Gromov-Witten-Floer theory and mirror symmetry

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– based on joints works with K. Fukaya, H. Ohta & K. Ono –

Outline

2*d* Open-closed TQFT

According to Lazaroiu, 2*d* open/closed TQFT is equivalent to the following data:

1. A: commutative Frobenius algebra —'closed sector'—
2. $\mathcal{C}$: Calabi-Yau category, i.e., —'open sector'—
3. Existence of bulk/boundary map

$$A \xrightarrow{\iota_{\mathcal{E}}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{E},\mathcal{E}), \quad \mathrm{Hom}_{\mathcal{C}}(\mathcal{E},\mathcal{E}) \xrightarrow{\iota^{\mathcal{E}}} A \quad \forall \mathcal{E} \in Ob(\mathcal{C})$$

   satisfying
   - $\mathrm{Im}\, \iota_{\mathcal{E}} \subset Z(\mathrm{Hom}_{\mathbb{C}}(\mathcal{E},\mathcal{E}))$
   - $\langle \iota_{\mathcal{E}}(a), \alpha \rangle_{\mathcal{C}} = \left\langle a, \iota^{\mathcal{E}}(\alpha) \right\rangle_A \quad \forall a \in A,\, \alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{E},\mathcal{E}).$
4. $\mathrm{tr}(\beta^m \alpha) = \langle \iota^{\mathcal{E}}(\alpha), \iota^{\mathcal{F}}(\beta) \rangle_A, \quad \alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{E},\mathcal{E}), \beta \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{F},\mathcal{F}),$

$$\beta^m \alpha := \beta \circ (\cdot) \circ \alpha : \mathrm{Hom}_{\mathcal{C}}(\mathcal{E},\mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{E},\mathcal{F}).$$

A-model: Gromov-Witten-Floer theory

The A-model on compact (X, ω) consists of

1. Open sector: The Fukaya category $\mathcal{C}_X = \mathcal{F}uk(X, \omega)$,
    - Objects: Lagrangian submanifold L
    - Morphisms: Floer cohomology $HF(L, K)$
    - Products: product $\mathfrak{m}_2$ (or $\mathfrak{m}_2^{\mathbf{b},b}$) and higher products $\mathfrak{m}_k$ (or $\mathfrak{m}_k^{\mathbf{b},b}$)
    - Pairing: Poincare pairing satisfying cyclicity

$$\langle \mathfrak{m}_k(x_1, \cdots, x_k), x_0 \rangle = (-1)^{|x_0|'(\sum_{i=1}^{k} |x_i|')} \langle \mathfrak{m}_k(x_0, \cdots, x_{k-1}), x_k \rangle$$

2. Closed sector: $QH^*(X, \omega)$ (or $A_X = HH^*(\mathcal{C}_X)$)
    - Product: quantum product $\cup$ or $\cup^{\mathbf{b}}$
    - pairing: Poincare pairing
3. Boundary/bulk maps (or open/closed maps)
    - Bulk-boundary map: $\widehat{\mathfrak{q}} : QH^*(X) \to HH^*(\mathcal{C}_X)$,
    - Boundary-bulk map: $\widehat{\mathfrak{p}} : HH_*(\mathcal{C}_X) \to QH^*(X)$.

Landau-Ginzburg B-model: matrix factorization

A B-model on Calabi Yau (noncompact) manifold Y associated with a Landau-Ginzburg potential consists of a holomorphic function W on Y

1. Open sector: $\mathcal{C}_Y = MF(W)$
   - Objects: a pair $\mathcal{E} = (E, Q)$ with $E = E^0 \oplus E^1$, $Q : E \to E$ satisfying $Q^2 = W \cdot Id$
   - Morphisms: $\mathbb{Z}/2$-graded morphisms $\Phi : \mathcal{E} \to \mathcal{F}$ respecting factorization
   - Products: $\mathrm{Hom}_{\mathcal{C}_Y}(\mathcal{E}, \mathcal{F}) \otimes \mathrm{Hom}_{\mathcal{C}_Y}(\mathcal{F}, \mathcal{G}) \mapsto \mathrm{Hom}_{\mathcal{C}_Y}(\mathcal{E}, \mathcal{G})$
   - Pairing: When E is trivial bundle, the paring can be written as

$$\langle f, g \rangle_{KL} = \frac{1}{(2\pi i)^n n!} \oint \frac{Tr(fg(dQ)^{\wedge n})}{\partial_{x_1} W \partial_{x_2} W \cdots \partial_{x_n} W}$$

