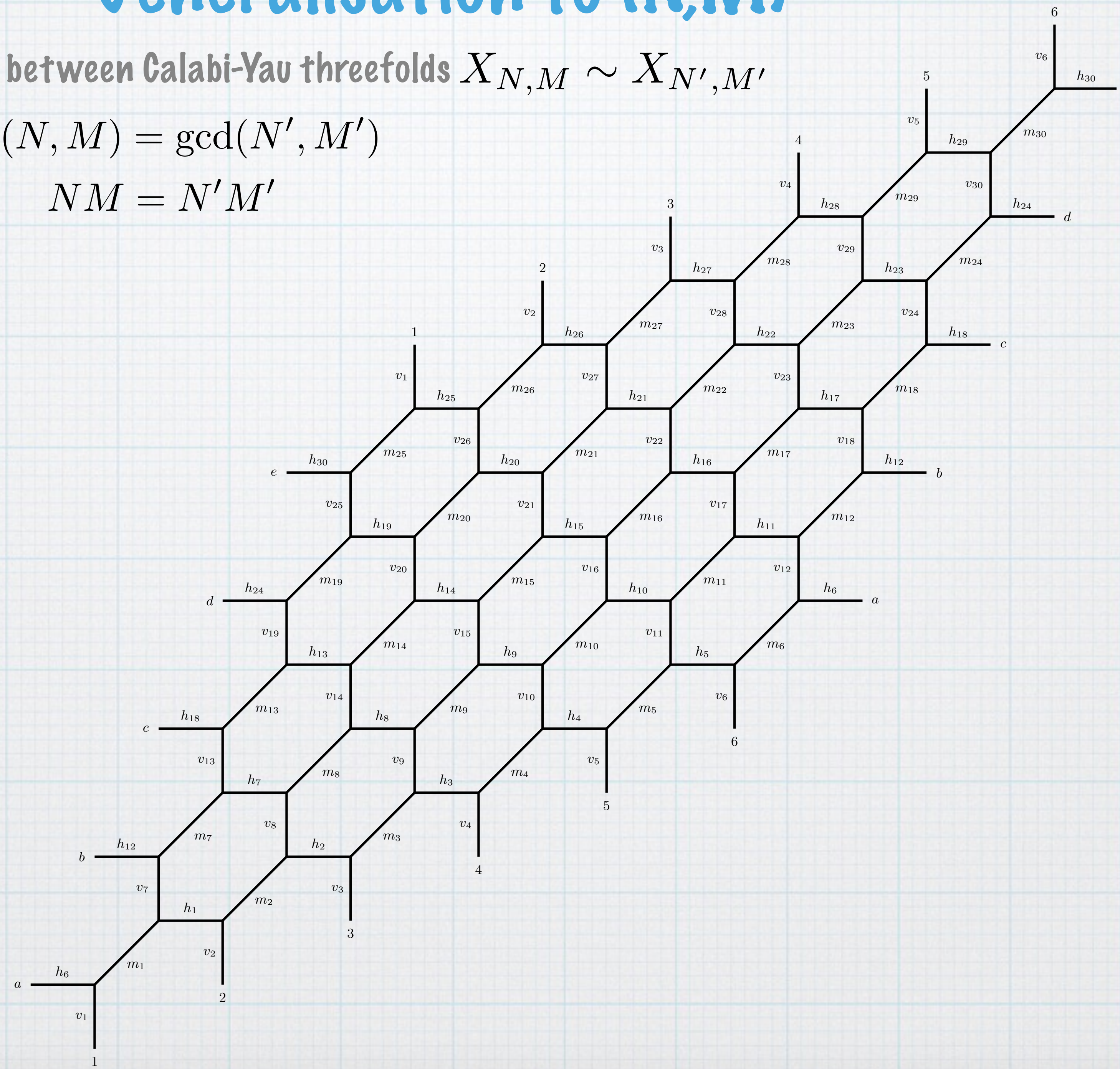
Generalisation to (N,M)

