Painlevé/Gauge theory correspondence on the torus

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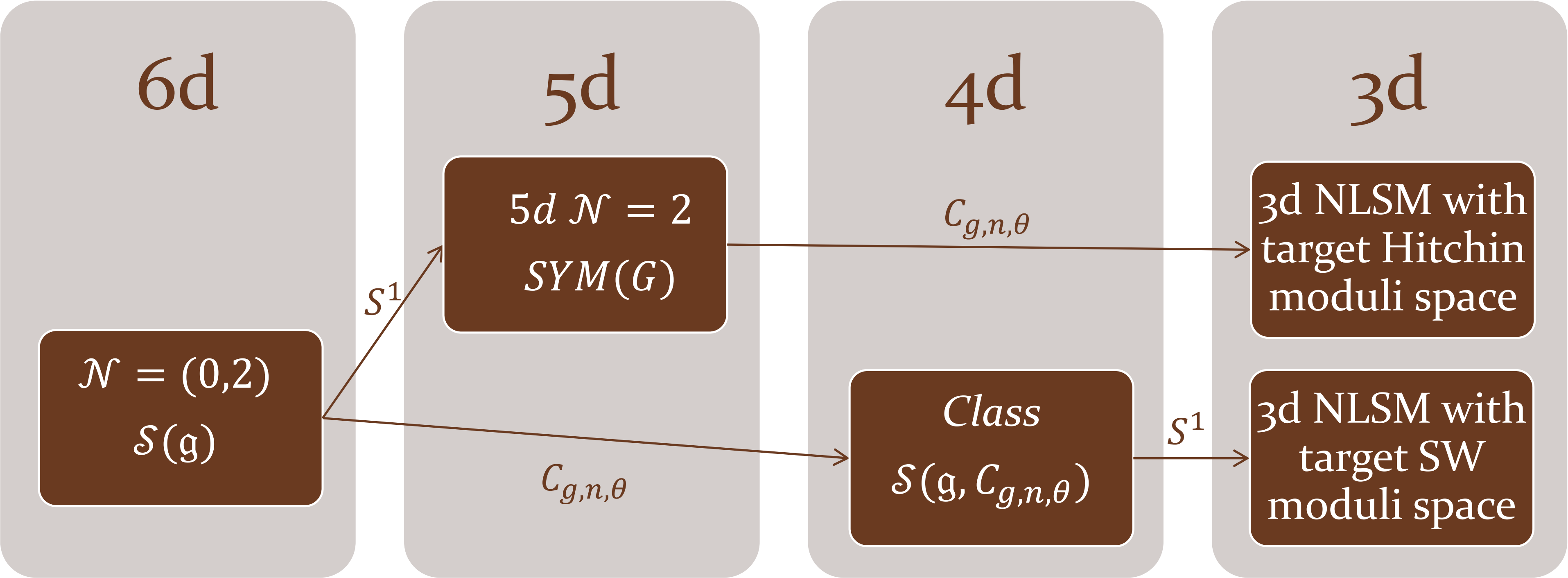
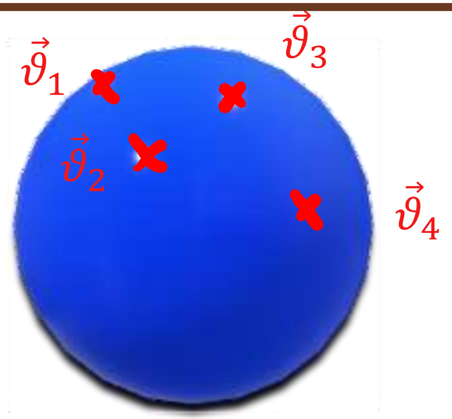
Class S theories and Hitchin Systems I

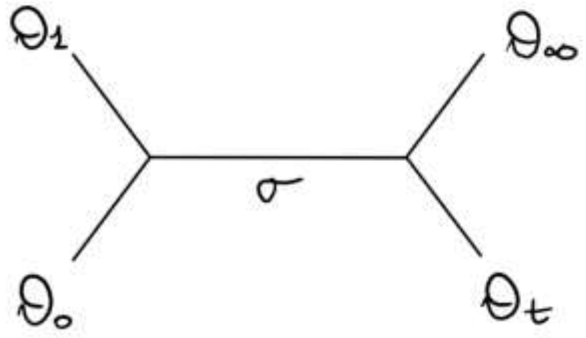


Class S theories and Hitchin system

- Data for the 4d theory: $C_{g,n,\theta}$: g, n determines the gauge and matter content; θ_k, σ determine masses of hypermultiplets and vector vev