2. Closed sector: $\mathrm{Jac}(W_Y)$ (or $A_Y = HH^*(D^b(\mathrm{Coh}(Y)))$
   - Product: natural product on $HH^*(D^b(\mathrm{Coh}(Y))$ (or $\mathrm{Jac}(W_Y)$)
   - Pairing: Serre duality induced pairing $\langle \cdots, \cdots \rangle$
3. Boundary/bulk maps or open/closed maps
   - Bulk-boundary map: $\iota_{\mathcal{E}} : A_Y \to \mathrm{Hom}_{\mathcal{C}_Y}(\mathcal{E}, \mathcal{E})\ a \mapsto \circ a \circ_h \mathrm{id}_{\mathcal{E}})$
   - Boundary-bulk map: $\iota^{\mathcal{E}} : \mathrm{Hom}_{\mathcal{C}_Y}(\mathcal{E}, \mathcal{E}) \to A_Y$

Boundary deformations

Let (X,ω) be closed symplectic manifolds and $\mathcal{F}_X = \mathcal{F}uk(X,\omega)$. The A_∞ Mauer-Cartan equation is

$$\sum_{i=0} \mathfrak{m}_i(b,\cdots,b) \equiv 0 \mod \Lambda_0\langle \mathbf{e}\rangle$$

for $b \in HF^{\text{odd}}(L;\Lambda_+)$. Denote by $\widetilde{\mathfrak{M}}_{\text{weak}}(L)$ the set of solutions and by $\mathfrak{M}_{\text{weak}}(L)$ the set of gauge equivalence class of solutions.

Definition

$L \subset X$ is called weakly unobstructed Lagrangian submanifold if $\mathfrak{M}_{weak}(L) \neq \emptyset$.

This enables us to define the FOOO-potential function

Definition

The potential function $\mathfrak{PO}_L : \mathfrak{M}_{weak}(L) \to \Lambda_0$ is given by

$$\mathfrak{PO}_L(b) = \sum_{i=0} \mathfrak{m}_i(b,\cdots,b)/\mathbf{e}.$$

1. Enhanced objets: $(L, b) \in Ob(\mathcal{F}_X)$, $b \in C^1(L; \Lambda_0)$ satisfying the A_∞ Mauer-Cartan equation
2. Deformed morphisms: $x \in CF^*((L, b_L), (K, b_K))$ (or $HF^*((L, b_L), (K, b_K))$ where

$$CF^*((L, b_L), (K, b_K)) = \begin{cases} \Lambda_0 \langle L \cap K \rangle & \text{for } L \pitchfork K \\ \Omega^*(L; \Lambda_0) & \text{for } K = L \end{cases}$$

3. Deformed products: For $b \in \mathfrak{M}_{\text{weak}}(L)$, the boundary deformed $\mathfrak{m}_k^b : CF(L, L)^{k\otimes} \to CF(L, L)$ is defined by

$$\mathfrak{m}_k^b(x_1, x_2, \cdots, x_k) = \sum \mathfrak{m}_*(b, \cdots, b, x_1, b, \cdots, b, x_k, b, \cdots, b).$$

The deformed $\mathfrak{m}^b = \{\mathfrak{m}_k^b\}$ still satisfies A_∞-relation and $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$ in addition.

Bulk deformations

Let $\mathbf{b} \in H^{\text{even}}(X, \Lambda_+)$. Consider the bulk-boundary (or closed-open) map

$$q^L_{\ell,k} : E_\ell(H(X)) \otimes B_k(H^*(L) \to H^*(L)$$

using the moduli space $\mathcal{M}_{k+1,\ell}(\beta)$ and setting

$$q^L_{k,\ell;\beta}(y_1, \cdots, y_\ell; x_1, \cdots, x_k) = (ev_0)_!(ev^+_\ell \times ev_+)^*(\mathbf{y} \times \mathbf{x})$$

where $\mathbf{y} = (y_1, \cdots, y_\ell)$, $\mathbf{x} = (x_1, \cdots, x_k)$ and $q^L_{k,\ell} = \sum_{\beta \in \Pi} q_{\ell,\ell;\beta} T^{\omega(\beta)}$.