Conjecture: dualities between Calabi-Yau threefolds $X_{N,M} \sim X_{N',M'}$

for $\gcd(N,M) = \gcd(N',M')$

$$NM = N'M'$$

example: $X_{6,5}$



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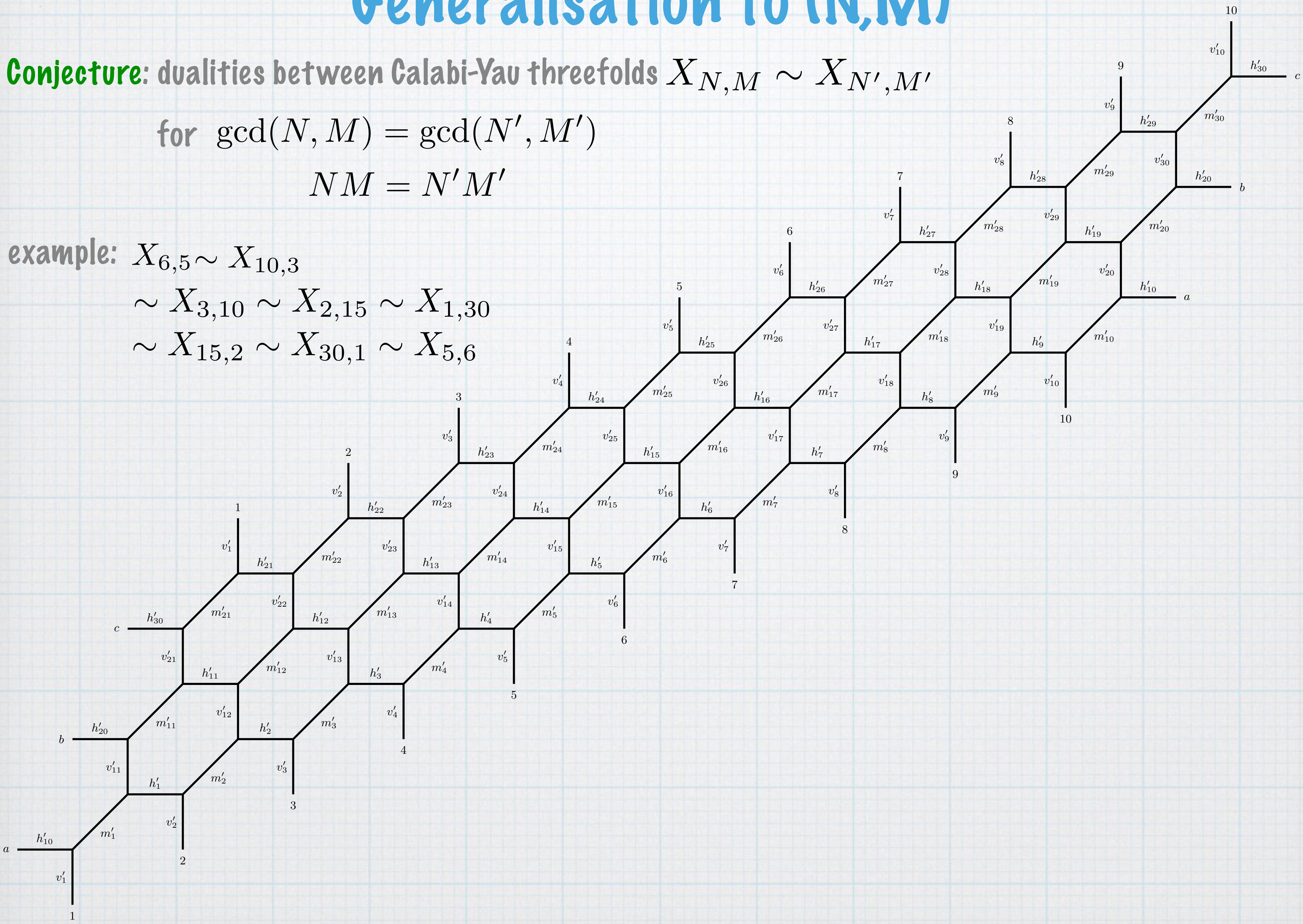
example: $X_{6,5} \sim X_{10,3}$