Weak coupling regime
Degeneration limit of RS



- Hitchin system: Lax pair (L, M) meromorphic matrix-valued differentials on $C_{g,n}$
- $tr(L^k) = u_k$ Hamiltonians of integrable system/Coulomb branch parameters; double pole coefficients of $tr(L^2)$ are θ_k's
- Σ: $\det(L - \lambda I)$ is the SW curve

Isomonodromic deformations and class S theories

Complex structure moduli of $C_{g,n}$

$\Updownarrow$

Gauge couplings (or dynamical energy scale in the asymptotically free cases) in weakly coupled frames

- Deforming this picture by varying these moduli does not change the physical theory: the resulting deformation equations are RG equations (asymptotically free) or conformal manifold equations (CFTs)
- These deformations are well-known from the point of view of integrable systems: they are **isomonodromic deformations** of the Hitchin system
- These flows are Hamiltonian, the generating function for these Hamiltonians is called the **tau function** $\mathcal{T} = Z^D = \sum_n Z(a+n)e^{in\eta}$ of the corresponding gauge theory in the self-dual Omega background

Topological Strings and Hitchin Systems

- Open CY3 X ⟶ Mirror curve Σ_X^{TS} (nontrivial part of the mirror CY3)
- 4d limit of Σ_X^{TS} is the spectral curve Σ_{X}: $\det(L - \lambda I) = 0$.
- The isomonodromic deformations of the Hitchin system turn out to be closely related to the quantization of the spectral curve.
- The differential equations are exactly satisfied by the dual gauge theory partition function in self-dual Omega background (unrefined TS) [GIL2013,ILT2015,GM2016, BLMST2017]
- If one considers the problem before the 4d limit, the "deformation" equations are really difference equations (q-Painlevé) satisfied by the TS partition function [BGT2018,BGM2018]
- The tau function can be written as a Fredholm determinant [Z1994,GL2018,CGL2018]: nonperturbative definition of TS!

Isomonodromic deformations and quantum mirror curves

- BGT ←→ GHM: inverse of quantum mirror curve is a trace class operator (see A. Grassi's talk)
  - One can show its Fredholm determinant in the 4d limit is tau function of an isomonodromic problem (pure gauge, 4d and 5d)
  - By other constructions ([BLMST]) one knows that this tau function is the NO dual partition function of a gauge theory. The Fredholm determinant gives a nonperturbative representation of the TS partition function.
- CPT (see J. Teschner's talk):
  - The free fermion representation of the partition function provides the bridge between TS and integrable hierarchies, again the Fredholm determinant provides nonperturbative definition
  - Follows the approach of [ADKMV2005,DHSV2008]: free fermion conformal blocks correspond to B-model amplitudes with possible insertion of B-branes.
- The two approaches are complementary, in the sense that BGT probes the magnetic phase of the 4d gauge theory, while CPT the electric phase: two different 4d limits of Topological Strings.

Goals

- Extend this correspondence to the (differential, 4d) case of $g=1$ with n regular punctures
- 4d gauge theory: circular quiver gauge theories, with matter in the adjoint representation of the gauge group (6d theory little strings, see S. Hohenegger's talk)
- Mathematical physics: explicit solution of isomonodromic problems on higher genus Riemann surfaces is still an open problem

Results I

- On the torus, there is an extra factor between tau function and NO dual partition function/free fermion chiral correlator:<

$$\mathcal{T}_{sphere} = Z^D$$

- One puncture case:

$$\mathcal{T} = \frac{Z^D}{\left(\frac{\theta_1(Q)^2}{\eta(\tau)^2}\right)} = \frac{Z_0^D}{\left(\frac{\theta_3(2Q)}{\eta(\tau)}\right)} = \frac{Z_{1/2}^D}{\left(\frac{\theta_2(2Q)}{\eta(\tau)}\right)}$$

$$\frac{d^2Q}{d\tau^2} = (2\pi i m)^2 \wp'(2Q|\tau)$$ (Manin's PVI special case)

Results I

- On the torus, there is an extra factor between tau function and NO dual partition function/free fermion chiral correlator:

$$\mathcal{T} = \frac{Z^D}{Z_{twist}(Q)} \qquad\qquad Z_{twist} = \prod_{i=1}^{N} \frac{\theta_1(Q_i)}{\eta(\tau)}$$