1. Closed sector: When $k = -1$, this deforms $QH^*(X)$ and define the big quantum cohomology as a Frobenius manifold $(QH^*_{\mathbf{b}}(X), \cup^{\mathbf{b}})$.
2. Open sector: Let L be any compact Lagrangian submanifold. We first deform $\mathfrak{m}^L$ by setting

$$\mathfrak{m}^{\mathbf{b}}_k = \sum_{\ell \geq 0, \beta} T^{\omega(\beta)} \mathfrak{q}_{\ell,k;\beta}(\mathbf{b}^{\otimes \ell}; x_1, \cdots, x_k).$$

Boundary deformation of toric fibers

Theorem (FOOO)

Let (X, ω) toric manifold, the moment projection $\pi : X \to \mathbb{R}^n$ and $P = \operatorname{Im} \pi$. Then

1. *all toric fibers $\pi^{-1}(u) := L_u$ for $u \in \operatorname{Int} P$ are weakly unobstructed (Cho-O). In particular $\mathfrak{M}_{weak}(L) \neq \emptyset$.*
2. *there exists a natural embedding $i_L : H^1(L; \Lambda_+) \hookrightarrow \mathfrak{M}_{weak}(L)$.*

Denote by $\mathfrak{PO}^u = \mathfrak{PO}_{L(u)}$. The function extends with the coefficient Λ_+ to Λ_0, by twisting non-unitary line bundle on $L(u)$ (Cho).
Fix an integral basis $\{\mathbf{e}_i\}_{i=1}^n$ of $H^1(L(u); \mathbb{Z})$, and x_i^u the associated coordinates of $H^1(L(u); \Lambda_+)$. The function $\mathfrak{PO}_{L(u)}$ factors through $y_i^u := e^{x_i^u} \in \Lambda^*$. Then $H^1(L; \Lambda^*) \cong (\Lambda^*)^n$. Denote
$W^u := \mathfrak{PO}^u \circ i_{L(u)} : H^1(L(u); \Lambda^*) \to \Lambda_0$. In coordinates, we have

$$W^u = W^u(y_1^u, \cdots, y_n^u) = \mathfrak{PO}^u \circ i_{L(u)}(x_1^u, \cdots, x_n^u).$$

Bulk/boundary deformations of open sector

Let $\mathbf{b} \in H^{\text{even}}(X;\Lambda_0)$ and represent it by a T^n-equivariant cycles. Denote by $\mathcal{A}(\mathbb{Z})$ the Chow group generated by the invariants cycles. Consider the deformed $\mathfrak{m}^{\mathbf{b}} = \{\mathfrak{m}^{\mathbf{b}}_k\}_{k=0}^{\infty}$, the associated Mauer-Cartan equation

$$\mathfrak{m}_0^{\mathbf{b},b} := \sum_{i=0} \mathfrak{m}_i^{\mathbf{b}}(b,\cdots,b) \equiv 0 \mod \Lambda_0\langle \mathbf{e}\rangle$$

and denote by $\mathfrak{M}_{\text{weak}}(L(u),\mathbf{b};\Lambda_0)$ the set of gauge equivalence class of solutions. Then we obtain a family of deformed potentials

$$\mathfrak{PO}^u : \widehat{\mathfrak{M}}_{\text{weak,def}}(L(u)) \to \Lambda_+, \quad \mathfrak{PO}^u(\mathbf{b},b) = \mathfrak{m}_0^{\mathbf{b},b}/\mathbf{e}_{L(u)}$$

where $\widehat{\mathfrak{M}}_{\text{weak,def}}(L(u)) := \sqcup_{\mathbf{b}} \widehat{\mathfrak{M}}_{\text{weak}}(L(u),\mathbf{b};\Lambda_0)$.