$\sim X_{3,10} \sim X_{2,15} \sim X_{1,30}$

$\sim X_{15,2} \sim X_{30,1} \sim X_{5,6}$



Consequences for General Configuration (N,M)

Extended moduli space of $X_{N,M}$:

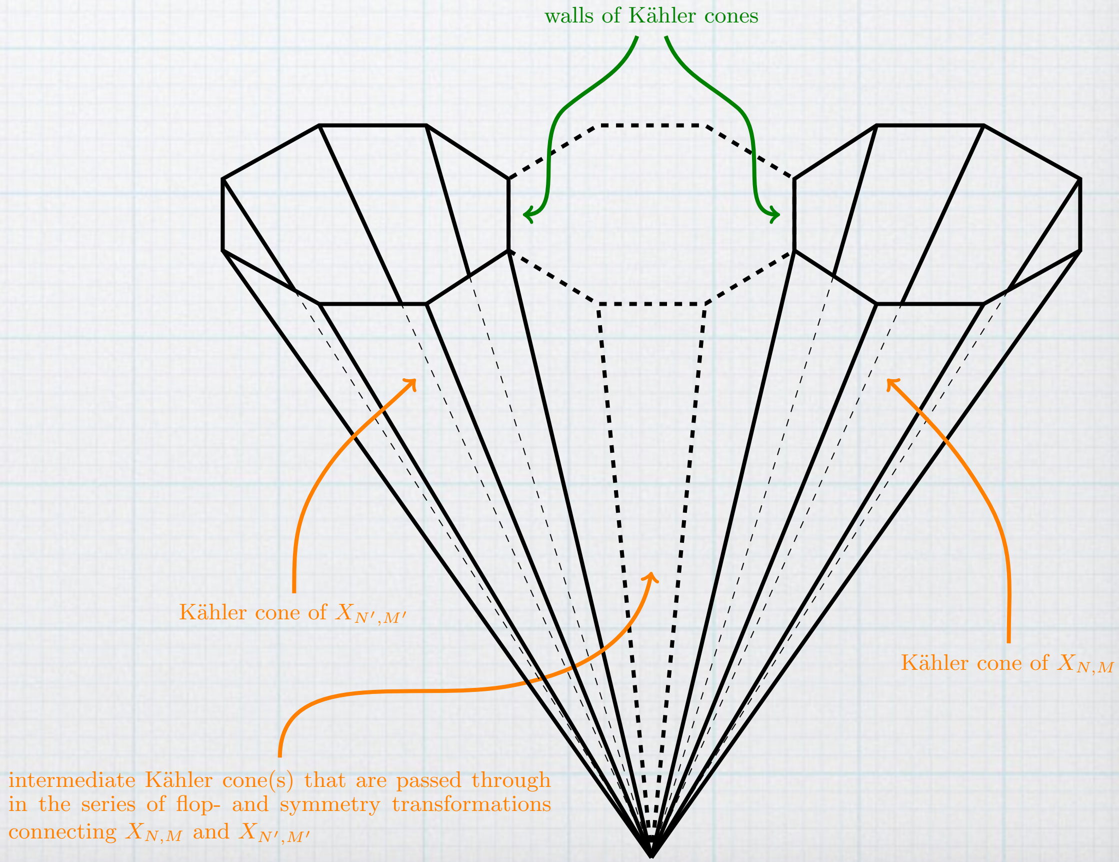
$$X_{N,M} \sim X_{N',M'} \qquad \text{for} \qquad \begin{array}{c} NM = N'M' \\ \gcd(N,M) = \gcd(N',M') \end{array}$$

Partition function invariant

[SH, Iqbal, Rey 2016]

$$\mathcal{Z}_{N,M}(\{h\},\{v\},\{m\},\epsilon_{1,2}) = \mathcal{Z}_{N',M'}(\{h'\},\{v'\},\{m'\},\epsilon_{1,2})$$

(partial) proves: [Bastian, SH, Iqbal, Rey 2017]
[Haghighat, Sun 2018]



