Partition function from quantum curves

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The problem

Let us consider CY Y_Σ of "class Σ", local CY of the form

$$Y_\Sigma = \left\{ (\eta_1, \eta_2, \eta_3) \in K_C^{\oplus 3} \,|\, \eta_1^2 + \eta_2^2 + \eta_3^2 = q \right\} \subset \mathrm{Tot}(K_C^{\oplus 3}),$$

with q: quadratic differential on C, or in local coordinates:

$$uv - P_\Sigma(x, y) = 0, \qquad \text{with} \quad P_\Sigma(x, y) = y^2 - q(x).$$

Problem: Define and compute **topological string partition function** $Z^{\mathrm{top}}(\mathbf{t}; \lambda)$, where $\mathbf{t}$: Special coordinates for complex structure moduli of Y_Σ.

Motivation

Geometric engineering of $N = 2$, $d = 4$ gauge theories of class $\mathcal{S}$ (partition functions etc.)

Perspectives

Beyond instanton calculus (Sicilian quivers, non-Lagrangian theories)

Clues from string dualities (Dijkgraaf-Hollands-Sulkowski-Vafa)

Chain of dualities $\rightsquigarrow$

$Z^{\rm top}(\mathbf{t};\lambda) \sim Z^{\rm D0-D2-D6}$ related to dual partition fct. $Z^{\rm D}(\tau,\mathbf{t};\lambda) \sim Z^{\rm D0-D2-D4-D6}$

$$Z^{\rm D}(\tau,\mathbf{t};\lambda) = \sum_{\mathbf{p}\in H^2(X,\mathbb{Z})} e^{\mathbf{p}\cdot\tau} Z^{\rm top}(\mathbf{t}+\lambda\mathbf{p},\lambda).$$

Furthermore:

$Z^{\rm D}(\tau,\mathbf{t};\lambda)$ can be represented as the partition function of **free fermions** on a non-commutative deformation of $\Sigma = \{(x,y); P_\Sigma(x,y)=0\}$, the **"quantum curve"**

$$P_\Sigma\left(x\,,\, \frac{\lambda}{\rm i}\frac{\partial}{\partial x}\right)\psi(x) = 0.$$

$\rightsquigarrow$ elegant non-perturbative definition of $Z_{\rm top}(\mathbf{t},\lambda)$ if we knew exactly

(A) how to **define** $Z^{\rm D}(\tau,\mathbf{t};\lambda)$,

(B) how to **expand** $Z^{\rm D}(\tau,\mathbf{t};\lambda)$.

(A) How to define $Z^{\mathrm{D}}(\tau, \mathbf{t}; \lambda)$? – Proposal (CPT):

$Z^{\mathrm{D}}(\tau, \mathbf{t}; \lambda)$ is the **tau-function** $\mathcal{T}(\mu; \mathbf{z})$ associated to isomonodromic deformations of

quantum curve $\left[\lambda^2 \partial_x^2 - q_\lambda(x)\right]\psi(x) = 0,$ for $C = C_{0,n}$:

$$q_\lambda(x) = \sum_{r=1}^{n} \left(\frac{a_r^2}{(x - z_r)^2} + \frac{H_r}{x - z_r} \right) + \lambda \sum_{k=1}^{d} \left(\frac{y_k}{x - x_k} - \frac{3\lambda}{4(x - x_k)^2} \right).$$

Motivated by **connections**

integrability $\rightleftharpoons^1$ **free fermions** $\rightleftharpoons^2$ **CFT**.

To answer (B), **mainly need:**

How are variables $\tau, \mathbf{t}, \lambda$ related to arguments $\mu, \mathbf{z}$ of $\mathcal{T}(\mu; \mathbf{z})$?

