Solving refined BPS invariants with blowup equations

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Based on:
1609.05914: Grassi, JG
1701.00764: JG, Huang, Kashani-Poor, Klemm
1811.02577: JG, Haghighat, Sun, Wang
1905.00864: JG, Klemm, Sun, Wang

Refined topological string theory

- Consider M-theory compactified on a non-compact Calabi-Yau threefold X, refined topological string theory computes
  - Refined BPS invariants $N^{\beta}_{j_L,j_R}$: numbers of BPS states of M2 branes wrapping curve class $\beta \in X$ with spins j_L, j_R in remaining $S^1 \times \mathbb{R}^4$

  which are assembed into partition function $Z(\mathbf{t}, \epsilon_1, \epsilon_2)$.
- $N^{\beta}_{j_L,j_R}$ are non-negative, and display checkerboard pattern for fixed β, e.g.

$2j_L/2j_R$	0	1	2	3	4	5	6	7	8
0				1		3		2	
1							1		1

  which is characterised by $\mathbf{r} \in (\mathbb{Z})^{b_2}$ such that

$$2j_L + 2j_R + 1 \equiv \mathbf{d} \cdot \mathbf{r} \bmod 2$$

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Computing refined BPS invariants

- Computational methods: torus localisation, refined topological vertex, refined holomorphic anomaly equations, modular bootstrap
- Well-known examples
  - Canonical bundles over $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_n$.
  - Resolution of $\mathbb{C}^3/\mathbb{Z}_5, \mathbb{C}^3/\mathbb{Z}_6$.
  - Canonical bundle over $\frac{1}{2}K3$.
  - ...

  X is either toric or with small b_4
- Our proposal: A universal computational method (blowup equations)
  - Applicable to all these models and beyond
  - Requiring minimum input data

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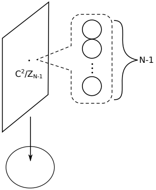
Geometric engineering

- $X_{N,m}$: fibration of resolved $\mathbb{C}^2/\mathbb{Z}_{N-1}$ singularity over $\mathbb{P}^1$.



- 5d $N = 1$ Super-Yang-Mills with $G = SU(N)$ and Chern-Simons level m on $S^1 \times_{\epsilon_1,\epsilon_2} \mathbb{R}^4$
  1. $N - 1$ vector moduli $\mathbf{m}$ and instanton counting parameter $\mathfrak{q}$.
  2. Partition function

$$Z(\mathbf{m}, \mathfrak{q}, \epsilon_{1,2}) = Z^{\text{cls}}(\mathbf{m}, \epsilon_{1,2}) Z^{\text{1-loop}}(\mathbf{m}, \epsilon_{1,2})(1 + \sum \mathfrak{q}^k Z_k(\mathbf{m}, \epsilon_{1,2}))$$

  3. Z_k is integral over moduli space $\mathcal{M}(k, N)$ of k instantons in $\mathbb{R}^4$.

$$Z(\mathbf{t}, \epsilon_1, \epsilon_2) = Z(\mathbf{m}, \mathfrak{q}, \epsilon_1, \epsilon_2)$$

Gottsche-Nakajima-Yoshioka blowup equations

- Göttsche-Nakajima-Yoshioka
  - put 5d $SU(N)$ SYM on $S^1 \times_{\epsilon_1,\epsilon_2} Bl_1(\mathbb{C}^2)$ with mag. flux k through exc'l divisor E, and
  - compute correlation function of operator $\mu(E)^d$ on $\mathcal{M}(k,N)$ ass'd to $\mathcal{O}(dE) \to Bl_1(\mathbb{R}^4)$ in two different ways.