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Weak coupling regions within given Kähler cone:

quiver gauge theories with gauge groups

$$G_{\text{hor}} = [U(M)]^N$$

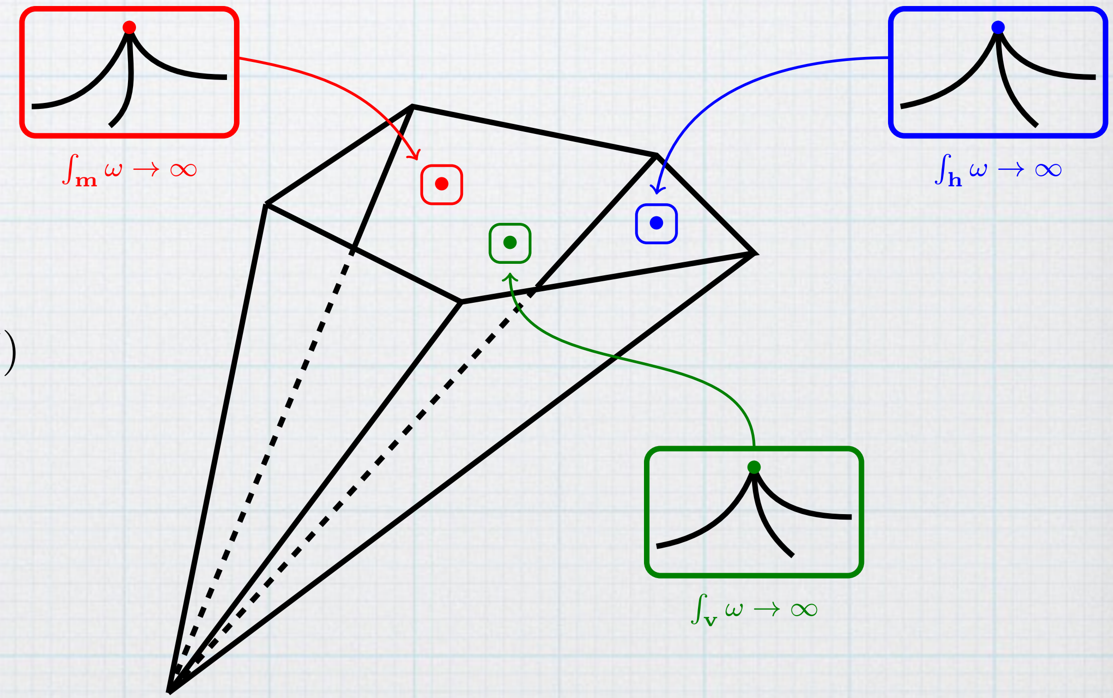
T-duality

$$G_{\text{vert}} = [U(N)]^M$$

$$G_{\text{diag}} = [U(NM/k)]^k \quad \text{for} \quad k = \gcd(N,M)$$

represent low energy limits of LSTs

triality of LSTs

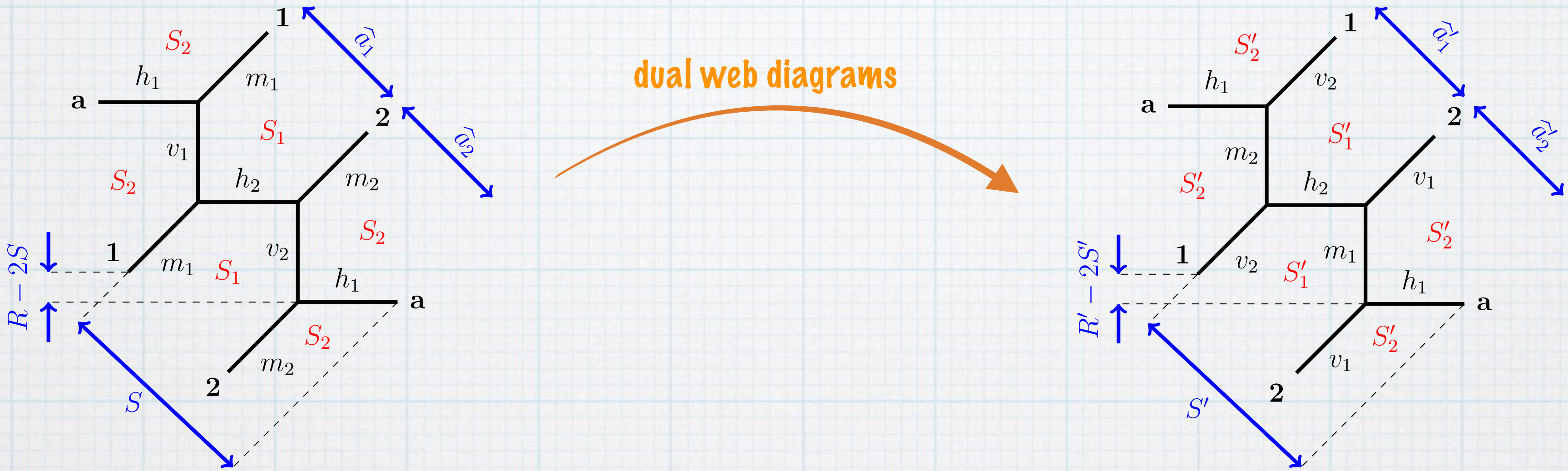


Dihedral Symmetries of Configuration (N,1)

Web of dualities among different theories can be turned into symmetries for individual theories

[SH, Bastian 2018]

Example (N,M)=(2,1):



$$\widehat{a}_1 = v_1 + h_2\,, \qquad \widehat{a}_2 = v_2 + h_1\,,$$
$$S = h_2 + v_2 + h_1\,, \qquad R - 2S = m_1 - v_2\,.$$

$$\widehat{a}'_1 = m_1 + h_1\,, \qquad \widehat{a}'_2 = m_2 + h_2\,,$$