- One puncture case:

$$\mathcal{T} = \frac{Z^D}{\left(\frac{\theta_1(Q)^2}{\eta(\tau)^2}\right)} = \frac{Z_0^D}{\left(\frac{\theta_3(2Q)}{\eta(\tau)}\right)} = \frac{Z_{1/2}^D}{\left(\frac{\theta_2(2Q)}{\eta(\tau)}\right)}$$

$$\frac{d^2 Q}{d\tau^2} = (2\pi i m)^2 \wp'(2Q|\tau)$$ (Manin's PVI special case)

Results II

- The last equation gives us an exact relation between the Painlevé transcendant and the two Fourier transforms (sectors with different fermion charges), that gives an exact relation between the UV and IR coupling of the gauge theory

$$\frac{Z_0^D}{Z_{1/2}^D} = \frac{\theta_3(2Q(\tau)|2\tau)}{\theta_2(2Q(\tau)|2\tau)}$$

$$\frac{\theta_2(2\tau_{SW})^2}{\theta_3(2\tau_{SW})^2} + \frac{\theta_3(2\tau_{SW})^2}{\theta_2(2\tau_{SW})^2} = \frac{\theta_2(2\tau)^2}{\theta_3(2\tau)^2} + \frac{\theta_3(2\tau)^2}{\theta_2(2\tau)^2} - \frac{\pi^2\mu^2}{u - 4\pi i\mu^2\partial_\tau \log\theta_2(\tau)} \frac{\theta_4(2\tau)^8}{\theta_2(2\tau)^2\theta_3(2\tau)^2}$$

- Fredholm determinant representation of the (dual) partition function:

$$Z^D = N(a,m,a)q^{a^2}\det(1+K)$$

Physical motivation

Mathematical Formulation of the problem

CFT approach to Isomonodromic deformations on the torus

Conclusions and Outlook

Riemann Hilbert Problem and linear system on the sphere

Riemann Hilbert Problem

$$\begin{cases} Y(z \sim z_k) = G_k (z - z_k)^{-\vec{\vartheta}_k} C_k \\ \det Y(z_0) \neq 0 \\ Y(z_0) = id_N \end{cases}$$

- $\vec{\vartheta}_k$ are the local monodromy exponents
- Kernel:

$$K(z, z_0) = \frac{Y^{-1}(z_0) Y(z)}{z - z_0}$$

⟺

Linear system

$$\begin{cases} \partial_z Y(z) = L(z) Y(z) \\ L(z) = \sum_{k=1}^{n} \frac{A_k}{z - z_k} \\ Y(\gamma_k \cdot z) = Y(z) M_k \\ Y(z_0) = id_N \end{cases}$$

$$A_k = C_k \boldsymbol{\theta_k} C_k^{-1}$$

$$M_K = C_k e^{2\pi i \boldsymbol{\theta_k}} C_k^{-1}$$

Isomonodromic deformations: Deformations of the above linear systems that preserve the conjugacy classes of the monodromies (moving the points z_k)

Isomonodromic tau function: $\partial_{z_k} \log \mathcal{T} = H_k, \qquad H_k = \frac{1}{2\pi i} \oint_{\gamma_k} \frac{1}{2} tr L^2 \, dz$

Linear Systems on the Torus

On the torus there cannot be a function with just one simple pole, so one cannot directly generalize $L(z)$.
There is more than one possibility:

- Take $L(z)$ to have N additional singularities with residues of rank 1 [Krichever 2002];
- Consider $L(z)$ to be a section of a nontrivial bundle on the torus [Korotkin, Takasaki 1997-2003]. ⇐

Flat $SL(N,\mathbb{C})$ bundles on the torus can be classified by an element $e^{\frac{2\pi ik}{N}}$ of their center $\mathbb{Z}_N$, i.e. by an integer $k=0,\dots,N-1$ through their transition functions [Levin et al. 2014]

$$L(z+1)=T_A L(z) T_A^{-1}, \qquad L(z+\tau)=T_B L(z) T_B^{-1}, \qquad T_A T_B T_A^{-1} T_B^{-1} = e^{\frac{2\pi ik}{N}}$$

Both twists and monodromies

$$\partial_z Y = LY \quad \Rightarrow \quad Y \to TYM$$

Linear System on the Torus

Bundles with $k = 0$

$$T_A = 1, \qquad T_B = diag\left(e^{2\pi i Q_1}, \dots, e^{2\pi i Q_N}\right), \qquad \sum Q_i = 0$$

It is possible to go from the Lax matrix of a bundle to another by means of a singular gauge transformation (**Hecke modification of the bundle**)