They find following equ'ns for partition function Z on $S^1 \times_{\epsilon_1,\epsilon_2} \mathbb{C}^2$

[Göttsche-Nakajima-Yoshioka,'06]

$$\sum_{\mathbf{n}} Z(\mathbf{m} + \epsilon_1 \mathbf{n}, \mathfrak{q} e^{\epsilon_1 (d + m(-1/2 + k/N) - N/2)}, \epsilon_1, \epsilon_2 - \epsilon_1)$$
$$\times Z(\mathbf{m} + \epsilon_2 \mathbf{n}, \mathfrak{q} e^{\epsilon_2 (d + m(-1/2 + k/N) - N/2)}, \epsilon_1 - \epsilon_2, \epsilon_2)$$
$$= \begin{cases} 0 & (k,d) \text{ in interior of } \square \\ \Lambda(\mathfrak{q}, \epsilon_1, \epsilon_2) Z(\mathbf{m}, \mathfrak{q}, \epsilon_1, \epsilon_2) & (k,d) \text{ on boundary of } \square \end{cases}$$

where

① $\mathbf{n}$ runs over $\mathbf{n} = (n_I) \in \mathbb{Q}^N$ with $\sum n_I = 0$, $n_I \equiv -k/N \mod 1$.
② $\square = \{k, d \in \mathbb{Z} \, : \, 0 \leq k, d \leq N\}$.

- Z_k can be computed recursively from the blowup equations. [Keller-Song'12]

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Generalised blowup equations

Consider a local Calabi-Yau threefold X

- $b_2 = \dim H_2(X,\mathbb{Z})$, $\quad b_4 = \dim H_4(X,\mathbb{Z})$;
- $\mathbf{C} = (C_{ij}) = (\Sigma_i \cdot D_j)$, $\qquad \Sigma_i \in H_2(X,\mathbb{Z}),\ D_j \in H_4(X,\mathbb{Z})$;
- $\mathbf{t} = (t_i) = (\mathrm{vol}(\Sigma_i))$, among which $\mathbf{t}_m = (t_{m_i})$ for curves not intersecting compact surfaces;
- The checkerboard pattern of non-vanishing $N^{\mathbf{d}}_{j_L,j_R}$ can be characterised by $\mathbf{r} \in (\mathbb{Z})^{b_2}$ satisfying

$$2j_L + 2j_R + 1 \equiv \mathbf{d}\cdot\mathbf{r} \bmod 2\ .$$

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Generalised blowup equations

There *exist* $\mathbf{r} \in \mathbb{Z}^{b_2}$ subject to checkerboard pattern condition such that refined topological string partition function satisfies [Grassi-JG,'16][JG-Huang-Kashani Poor-Klemm,'17][Huang-Sun-Wang,'17]

$$\sum_{\mathbf{n}\in\mathbb{Z}^{b_4}} (-1)^{|\mathbf{n}|} Z(\mathbf{t}+\epsilon_1\mathbf{R},\epsilon_1,\epsilon_2-\epsilon_1)\cdot Z(\mathbf{t}+\epsilon_2\mathbf{R},\epsilon_1-\epsilon_2,\epsilon_2)$$

$$=\Lambda(\mathbf{t}_m,\epsilon_1,\epsilon_2,\mathbf{r}) Z(\mathbf{t},\epsilon_1,\epsilon_2)\ , \qquad \mathbf{R} = \mathbf{C}\cdot\mathbf{n}+\mathbf{r}/2\ .$$

- Different $\mathbf{r}$ give rise to different equations.
- Nontrial: $\Lambda(\mathbf{t}_m,\epsilon_1,\epsilon_2,\mathbf{r})$ depends on $\mathbf{t}_m$ only.
- *Unity* (*Vanishing*) equations if Λ does not (does) vanish identically.

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- Justification
  - Blowup equations have a universal form with seemingly no constraint on the type of Calabi-Yau threefold.
  - If X is toric, vanishing equations are consistency conditions for the quantisation of mirror curves. ⇒ Alba's talk

Aside: quantisation of mirror curves

- Mirror curve of a toric Calabi-Yau threefold X can be promoted to an operator (quantum mirror curve), i.e. for local $\mathbb{P}^1 \times \mathbb{P}^1$

  $$\left(\mathrm{e}^{\mathsf{x}} + m\mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}} + u\right)\Psi(x) = 0$$

  with $[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$. The eigenstate equation cuts out a divisor $\mathcal{D}$ in complex moduli space $\mathcal{M}$, which is solved by [Grassi-Hatsuda-Marino,'14][Codesido-Grassi-Marino,'15].
- Polytope of X defines a quantum cluster integrable system with b_4 Hamiltonians. The discrete spectrum $\mathcal{S}$ (S-dual) is solved by [Wang-Zhang-Huang,'15][Hatsuda-Marino,'15][Franco-Hatsuda-Marino,'15].
- Quantum mirror curve is quantum Baxeter equation of quantum cluster integrable system, with complex moduli identified with Hamiltonians (and Casimirs). The spectrum $\mathcal{S}$ must lie within $\mathcal{D}$. [Sun-Wang-Huang,'16]
- A necessary condition is the existence of b_4 vanishing blowup equations in the $\epsilon_1 \to 0$ limit. [Grassi-JG,'16]