$$S' = h_2 + m_1 + h_1\,, \qquad R' - 2S' = v_2 - m_1\,.$$

Implies the following symmetry of the partition function:

$$\begin{pmatrix} \widehat{a}_1 \\ \widehat{a}_2 \\ S \\ R \end{pmatrix} = G_1 \cdot \begin{pmatrix} \widehat{a}'_1 \\ \widehat{a}'_2 \\ S' \\ R' \end{pmatrix}$$

where

$$G_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

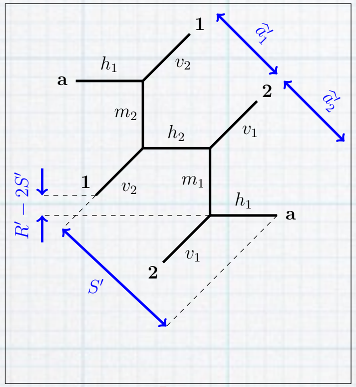
$$\det G_1 = 1$$
$$G_1 \cdot G_1 = \mathbb{1}_{4\times 4}$$

Generalising to include other duality transformations:

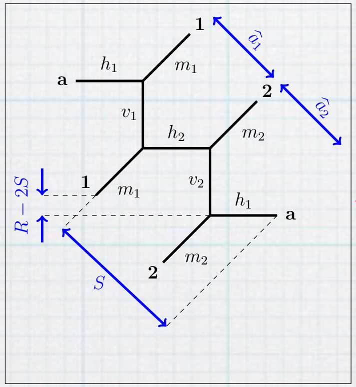
$$G_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 2 & -4 & 1 \end{pmatrix}$$

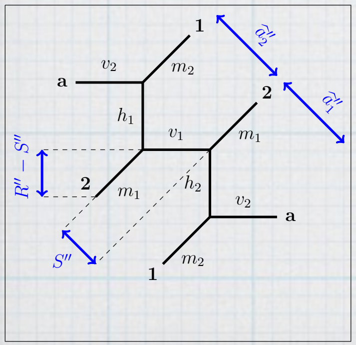
$$G_3 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 \\ 2 & 2 & -4 & 1 \end{pmatrix}$$