[1] DHSV: Generalised Krichever construction: quantum curve $\Sigma_\lambda(\tau) \mapsto$ state $|\Sigma_\lambda(\tau)\rangle$
Free fermion partition function: Matrix element $Z^{\mathrm{D}}(\tau, \mathbf{t}; \lambda) = \langle\, \mathbf{t}\, | \Sigma_\lambda(\tau)\rangle$

[2] CPT: Conformal Ward identities $\Rightarrow \langle\, \mathbf{t}\, | \Sigma_\lambda(\tau)\rangle = \mathcal{T}(\mu; \mathbf{z})$

How to define the tau-functions I

Consider λ-connections $\nabla_\lambda = \lambda\partial_x - A(x)$.

Locally gauge equivalent to **oper** $\lambda\partial_x - \begin{pmatrix} 0 & q(x) \\ 1 & 0 \end{pmatrix} \quad \rightleftharpoons \quad$ **quantum curve**.

Example $g = 0$**:** $$A(x) = \sum_{r=1}^{n} \frac{A_r}{x - z_r}, \qquad A_r \in \mathfrak{sl}_N, \qquad \sum_{r=1}^{n} A_r = 0.$$

Poisson-structure (Goldman-Atiyah-Bott)

$$\{\, A(x) \overset{\otimes}{,} A(y) \,\}_{\mathrm{GAB}} = \frac{1}{x - y}\big[\, \mathsf{P} \,,\, A(x) \otimes 1 + 1 \otimes A(y) \,\big].$$

Hamiltonians:

$$H_r(\mathbf{z}) = \sum_{s \neq r} \frac{\mathrm{tr}(A_r A_s)}{z_r - z_s}, \qquad \mathbf{z} = (z_1, \ldots, z_n).$$

Isomonodromic deformations (Schlesinger system):

$$\frac{\partial}{\partial z_r} A_s = \{\, A_s \,,\, H_r \,\}_{\mathrm{GAB}} \quad \Leftrightarrow \quad \text{Monodromy } \mu \text{ of } \nabla_\lambda \text{ is constant.}$$

How to define the tau-functions II

Isomonodromic tau-function, classical definition (Sato-Miwa-Jimbo):

$$\frac{\partial}{\partial z_r} \log \mathcal{T}(\mu, \mathbf{z}) = H_r(\mu, \mathbf{z}),$$

where $\mathbf{z} = (z_1, \ldots, z_n)$,

μ: monodromy data (representation $\pi_1(C) \to \mathrm{GL}(N, \mathbb{C})$),

H_r: Schlesinger-Hamiltonians.

Longstanding problems:

- Calculate series expansions of $\mathcal{T}(\mu, \mathbf{z})$ around singular points.
- How to fix dependence on monodromy parameters μ?

How to compute the tau-functions I

Explicit formula:

(conjectured by Gamayun-Iorgov-Lisovyy[3],

proofs by Iorgov-Lisovyy-J.T.[4], Bershtein-Shchechkin[5], Gavrylenko-Lisovyy[6])

$$\mathcal{T}(\sigma,\tau\,;\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^{n-3}} e^{i(\mathbf{n},\tau)}\,\mathcal{G}(\,\sigma+\mathbf{n}\,;\mathbf{z}\,),$$

where:

[3]Inspired by/using results of Sato-Miwa-Jimbo, Moore, Moore-Nekrasov-Shatashvili, Nekrasov, Alday-Gaiotto-Tachikawa

[4]CFT: Monodromy of $\mathfrak{Vir}$-degenerate fields, bosonic version of Sato-Miwa-Jimbo solution of Riemann-Hilbert problem

[5]VOA duality (Bershtein-Feigin-Litvinov) $\rightsquigarrow$ bilinear equations of Hirota type, related to Nakajima-Yoshioka blow-up

[6]Combinatorial expansion of Fredholm determinants; Cafasso-Gavrylenko-Lisovyy: Relation with Sato-Segal-Wilson

How to compute the tau-functions II

For $n = 4$ and **suitable** choice of τ: $\mathcal{G}(\,\sigma\,,\,\underline{\theta}\,;\,z\,)$ can be factorised as

$$\mathcal{G}(\,\sigma\,,\,\underline{\theta}\,;\,z\,) = M(\sigma,\theta_4,\theta_3)M(\sigma,\theta_2,\theta_1)\mathcal{F}(\,\sigma\,,\,\underline{\theta}\,;\,z\,),$$

using the following notations:

- The functions $M(\theta_3,\theta_2,\theta_1)$ are defined as

$$M(\theta_3,\theta_2,\theta_1) \;=\; \frac{\prod_{\epsilon=\pm} G(1+\theta_3+\epsilon(\theta_2+\theta_1))G(1+\theta_3+\epsilon(\theta_2-\theta_1))}{G(1+2\theta_3)G(1-2\theta_2)G(1-2\theta_1)},$$

  where $G(p)$ is the Barnes G-function that satisfies $G(p+1) = \Gamma(p)G(p)$.

- $\mathcal{F}(\,\sigma\,,\,\underline{\theta}\,;\,z\,)$ **Conformal blocks / instanton partition functions (AGT)**.

- (σ,τ): **very special** parameters for monodromy data, introduced later.

How to use the tau-functions I

Magic formula:

$$\mathcal{T}\big(\sigma,\tau\,;\,\mathbf{z}\big)=\sum_{\mathbf{n}\in\mathbb{Z}^{n-3}}e^{i(\mathbf{n},\tau)}\,\mathcal{G}(\,\sigma+\mathbf{n}\,;\,\mathbf{z}\,),$$

can be compared to:

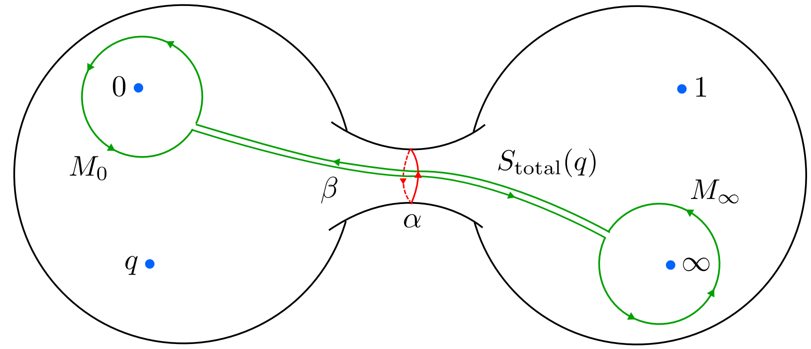
$$Z^{\mathrm{D}}(\tau,\mathbf{t};\lambda)=\sum_{\mathbf{p}\in H^2(X,\mathbb{Z})}e^{\mathbf{p}\cdot\tau}Z^{\mathrm{top}}(\mathbf{t}+\lambda\mathbf{p},\lambda).$$

This further supports identification $Z^{\mathrm{D}}(\tau,\mathbf{t};\lambda)\equiv\mathcal{T}\big(\sigma,\tau\,;\,\mathbf{z}\big)$

Main problem:

Which **monodromy parameters** τ do the job?

Coordinates for monodromy data



Define a system of Darboux coordinates:

Use pants decomposition to reduce to $C_{0,4}$ or $C_{1,1}$:

Pants decomposition $\rightsquigarrow$ Factorisation of holonomy:

$$L_\alpha = \mathrm{tr}(M_0 M_q) = 2\cos(2\pi\sigma),$$

$$L_\beta = \mathrm{tr}(T^{-1} M_0 T M_\infty) = \mathrm{tr}\left(T^{-1}\begin{pmatrix} * & \mu_0^+ \\ \mu_0^- & * \end{pmatrix} T \begin{pmatrix} * & \mu_\infty^+ \\ \mu_\infty^- & * \end{pmatrix}\right)$$
$$= e^{2\pi i\tau}\mu_0^-\mu_\infty^+ + e^{-2\pi i\tau}\mu_0^+\mu_\infty^- + N_0,$$

where $T = \begin{pmatrix} e^{\pi i\tau} & 0 \\ 0 & e^{-\pi i\tau} \end{pmatrix}$ and $\mu_0^\pm$, $\mu_\infty^\pm$ and N_0 depend on σ but **not** on τ.