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Checking blowup equations

- Semiclassical data Z^{cls} as input which includes
  1. integral basis of $H_2(X,\mathbb{Z}), H_4(X,\mathbb{Z})$, and curve-divisor intersection matrix C
  2. triple intersection numbers a_{ijk}
  3. evaluation of $c_2(TX)$ along divisors b_i
- Condition on Λ determines r and consequently Λ
- Constraint equations of refined BPS invariants are extracted, which are verified to high orders.
- Verified geometries:
  - $X_{N,m}$, canonical bundles over $\mathbb{P}^2, \mathbb{F}_n, Bl_3(\mathbb{P}^2)$, resolved $\mathbb{C}^3/\mathbb{Z}_5$.
  - Elliptic fibration over $\mathcal{O}_{\mathbb{P}^1}(-\mathfrak{n})$ with $\mathfrak{n} = 1, 3, 4$.

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Solving blowup equations

The constraint equations can also be used to solve refined BPS invariants

- Toric Calabi-Yau threefolds: empirically the equations are enough to solve all invariants [Huang-Sun-Wang,'17]
  - Resolved conifold, canonical bundle over $\mathbb{P}^2$, resolved $\mathbb{C}^3/\mathbb{Z}_5$.
  - Both unity and vanishing equations are present.
- Elliptic Calabi-Yau threefolds: still soluble but sometimes additional inputs are needed [JG-Haghighat-Sun-Wang,'18][JG-Klemm-Sun-Wang,'19]
  - Sometimes either unity or vanishing equations are missing.

Minimal noncompact elliptic Calabi-Yau threefolds

- Elliptic fibration over $\mathcal{O}_{\mathbb{P}^1}(-\mathfrak{n})$ with minimum singularity

$\mathfrak{n}$	1	2	3	4	5	6	7	8	12
fiber	I_0	I_0	IV	I_0^*	IV_{ns}^*	IV^*	III^*	III^*	II^*
G_{min}	–	–	$SU(3)$	$SO(8)$	F_4	E_6	$E_7 \oplus \frac{1}{2}56$	E_7	E_8

- Why are they interesting?
  - Simpliest noncompact elliptic Calabi-Yau threefolds.
  - Engineer 6d SCFTs by F-theory compactification.
  - Refined BPS invariants not known for $\mathfrak{n} \geq 5$.

With blowup equations and some additional easy input, *all* BPS invariants can be computed for $\mathfrak{n} = 3, 4, 5, 6, 8, 12$.

- BPS invariants for $\mathfrak{n} = 7$ can also be computed.

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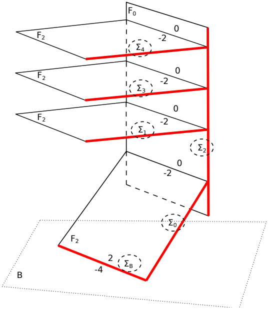
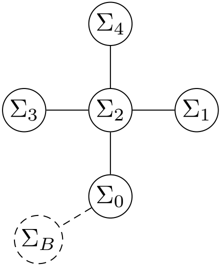
Elliptic fibration X over $\mathcal{O}_{\mathbb{P}^1}(-\mathfrak{n})$





- Decomposition of partition function

$$Z = Z^{\text{cls}} Z^{\text{1-loop}} \left(1 + \sum Q_b^k \mathbb{E}_k(\tau, \mathbf{m}, \epsilon_1, \epsilon_2)\right)$$

with

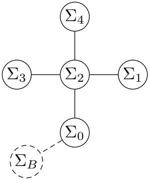
$$\tau = \sum_{I=0}^{r} a_I t_I \ , \quad \mathbf{m} = (m_i) = (t_i) \ , \quad Q_b = \mathrm{e}^{t_b} \ .$$