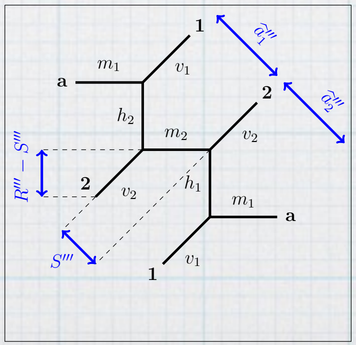


G_1 G_1



G_2 G_2 G_3 G_3





	$\mathbb{1}_{4\times4}$	G_1	G_2	G_3
$\mathbb{1}_{4\times4}$	$\mathbb{1}_{4\times4}$	G_1	G_2	G_3
G_1	G_1	$\mathbb{1}_{4\times4}$	G_3	G_2
G_2	G_2	G_3	$\mathbb{1}_{4\times4}$	G_1
G_3	G_3	G_2	G_1	$\mathbb{1}_{4\times4}$

Group Structure:

$$\{\mathbb{1}_{4\times4}, G_1, G_2, G_3\} \cong \mathrm{Dih}_2$$

Generalisation to (N,1): Symmetry group

$$\mathbb{G}(N) \times \mathrm{Dih}_N$$

'shuffling' of roots

where

$$\mathbb{G}(N) \cong \begin{cases} \mathrm{Dih}_3 & \text{if} \quad N = 1\,, \\ \mathrm{Dih}_2 & \text{if} \quad N = 2\,, \\ \mathrm{Dih}_3 & \text{if} \quad N = 3\,, \\ \mathrm{Dih}_\infty & \text{if} \quad N \geq 4\,. \end{cases}$$

Explicitly

$$\mathbb{G}(N) \cong \left\langle \{\mathcal{G}_2(N), \mathcal{G}_2'(N) \middle| (\mathcal{G}_2(N))^2 = (\mathcal{G}_2'(N))^2 = (\mathcal{G}_2(N) \cdot \mathcal{G}_2'(N))^n = \mathbb{1}\} \right\rangle$$

$$n = \begin{cases} 3 & \text{for} \quad N = 1, 3 \\ 2 & \text{for} \quad N = 2 \\ \infty & \text{for} \quad N \geq 4 \end{cases}$$

with the $(N+2) \times (N+2)$ matrices

$$\mathcal{G}_2(N) = \begin{pmatrix} & & & 0 & 0 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & 0 & 0 \\ 1 & \cdots & 1 & -1 & 0 \\ N & \cdots & N & -2N & 1 \end{pmatrix}$$

and

$$\mathcal{G}_2'(N) = \begin{pmatrix} & & & -2 & 1 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & -2 & 1 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Conclusions and Further Directions

Studied dualities in a class of Little String Orbifolds:

* efficiently described by dual F-theory compactification on a class of toric CY3folds $X_{N,M}$
* partition function $\mathcal{Z}_{N,M}$ compute as topological string partition function on $X_{N,M}$
* Kähler cone of $X_{N,M}$ contains three weak coupling regions in which web diagram decomposes into parallel strips
* weak coupling regions give rise to different (but equivalent) expansions of $\mathcal{Z}_{N,M}$ that can be interpreted as instanton partition functions, realising a **triality** of 5dim quiver gauge th.:

$$G_{\text{hor}} = [U(M)]^N \quad \Longleftrightarrow \quad G_{\text{vert}} = [U(N)]^M \quad \Longleftrightarrow \quad G_{\text{diag}} = [U(\tfrac{MN}{k})]^k \quad \text{for} \quad k = \gcd(N,M)$$

* further dualities: $[U(M)]^N \Longleftrightarrow [U(M')]^{N'}$ for $\begin{array}{l} NM = N'M' \\ \gcd(N,M) = \gcd(N',M') \end{array}$
* implies (dihedral) symmetries of the partition function

Future directions:

* study implications of triality on W-algebras associated with AGT dual theories
* Generalisation to other LSTs than A-series
* study extended web of dualities by considering further weak coupling regions in the extended moduli space of $X_{N,M}$