Coordinates (σ, τ): Darboux-coordinates for moduli space $(\mathcal{M}_{\mathrm{flat}}(C), \Omega_{\mathrm{GAB}})$ (related to work of `Nekrasov, Rosly, Shatashvili`).

How exact WKB determines the right coordinates I

Holonomy of quantum curve $\lambda\partial_x - A_\lambda(x)$, $A_\lambda(x) = \varphi(x) + \mathcal{O}(\lambda)$, for $\lambda \to 0$,

$$\lambda\partial_x - A_\lambda(x) \underset{\text{gauge}}{\sim} \lambda\partial_x - \begin{pmatrix} 0 & q(x) \\ 1 & 0 \end{pmatrix} + \mathcal{O}(\lambda)\,,$$

where $q(x) = \mathrm{tr}(\varphi^2)$: quadratic differential defining Σ!

Exact WKB: Construction of coordinates (σ, τ) as **quantum periods**, deformations of period coordinates $a = \int_\alpha \sqrt{q}$, $a^{\mathrm{D}} = \int_\beta \sqrt{q}$ by Borel summation of λ-expansion, "controlled" by $q(x)$.

Stokes phenomenon: Result is piecewise analytic over $\mathfrak{Q}$: space of pairs (C, q), analytic in cells distinguished by type of Stokes graph defined by q, having arcs

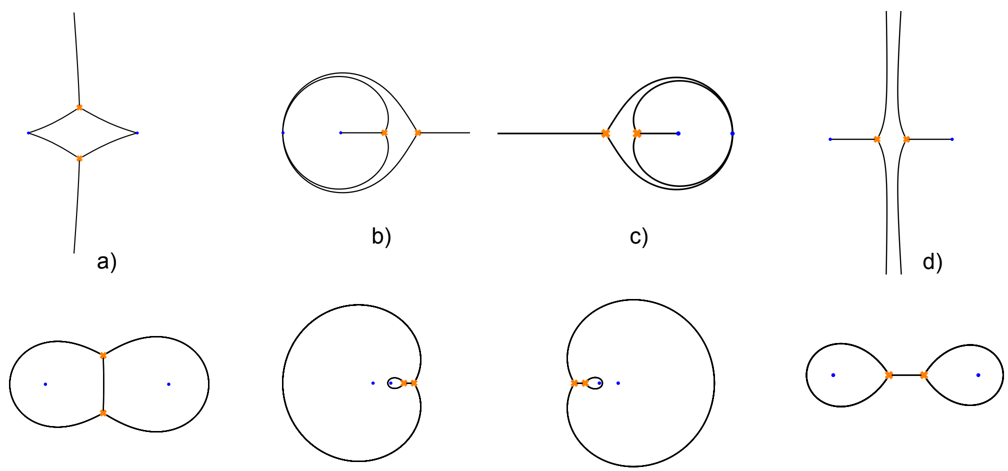
$$\mathrm{Im} \int_b^x \sqrt{q} = 0, \qquad b: \text{ branch point of } \pi : \Sigma \to C.$$

Note: $\mathfrak{Q}$ represents space of complex structures of Σ, **period coordinates** $\mathbf{t} = (a, \mathbf{z})$.

How exact WKB determines the right coordinates II

- **Construction:** Generic quadratic differential determines pair $\mathsf{P} = (\wp, \mathsf{t})$, where $\wp$: pants decomposition, t: Stokes graph in each pair of pants.

- To each $\mathsf{P} = (\wp, \mathsf{t})$ associate coordinates $(\sigma_{\mathsf{P}}, \tau_{\mathsf{P}})$ by the exact WKB method.
  Indeed, pants decomposition $\rightsquigarrow$ reduction to RH problem on $C_{0,3}$. Exact WKB $\rightsquigarrow$ canonical solutions to RH problem on $C_{0,3}$ depending on Stokes graph (Aoki, Takahashi, Tanda)



Outcome: Cell decomposition of $\mathfrak{Q}$, cells labelled by $\mathsf{P} = (\wp, \mathsf{t})$, coordinates $(\sigma_{\mathsf{P}}, \tau_{\mathsf{P}})$.