Elliptic fibration X over $\mathcal{O}_{\mathbb{P}^1}(-\mathfrak{n})$

- Additional input $Z^{\text{1-loop}}$
  The only BPS states supported on curves in fiber are vector multiplets

$$N^{\beta}_{0,1/2} = 1 \,, \quad \beta \in \Delta_+(\hat{\mathfrak{g}})$$

sphere-tree in fiber



- $\mathbb{E}_k$ encode BPS invariants wrapping base curve with degree k
  - $\mathbb{E}_k$ is quotient of Weyl invariant Jacobi forms for $SU(2)_L \times SU(2)_R \times G$. ⇒ Albrecht's talk
  - $\mathbb{E}_k$ has modular weight $w_{\mathbb{E}_k} = 0$ and index polynomial

$$f_{\mathbb{E}_k} = -\frac{\mathfrak{n}k^2+(2-\mathfrak{n})k}{2}\,\epsilon_L^2 + \frac{\mathfrak{n}k^2-(2-\mathfrak{n}+2h^\vee_G)k}{2}\,\epsilon_R^2 - \mathfrak{n}k\,\mathbf{m}\cdot\mathbf{m}$$

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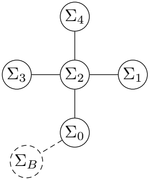

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Elliptic blowup equations

$$\begin{aligned}&\sum_{\omega\in\phi_\lambda(Q^\vee),k_{1,2}\in\mathbb{N}}^{\frac{1}{2}||\omega||^2+k_1+k_2=k}\theta_i^{[a]}\left(\mathfrak{n}\tau,\mathfrak{n}((\tfrac{\mathfrak{n}-2}{\mathfrak{n}}-\tfrac{||\omega||^2}{2}-k_1)\epsilon_1+(\tfrac{\mathfrak{n}-2}{\mathfrak{n}}-\tfrac{||\omega||^2}{2}-k_2)\epsilon_2+\mathbf{m}\cdot\omega)\right)\\ &\qquad\times(-1)^{|\phi_\lambda^{-1}(\omega)|}\cdot A_\omega(\mathbf{m})\cdot\mathbb{E}_{k_1}(\tau,\mathbf{m}-\epsilon_1\omega,\epsilon_1,\epsilon_2-\epsilon_1)\cdot\mathbb{E}_{k_2}(\tau,\mathbf{m}-\epsilon_2\omega,\epsilon_1-\epsilon_2,\epsilon_2)\\ &\qquad\qquad=\begin{cases}\theta_i^{[a]}(\mathfrak{n}\tau,(\mathfrak{n}-2)(\epsilon_1+\epsilon_2))\cdot\mathbb{E}_k(\tau,\mathbf{m},\epsilon_1,\epsilon_2) & \text{fixed } k\in\mathbb{N}\\ 0 & \text{fixed } k\notin\mathbb{N}\end{cases}\end{aligned}$$

- $\theta_i^{[a]}$ with $i=4$ if $\mathfrak{n}$ is odd and $i=3$ if $\mathfrak{n}$ is even and $a=\frac{1}{2}-\frac{\ell}{\mathfrak{n}}$, $\ell=0,1,\ldots,\mathfrak{n}-1$.
- $A_\omega(\mathbf{m})$ is a rational expression of η and θ_1.
- $\phi_\lambda : Q^\vee \hookrightarrow P$ with $\phi_\lambda(\alpha^\vee)=\alpha^\vee+\lambda$ is induced by $\lambda\in P$.
- Number of embeddings: $|P:Q^\vee|$.