For the first nontrivial case of one puncture, $N = 2$, the Lax matrix is that of the **elliptic Calogero-Moser integrable system**:

$$L(z|\tau) = \begin{pmatrix} p & igx(Q,z) \\ igx(-Q,z) & -p \end{pmatrix} \qquad x(Q,z) = \frac{\theta_1(z-Q|\tau)\theta_1'(\tau)}{\theta_1(z|\tau)\theta_1(-Q)}$$

$$\frac{d^2Q}{d\tau^2} = (2\pi i m)^2 \wp'(2Q|\tau)$$

$$\partial_z Y(z|\tau) = L(z|\tau)Y(z|\tau)$$

$$\frac{1}{2}L^2 = (Y^{-1}\partial_z Y)^2$$

GOAL ONE: Find a matrix function Y with given singular behavior and monodromies around points z_k, monodromies and twists around A- and B-cycle.

GOAL TWO: Obtain from Y the tau function generating isomonodromic Hamiltonians. (deautonomization of quadratic Hitchin Hamiltonians)

$$H_k = \frac{1}{2\pi i}\oint_{\gamma_k} dz\, tr\frac{1}{2}L^2(z) = \partial_{t_k}\log\mathcal{T}$$

$$H_\tau = \oint_A dz\, tr\frac{1}{2}L^2(z) = 2\pi i\partial_\tau \log\mathcal{T}$$

Physical motivation

Mathematical Formulation of the problem

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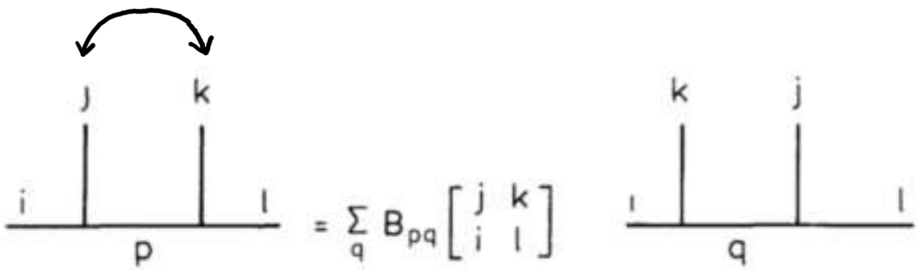
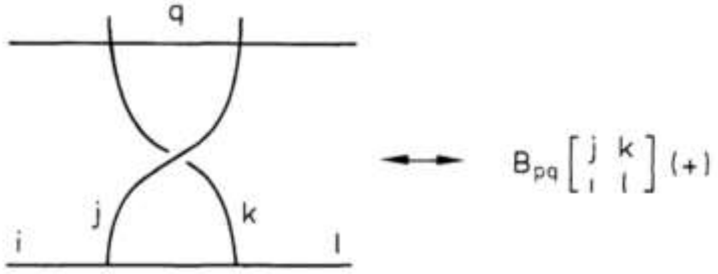
Conclusions and Outlook

Main tool: degenerate braiding and fusion I

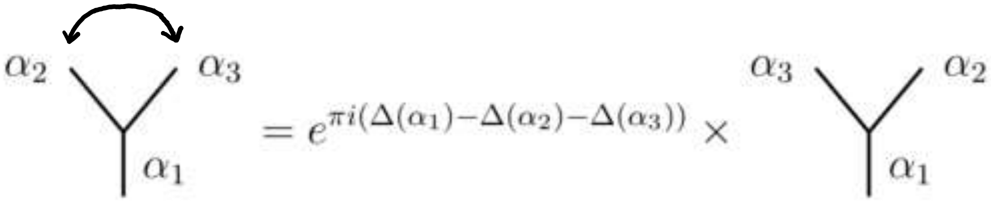
For the 1-punctured torus we need only the following braiding move (analytic continuation around a point):



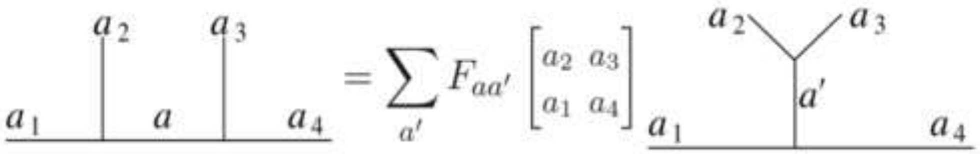


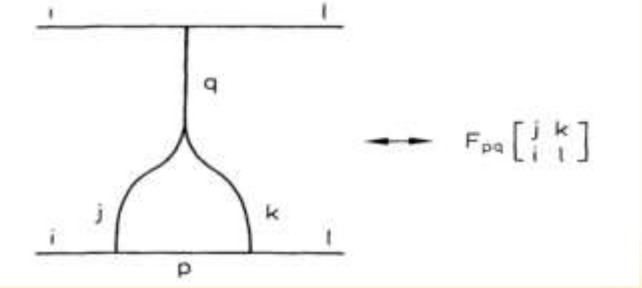
In general [Moore,Seiberg 1989; Alday et al. 2010; Drukker, Teschner et al. 2010]