Definition (b-weakly unobstructed Lagrangian)

is a pair $((L(u),b)$ *with* $b \in \mathfrak{M}^{\mathbf{b}}_{weak}(L(u))$.

Definition ($\mathbf{b}$-deformed Fukay category)

$\mathcal{F}uk^{\mathbf{b}}(X, \omega)$ is an A_∞ category consisting of

1. *Objects: $(L(u), b)$ with $b \in \mathfrak{M}^{\mathbf{b}}_{weak}(L(u))$*
2. *Morphisms: $CF^{\mathbf{b}}((L_0, b_0), (L_1, b_1))$ or its cohomology*
3. *Products: For $\kappa_i := (L_i, b_i)$, $i = 0, \ldots, k$,*
   $\mathfrak{m}_k^{\mathbf{b};b} : CF^{\mathbf{b}}(\kappa_0, \kappa_1) \otimes \cdots \otimes CF^{\mathbf{b}}(\kappa_{k-1}, \kappa_k) \to CF^{\mathbf{b}}(\kappa_0, \kappa_k)$

Proposition

For each $u \in \operatorname{Int} P$, there exists a canonical inclusions

- $\mathcal{A}(\Lambda_+) \times H^1(L(u); \Lambda_+) \to \widehat{\mathfrak{M}}_{weak,def}(L(u))$
- $H^{even}(X, \Lambda_+) \times H^1(L(u); \Lambda_+) \to \mathfrak{M}_{weak,def}(L(u))$.

We now define a **b**-family of functions

$$\mathfrak{PO}^u : \bigsqcup_{\mathbf{b} \in \mathcal{A}(\Lambda_+)} \mathfrak{M}^{\mathbf{b}}_{\mathrm{weak}}(L(u)) \to \Lambda_0$$

Bulk deformation of Landau-Ginzburg potential

By restricting $\mathfrak{PO}^u$ to $H^*(X;\Lambda_+) \times H^*(L(u);\Lambda_0)$, we obtain

$$W^u : \mathcal{A}(\Lambda_+) \times H^1(L(u);\Lambda_+) \to \Lambda_0.$$

We fix a basis of $\mathcal{A}_{\mathbb{Z}}$ $\{[D_a]\}_{a=1}^B$ of toric divisors and denote the asscociated coordinate variables by $w_1, \ldots, w_B$. Trivialize the torus fibration $\Phi : \pi^{-1}(\mathrm{Int}\, P) \to \mathrm{Int}\, P \times T^n$ and induce an isomorphism $\psi_u : H^*(T^n, \mathbb{Z}) \to H^*(L(u), \mathbb{Z})$. We fix a basis $\{e_1, \cdots, e_n\}$ of $H^1(T^n; \mathbb{Z})$ and make the change of the variables

$$y_i = T^{-u_i} y_i^u \text{ or equivalently } y_i^u = T^{u_i} y_i$$

so that the new coordinates $\{y_1, \cdots, y_n\}$ satisfies $\mathfrak{v}^u(y_i) = 0$ for the valuation $\mathfrak{v}^u$: We set $\mathfrak{v}^u(T^\lambda w_j) := \lambda =: \mathfrak{v}^u(T^\lambda y_i)$. Then we have

$$u = (\mathfrak{v}^u(y_1), \cdots, \mathfrak{v}^u(y_n)).$$

Construction of mirror

We obtain a function $\widetilde{W} : \mathcal{A}(\Lambda_+) \times H^1(T^n; \Lambda_+) \to \Lambda_0$ which descends to (and extends to)

$$W : H^*(x; \Lambda_+) \times H^1(T^n; \Lambda^*) \cong H^*(x; \Lambda_+) \times (\Lambda^*)^n \to \Lambda_0,$$

with $W = W(w_1, \ldots, w_B; y_1, \ldots, y_n)$.

Then $W_{\mathbf{b}} := W(\mathbf{b}; \cdot) : (\Lambda^*)^n \to \Lambda_0$ defines a deformation family of potential functions defined on $(\Lambda^*)^n$ parameterized by $\mathbf{b} \in H^*(X, \Lambda_+)$.