$\mathfrak{n}$	3	4	5	6	8	12
G	$SU(3)$	$SO(8)$	F_4	E_6	E_7	E_8
$\|P:Q^\vee\|$	3	4	1	3	2	1
$\#(\mathbf{r})$	9	16	5	18	16	12

Elliptic blowup equations

$$\sum_{\substack{\frac{1}{2}||\omega||^2+k_1+k_2=k\\ \omega\in\phi_\lambda(Q^\vee),k_{1,2}\in\mathbb{N}}} \theta_i^{[a]}\left(\mathfrak{n}\tau,\mathfrak{n}\left(\left(\frac{\mathfrak{n}-2}{\mathfrak{n}}-\frac{||\omega||^2}{2}-k_1\right)\epsilon_1+\left(\frac{\mathfrak{n}-2}{\mathfrak{n}}-\frac{||\omega||^2}{2}-k_2\right)\epsilon_2+\mathbf{m}\cdot\omega\right)\right)$$
$$\times(-1)^{|\phi_\lambda^{-1}(\omega)|}\cdot A_\omega(\mathbf{m})\cdot\mathbb{E}_{k_1}(\tau,\mathbf{m}-\epsilon_1\omega,\epsilon_1,\epsilon_2-\epsilon_1)\cdot\mathbb{E}_{k_2}(\tau,\mathbf{m}-\epsilon_2\omega,\epsilon_1-\epsilon_2,\epsilon_2)$$
$$=\begin{cases}\theta_i^{[a]}(\mathfrak{n}\tau,(\mathfrak{n}-2)(\epsilon_1+\epsilon_2))\cdot\mathbb{E}_k(\tau,\mathbf{m},\epsilon_1,\epsilon_2) & \text{fixed } k\in\mathbb{N}\\ 0 & \text{fixed } k\notin\mathbb{N}\end{cases}$$

- $\theta_i^{[a]}$ with $i=4$ if $\mathfrak{n}$ is odd and $i=3$ if $\mathfrak{n}$ is even and $a=\frac{1}{2}-\frac{\ell}{\mathfrak{n}}$, $\ell=0,1,\ldots,\mathfrak{n}-1$.
- $A_\omega(\mathbf{m})$ is a rational expression of η and θ_1.
- $\phi_\lambda : Q^\vee\hookrightarrow P$ with $\phi_\lambda(\alpha^\vee)=\alpha^\vee+\lambda$ is induced by $\lambda\in P$.
- Number of embeddings: $|P:Q^\vee|$.

$\mathfrak{n}$	3	4	5	6	8	12
G	$SU(3)$	$SO(8)$	F_4	E_6	E_7	E_8
$\|P:Q^\vee\|$	3	4	1	3	2	1
$\#(\mathbf{r})$	9	16	5	18	16	12

Elliptic blowup equations

$$\sum_{\omega\in\phi_\lambda(Q^\vee),\,k_{1,2}\in\mathbb{N}}^{\frac{1}{2}||\omega||^2+k_1+k_2=k} \theta_i^{[a]}\left(\mathfrak{n}\tau,\mathfrak{n}\Big(\big(\tfrac{\mathfrak{n}-2}{\mathfrak{n}}-\tfrac{||\omega||^2}{2}-k_1\big)\epsilon_1+\big(\tfrac{\mathfrak{n}-2}{\mathfrak{n}}-\tfrac{||\omega||^2}{2}-k_2\big)\epsilon_2+\mathbf{m}\cdot\omega\Big)\right)$$
$$\times\,(-1)^{|\phi_\lambda^{-1}(\omega)|}\cdot A_\omega(\mathbf{m})\cdot\mathbb{E}_{k_1}(\tau,\mathbf{m}-\epsilon_1\omega,\epsilon_1,\epsilon_2-\epsilon_1)\cdot\mathbb{E}_{k_2}(\tau,\mathbf{m}-\epsilon_2\omega,\epsilon_1-\epsilon_2,\epsilon_2)$$
$$=\begin{cases}\theta_i^{[a]}(\mathfrak{n}\tau,(\mathfrak{n}-2)(\epsilon_1+\epsilon_2))\cdot\mathbb{E}_k(\tau,\mathbf{m},\epsilon_1,\epsilon_2) & \text{fixed } k\in\mathbb{N}\\ 0 & \text{fixed } k\notin\mathbb{N}\end{cases}$$

Sanity checks:

1. Every term has the same modular weight and index polynomial.
2. Valid in Q_τ expansion when known $\mathbb{E}_k$ are plugged in ($\mathfrak{n}=3,4$).