Remark: Analogous/related to assignment of Fock-Goncharov coordinates to WKB triangulations by Gaiotto, Moore, Neitzke and Allegretti.

How to use the tau-functions II

The results outlined above give:

- Cell decomposition of complex structure moduli space $\mathfrak{Q}$, cells $\mathfrak{Q}_{\mathsf{P}}$, $\mathsf{P} = (\wp, \mathsf{t})$,
- tau-functions $\mathcal{T}_{\mathsf{P}}(\sigma_{\mathsf{P}}, \tau_{\mathsf{P}}\,;\,\mathbf{z})$ defined in $\mathfrak{Q}_{\mathsf{P}}$.

To get topological string partition functions associated to $\mathfrak{Q}_{\mathsf{P}}$, use

$$\mathcal{T}_{\mathsf{P}}(\sigma_{\mathsf{P}}, \tau_{\mathsf{P}}\,;\,\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^{n-3}} e^{i(\mathbf{n},\tau_{\mathsf{P}})}\,\mathcal{G}_{\mathsf{P}}(\,\sigma_{\mathsf{P}} + \mathbf{n}\,;\,\mathbf{z}\,).$$

Proposal:

$$\mathcal{Z}^{\mathrm{top}}(a_{\mathsf{P}}\,,\,\mathbf{z}\,;\,\lambda) = \mathcal{G}_{\mathsf{P}}\left(\frac{1}{\lambda}a_{\mathsf{P}}\,;\,\mathbf{z}\,\right)$$

This defines $\mathcal{Z}^{\mathrm{top}}$ as piecewise analytic function on $\mathfrak{Q}$ (wall-crossing).

Checked whenever $\mathcal{Z}^{\mathrm{top}}$ is calculable using the topological vertex.

Summary

We have a proposal for a **non-perturbative**[7], **geometric** and **computable** definition of the topological string partition functions for class Σ.

Key ingredient: Exact WKB gives a **canonical** way to assign coordinate systems to cells of the moduli space of spectral curves Σ.

The proposal was successfully checked by topological vertex calculations.

Further relations to:

– the quantised Hitchin system (see next slide)

– the spectrum of BPS states (cells P determined by lightest hypers ...)

[7] manifest in representation as a Fredholm determinant (Cafasso-Gavrylenko-Lisovyy)

Relation to Hitchin's integrable system:

Diaconescu, Donagi, Pantev:

Intermediate Jacobian of $Y_\Sigma \simeq$ Hitchin's fibration

Use the isomonodromic tau-functions to define the functions $\mathcal{Z}_{\rm ff}(a,\theta;z;\lambda)$,

$$\mathcal{Z}_{\rm ff}(a,\theta;z;\lambda) := \mathcal{T}\left(\frac{1}{\lambda}a\,,\,\frac{1}{\lambda}a^{\rm D}+\theta\,;\,z\right).$$

The limit $\log\Theta_\Sigma(\theta,a) := \lim_{\lambda\to 0}\left[\log\mathcal{Z}_{\rm ff}(a,\theta;z;\lambda) - \frac{1}{\lambda^2}\mathcal{F}(a) - \mathcal{F}_1(a)\right]$ exists.

The function $\Theta_\Sigma(\theta,a)$ is the **theta function** on the Hitchin Prym

$$\Theta_\Sigma(\theta,a) = \sum_{\mathbf{n}\in\mathbb{Z}^{n-3}} e^{2\pi \mathrm{i}(\mathbf{n},\theta)}\, e^{\pi\mathrm{i}(\mathbf{n},\tau^\Sigma(a)\mathbf{n})}, \qquad \tau_{rs}^\Sigma = \frac{1}{2\pi\mathrm{i}}\frac{\partial^2\mathcal{F}}{\partial a^r\partial a^s}.$$