(*The $\mathbb{C}$-reduction to $(\mathbb{C}^*)^n$ of $W = W_0$ is the physicists' Landau-Ginzburg potential function (or the refined version of Hori-Vafa potential).*)

By now, we have constructed a mirror $((\Lambda^*)^n, W_{\mathbf{b}})$ of toric A-model using the open sector $\mathcal{F}uk^{\mathbf{b}}(X, \omega)$. In particular, the logarithmic derivative

$$y_i \frac{\partial W_{\mathbf{b}}}{\partial y_i} = \frac{\partial W_{\mathbf{b}}}{\partial x_i}$$

reflects information on Floer cohomology of the fiber of toric fibration.

Kodaira-Spencer map $\mathfrak{ks}_{\mathbf{b}}$

Recall the coordinate functions $w_1, \ldots, w_B$ of $\mathcal{A}(\Lambda_+)$ with respect to the basis $\{D_a\}$. For given $\mathbf{b} \in H^*(X, \Lambda_+)$ with its representative $\mathbf{f} \in \mathcal{A}(\Lambda_+)$ $[\mathbf{f}] = \mathbf{b}$, consider the function

$$\widetilde{\mathfrak{ks}}_{\mathbf{f}} : T_{\mathbf{f}}\mathcal{A}(\Lambda_+) \to \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}}; \quad \frac{\partial}{\partial w_i} \mapsto \frac{\partial \widetilde{W}_{\mathbf{f}}}{\partial w_i}$$

We define a Jacobian ring by

$$\mathrm{Jac}(W_{\mathbf{b}}) = \frac{\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}}}{\mathrm{Clos}_{d_{\overset{\circ}{P}}}\left(y_i \frac{\partial \mathfrak{PO}_{\mathbf{b}}}{\partial y_i} : i = 1, \ldots, n\right)}$$

where

$$\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}} := \bigcap_{u \in \mathrm{Int}\, P} \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{u}$$

Kodaira-Spencer map and Frobenuius manifold

Theorem (FOOO)

Equip $H(X;\Lambda_0)$ with the $\mathbf{b}$-deformed quantum product $\cup^{\mathbf{b}}$.

1. *The map $\widetilde{\mathfrak{ks}_{\mathfrak{f}}}$ induces a ring isomorphism*

$$\mathfrak{ks}_{\mathbf{b}} : H(X,\Lambda_0) \cong T_{\mathbf{b}} H(X,\Lambda_0) \to Jac(W_{\mathbf{b}}).$$

2. *If $\mathfrak{PO}_{\mathbf{b}}$ is a Morse function, then $\mathfrak{ks}_{\mathbf{b}}$ intertwines the Poincaré pairing on $H^*(X,\Lambda)$ and the residue pairing on $Jac(W_{\mathbf{b}}) \otimes_{\Lambda_0} \Lambda$.*

We call the map $\mathfrak{ks}_{\mathbf{b}} : H(X,\Lambda_0) \to \mathrm{Jac}(W_{\mathbf{b}})$ the Kodaira-Spencer map.

Remark

For the case of Fano X and $\mathbf{b} = 0$, similar result was known by Batyrev (by direct calculation) and Givental by his mirror theorem on toric completion. Our proof is more conceptual and involves open-closed Gromov-Witten-Floer theory.

Boundary-bulk map and generation criterion

Boundary-bulk map $\widehat{p} : HH_*(\mathcal{F}_X) \to \Omega^*(X)$ is induced by the collection of maps $\{p_k\}$ where when $\mathcal{F}_X = \{(L, b)\}$, $p_k : B_k^{\text{cyc}}\Omega^*(L) \to \Omega^*(X)$ is defined by setting

$$p_k(\alpha_1, \cdots, \alpha_k) := (ev_{\text{int}})_!(ev_\partial^*(\alpha_1 \wedge \cdots \alpha_k))$$

with the moduli space $\mathcal{M}_{k,1}(L, \beta)$ and its associated evaluation maps. This map has the natural categorical version defined for any finite collection $\mathbf{L} = \{(L_i, b_i)\}_{i \in I}$ with