Unity blowup equations

- If $\lambda \in Q^\vee$, then $\phi_\lambda(Q^\vee) = Q^\vee$, and k always a positive integer.

$$\sum_{\alpha^\vee \in Q^\vee, k_{1,2}\in\mathbb{N}}^{\frac{1}{2}||\alpha^\vee||^2+k_1+k_2=k} \theta_i^{[a]}\left(\mathfrak{n}\tau, \mathfrak{n}((\tfrac{\mathfrak{n}-2}{\mathfrak{n}} - \tfrac{||\alpha^\vee||^2}{2} - k_1)\epsilon_1 + (\tfrac{\mathfrak{n}-2}{\mathfrak{n}} - \tfrac{||\alpha^\vee||^2}{2} - k_2)\epsilon_2 + \mathbf{m}\cdot\alpha^\vee)\right)$$
$$\times (-1)^{|\alpha^\vee|} \cdot A_{\alpha^\vee}(\mathbf{m}) \cdot \mathbb{E}_{k_1}(\tau, \mathbf{m} - \epsilon_1\alpha^\vee, \epsilon_1, \epsilon_2 - \epsilon_1) \cdot \mathbb{E}_{k_2}(\tau, \mathbf{m} - \epsilon_2\alpha^\vee, \epsilon_1 - \epsilon_2, \epsilon_2)$$
$$= \theta_i^{[a]}(\mathfrak{n}\tau, (\mathfrak{n}-2)(\epsilon_1+\epsilon_2)) \cdot \mathbb{E}_k(\tau, \mathbf{m}, \epsilon_1, \epsilon_2)$$

- Number of equations

$\mathfrak{n}$	3	4	5	6	8	12
G	$SU(3)$	$SO(8)$	F_4	E_6	E_7	E_8
#(unity-**r**)	3	4	5	6	8	12

- Recursive relations

$$\theta_i^{[a]} \cdot \mathbb{E}_k(\tau, \mathbf{m}, \epsilon_1, \epsilon_2 - \epsilon_1) + \theta_i^{[a]} \cdot \mathbb{E}_k(\tau, \mathbf{m}, \epsilon_1 - \epsilon_2, \epsilon_2) + \theta_i^{[a]} \cdot \mathbb{E}_k(\tau, \mathbf{m}, \epsilon_1, \epsilon_2) = I_k^{[a]}(\mathbb{E}_{<k})$$

- $\mathbb{E}_k(\tau, \mathbf{m}, \epsilon_1, \epsilon_2)$ can be solved if there are three such equations.

Results

- Recursion formula

$$\mathbb{E}_k(\tau,\mathbf{m},\epsilon_1,\epsilon_2)=\sum_{k_0=\frac{1}{2}||\alpha^\vee||^2,k_{1,2}<k}^{k_0+k_1+k_2=k}(-1)^{|\alpha^\vee|}\frac{D^{\alpha^\vee}_{\{k_0,k_1,k_2\}}}{D_k}A_{\alpha^\vee}(\mathbf{m})$$
$$\times\,\mathbb{E}_{k_1}(\tau,\mathbf{m}-\epsilon_1\alpha^\vee,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{k_2}(\tau,\mathbf{m}-\epsilon_2\alpha^\vee,\epsilon_1-\epsilon_2,\epsilon_2)$$

where $D^{\alpha^\vee}_{\{k_0,k_1,k_2\}}, D_k$ are polynomials of $\theta_i^{[a]}$.

- Complete solution of $\mathbb{E}_k$ for all the $\mathfrak{n}=3,4,5,6,8,12$ geometries!
- Complete solution of refined BPS invariants for these geometries!

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Recursion formulas

One-string elliptic genus

$$\mathbb{E}_1 = \sum_{\alpha\in\Delta_l} \frac{D^{\alpha}_{\{1,0,0\}}}{D_1} \frac{\eta^4}{\theta_1(m_\alpha)\theta_1(m_\alpha-\epsilon_1)\theta_1(m_\alpha-\epsilon_2)\theta_1(m_\alpha-\epsilon_1-\epsilon_2)} \prod_{\substack{\beta\in\Delta\\ \alpha\cdot\beta=1}} \frac{\eta}{\theta_1(m_\beta)}$$