BRAIDING



FUSION

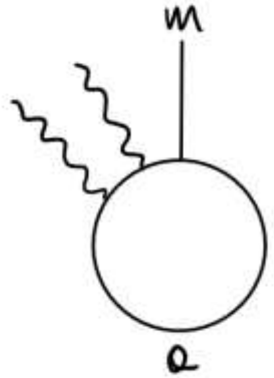




One Punctured Torus: Solution to the Linear System I

Chiral conformal block [ILT2015]:

$$[\phi] \otimes [V_m] = \left[V_{m+\frac{1}{2}}\right] \oplus \left[V_{m-\frac{1}{2}}\right], \qquad \phi_s, \qquad s = 1,2$$



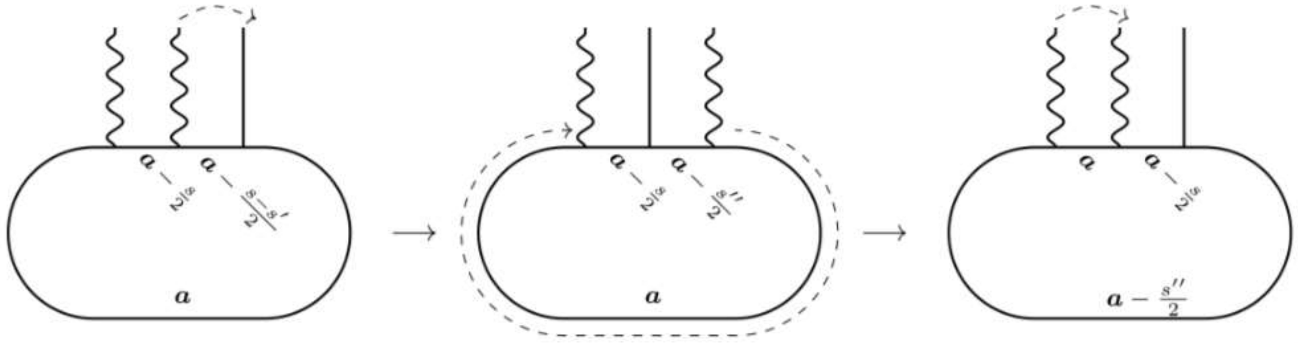
$$\Phi(z, z_0|\tau, a, m) = \frac{\langle V_m(1)\tilde{\phi}(z_0) \otimes \phi(z)\rangle}{\langle V_m\rangle} = \frac{\mathrm{tr}_{\mathcal{V}_a}\left(q^{L_0} V_m(1)\tilde{\phi}(z_0) \otimes \phi(z)\right)}{\mathrm{tr}_{\mathcal{V}_a}\left(q^{L_0} V_m(1)\right)}$$

- $\sum_n e^{in\eta}\Phi(a+n)$ would have given monodromies as in the sphere around 1 and A-cycle:

$$M_A \sim e^{2\pi i a} \qquad\qquad M_1 \sim e^{2\pi i m}$$

- But B-cycle has half-integer shifts!
- Necessary to add U(1) boson and consider free fermions

$$\psi(z) = \phi(z)e^{i\varphi(z)}$$



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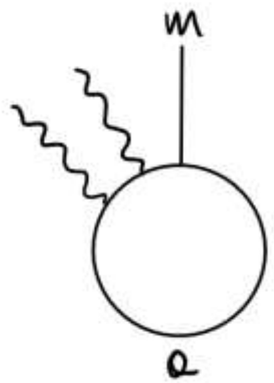

$$\Phi(z,z_0|\tau,a,m)=\frac{\langle V_m(1)\tilde{\phi}(z_0)\otimes\phi(z)\rangle}{\langle V_m\rangle}=\frac{\mathrm{tr}_{\mathcal{V}_a}\left(q^{L_0}V_m(1)\tilde{\phi}(z_0)\otimes\phi(z)\right)}{\mathrm{tr}_{\mathcal{V}_a}\left(q^{L_0}V_m(1)\right)}$$