$$CH_*(\mathcal{L}) = \oplus_{\vec{i}} B_{\vec{i}} CF(\mathcal{L})$$

and

$$B_{\vec{i}} CF(\mathcal{L}) := \oplus_{\vec{i}} CF(i_0, i_1) \otimes \cdots \otimes CF(i_{k-1}, i_k)$$

with $\vec{i} = (i_0, \ldots, i_k)$ and $i_\ell = (L_{i_\ell}, b_{i_\ell})$, where $\mathcal{L}$ is the A_∞ category split-generated by $\mathbf{L}$. The A_∞-map $\mathfrak{m}_k$ defines the bar differential on $CH_*(\mathcal{L})$ and so Hochschild homology $HH_*(\mathcal{L}, \mathcal{L})$.

Then the maps $\widehat{q}$, $\widehat{p}$ are dual to each other in that

$$\langle \widehat{p}(\mathbf{h}), \alpha \rangle_{\mathrm{PD}} = \langle \mathbf{h}, \widehat{q}(\alpha) \rangle_{\mathrm{HH}}$$

for $\mathbf{h} \in CH_*(\mathcal{L})$. Combining the property that Fukaya category $\mathcal{F}uk(X, \omega)$ is a Calabi-Yau category, we obtain

Proposition (AFOOO)

- *The map $\widehat{q}$ is a ring homomorphism and $\widehat{p}$ is a module homomorphism.*
- *They satisfy Cardy relation, i.e., $\langle \widehat{p}(\mathbf{x}), \widehat{p}(\mathbf{y}) \rangle_{PD} = Z(\mathbf{x}, \mathbf{y})$ where $Z : CH_*(\mathcal{L}) \times CH^*(\mathcal{L}) \to \Lambda_0$ is the trace map.*

Consider a finite collection **L** whose elements are pairs (L, b) for which

$$[b] \in \mathcal{M}_{\text{weak}}(L; \mathfrak{b}; \Lambda_0) \tag{1}$$

is a weak bounding cochain. Let $\mathcal{L}$ be the cyclic filtered A_∞ category over the Novikov field generated by the collection.

Theorem (Generation criterion)

If id_X lies in the image of $\widehat{\mathfrak{p}}^{\mathbf{b}} : HH_*(\mathcal{L}, \mathcal{L}) \to QH^*_{\mathfrak{b}}(X; \Lambda)$, then $\widehat{\mathfrak{p}}^{\mathbf{b}}$ is an isomorphism.

Theorem (AFOOO)

Let (X, ω) *is a compact toric manifold. Consider* $\mathrm{st} = 0$ *and* $\mathfrak{b} \in H^*(X; \Lambda_0^{\mathbb{C}})$. *Then take the set* $\mathcal{L}$ *of all pairs* (L, b) *where* L *is a torus orbit of* X *and* $b \in H^1(L; \Lambda_0^{\mathbb{C}})/2\pi\sqrt{-1}H^1(L; \mathbb{Z})$ *such that* $HF((L, b), (L, b); \Lambda) \cong H^*(L; \Lambda)$. *Then the map*

$$\widehat{\mathfrak{p}}^{\mathbf{b}} : HH_*(\mathcal{L}, \mathcal{L}) \to QH_{\mathfrak{b}}^*(X; \Lambda)$$

is surjective. In particular, the collection $\mathbf{L}$ *split-generates* $\mathcal{F}uk(X, \omega)$.

In this toric case, we can take $\mathbf{b} = 0$ and the the collection $\mathbf{L}$ to be

$$\mathbf{L} = \mathrm{Crit}\, W_{\mathbf{0}}$$

which can be explicitly described in terms of the moment polytope.

Open questions

- Extend our theorem to the level of mirror symmetry of higher residue pairing for the versal family $((\Lambda^*)^n, W_{\mathbf{b}})$ with $\mathbf{b} \in H^*(X, \Lambda_+)$.
- Investigate how the story changes for the general Fano manifolds, such as partial flag manifolds. Some progresses have been made by Nishino-Nohara-Ueda and Cho-Kim -O by considering the integrable system called Gelfand-Cetlin system.

Thank you for your attention!