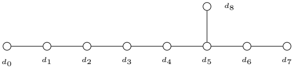
where $D^{\alpha}_{\{1,0,0\}} \propto \theta_1(m_\alpha-\epsilon_1)\theta_1(m_\alpha-\epsilon_2)$ and $D_1 \propto \theta_1(\epsilon_1)\theta_1(\epsilon_2)$. In the limit $\tau \to \mathrm{i}\infty$ (shrinking $S^1 \in T^2$), it reduces to the universal Z_1 formula for 5d SYM [Keller-Mekareeya-Song-Tachikawa,'11][Keller-Song,'12]

$$Z_1 = \frac{1}{(1-\mathrm{e}^{-\epsilon_1})(1-\mathrm{e}^{-\epsilon_2})} \sum_{\alpha\in\Delta_l} \frac{\mathrm{e}^{(h^\vee_G-1)m_\alpha/2}}{(1-\mathrm{e}^{-\epsilon_1-\epsilon_2+m_\alpha})(\mathrm{e}^{m_\alpha/2}-\mathrm{e}^{-m_\alpha/2})\prod_{\beta\cdot\alpha=1}(\mathrm{e}^{m_\beta/2}-\mathrm{e}^{-m_\beta/2})}$$

Refined BPS invariants

- Checks:
  - Reproduce BPS invariants of $\mathfrak{n} = 3, 4$ models.
  - Reproduce unrefined genus 0 GV invariants of all models.
  - Checkerboard pattern in all models manifest.
- Some BPS invariants of $\mathfrak{n} = 12$ theory with $G = E_8$

base d_b, fiber



$\mathbf{d} = (d_b, d_I)$ $(I = 0, 1, \ldots, 8)$	$\oplus N^{\mathbf{d}}_{j_L,j_R} \cdot (j_L, j_R)$
(1,0,0,0,0,0,0,0,0,0)	$(0, 1/2)$
(1,0,0,0,0,0,0,0,0,1)	$(0, 1/2)$
(1,0,0,0,0,0,1,0,0,1)	$(0, 1/2) \oplus (0, 3/2)$
(1,0,0,0,0,0,1,1,0,1)	$2(0, 1/2) \oplus (0, 3/2)$
(1,0,0,0,0,0,2,1,0,1)	$(0, 1/2) \oplus 2(0, 3/2) \oplus (0, 5/2)$
(2,0,0,0,0,3,0,0,0,0)	$(0, 5/2)$
(2,0,0,0,1,3,0,0,0,0)	$(0, 3/2) \oplus (0, 5/2)$
(2,0,0,0,0,4,0,0,0,0)	$(0, 5/2) \oplus (0, 7/2) \oplus (1/2, 4)$

Bonus: Schur indices of 4d SCFTs

- Non-critical strings in 6d SCFTs arise from D3-branes which wrap base $\mathbb{P}^1$ and which probe exotic 7-branes.
- k D3-branes probing exotic 7-branes of type G carry 4d $N = 2$ SCFT $H_G^{(k)}$ in worldvolume.

7-brane	IV	I_0^*	IV^*	III^*	II^*
D3-probe	(A_1, D_4) AD	$SU(2)\, N_f = 4$	MN E_6	MN E_7	MN E_8
$H_G^{(1)}$	$H^1_{SU(3)}$	$H^1_{SO(8)}$	$H^1_{E_6}$	$H^1_{E_7}$	$H^1_{E_8}$

Schur indices of 4d SCFT $H_G^{(k)}$ can be derived from $\mathbb{E}_k$ of 6d SCFT with gauge group G. [del Zotto-Lockhart,'18][JG-Klemm-Sun-Wang,'19]

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Conclusion

- A universal computational method for refined BPS invariants.
  - Applicable to a wide range of non-compact Calabi-Yau threefolds.
  - Requiring only semi-classical data (and some additional data) as input.
- Refined BPS invariants computed for minimal non-compact elliptic Calabi-Yau threefolds.

Outlook

- NHC $(-3,-2),(-3,-2,-2),(-2,-3,-2)$.
- Nonminimal elliptic Calabi-Yau threefolds.
- Little string theories.
- A proof of blowup equations?
- Compact Calabi-Yau threefolds?