- $\sum_n e^{in\eta}\Phi(a+n)$ would have given monodromies as in the sphere around 1 and A-cycle:

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- But B-cycle has half-integer shifts!
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Kernel:

$$K(z,z_0|\tau,a,m)\equiv\frac{\langle V_m(1)\bar{\psi}(z_0)\otimes\psi(z)\rangle}{\langle V_m\rangle}=Y^{-1}(z_0)\frac{\theta_1(z-z_0+\boldsymbol{Q}|\tau)\theta_1'(\tau)}{\theta_1(z-z_0)\theta_1(\boldsymbol{Q})}Y(z)$$

One-Punctured Torus: Tau Function

To get the tau function we expand in $z - z_0$

$$Z_{twist}(Q) = \frac{\theta_1(Q|\tau)^2}{\eta(\tau)^2}$$

$$\begin{cases} K(z, z_0) = Y^{-1}(z_0)\dfrac{\theta_1(z - z_0 + \boldsymbol{Q}|\tau)\theta_1'(\tau)}{\theta_1(z - z_0)\theta_1(\boldsymbol{Q})}Y(z) \\ \partial_z Y(z) = L(z)Y(z) \end{cases}$$

$$\Rightarrow \quad \mathrm{tr}\, K = \frac{2}{z - z_0} + (z - z_0)\left[\frac{1}{2} tr\, L^2(z) + 2\pi i \partial_\tau \log Z_{twist}(Q)\right]$$

Additional contribution due to the twists

Fermion OPE:

$$\mathrm{tr}\, K = \frac{\langle V_m T(z)\rangle}{\langle V_m\rangle} = \langle T\rangle + m^2[\wp(z - 1|\tau) + 2\eta_1(\tau)] + 2\pi i \partial_\tau \log\langle V_m\rangle$$

$$2\pi i \partial_\tau \log Z$$

$$2\pi i\, \partial_\tau \log \mathcal{T} = H_\tau = \oint_A \frac{1}{2} tr\left(L^2\right)dz$$

$$\Downarrow$$

Tau function:

$$\mathcal{T} = \frac{1}{Z_{\text{twist}}}\langle V_m(1)\rangle_D = \frac{1}{Z_{twist}}\sum_n e^{\frac{in\eta}{2}}\, \mathrm{tr}_{\mathcal{H}_a^{\left(\frac{n}{2}\right)}}\, q^{L_0} V_m(1) \propto \frac{1}{Z_{twist}} Z^D$$

Generalizations

The generalization from one to many punctures is straightforward: one only has to add a primary field insertion at every puncture's location.

We dealt with the case of an $SL(2)$ linear system. The picture can be extended to the case of $SL(N)$ Fuchsian systems by using a W_N conformal field theory of free fermions. [BDGT19xx.xxxx]

$SL(N)$ Kernel

$$Y^{-1}(z_0)\frac{\theta_1(z-z_0+\boldsymbol{Q}|\tau)\theta_1'(\tau)}{\theta_1(z-z_0)\theta_1(\boldsymbol{Q})}\mathrm{Y}(\mathrm{z}) = \frac{\left\langle V_1(z_1)\dots V_n(z_n)\bar{\psi}(z_0)\otimes\psi(z)\right\rangle}{\langle V_1\dots V_n\rangle}$$

$SL(N)$ Tau function

$$\mathcal{T} = \frac{1}{\prod_i \frac{\theta_1(Q_i|\tau)}{\eta(\tau)}}\langle V_1(z_1)\dots V_n(z_n)\rangle$$

Contents

Conclusions and Outlook

- We extended the link between isomonodromy deformations, two-dimensional CFT and four-dimensional $\mathcal{N} = 2$ gauge theories to the case of genus 1. New ingredients are needed because of nontriviality of bundles on the torus. [1901.10497]
- The construction can be extended to higher number of punctures and SL_N linear systems [BDGT, to appear]
- Higher genus extensions: what is the physical meaning of Z_{twist}? What is its analogue for $g > 1$?
- Connection with surface operators, B-branes and KZ equations/quantum isomonodromy (work in progress)
- Uplift to 5d: q-deformation of isomonodromy problem on the torus, connection with topological strings and quantum mirror curves [GHM2016; BGT 2017,2018; CPT 2018]
- Mathematical proof of Fredholm determinant expression following the approach of [Gavrylenko Lisovyy, Cafasso Gavrylenko Lisovyy 2018]

Thank you!