Laguerre Gaussian Mode Decomposing

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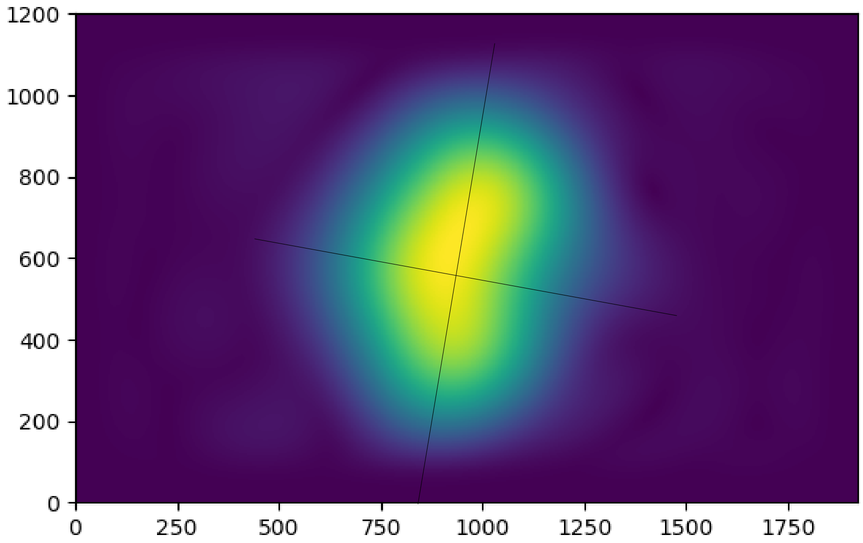
Goal

- Gober's model works only azimuthally symmetric
- Reshape the nearfield to be azimuthally symmetric
- Decompose the line out signal to variouse Laguerre Gaussian modes, and calculate each coefficients



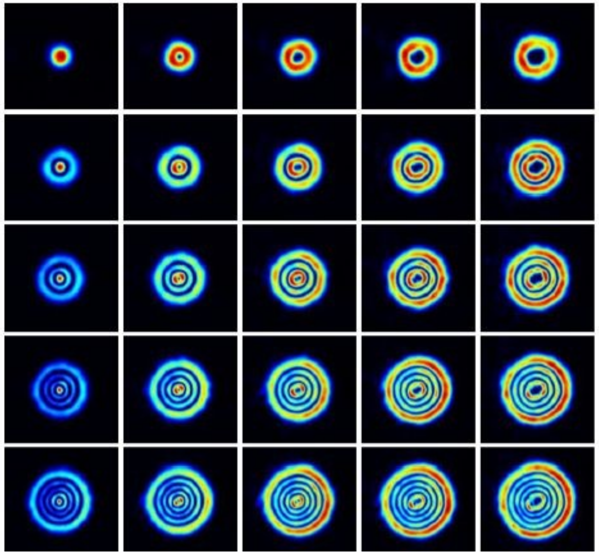
Near field intensity
Major axis and minor axis

Laguerre Gaussian modes

- Laguerre Gaussian solution

$$u(r,\phi) = \left(\sqrt{2}\, r\right)^{|l|} \exp(-r^2)\, L_p^{|l|}(2r^2) \exp(i\, l\, \phi) \exp\left(\frac{i\, r^2}{2}\right) \exp(i\,(2p + |l| + 1))$$

- Laguerre Polynomial $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) = \frac{1}{n!}\left(\frac{d}{dx} - 1\right)^n x^n,$

- 

Solutions to the paraxial wave equation in polar coordinates by separation of variables

Note that when l=0, phi vanish.
→ azimuthal symmetric

mode key **(p,l)**

(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)
(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)
(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)
(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)
(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Start from Discrete Fourier

- In our case,
  - x= x[r]
  - X= X[p]
  - W = u[r, p]
- x= WX
- W^-1 x =X

only N unique values. The summation needs only N terms to utilize all the unique values of X[k]. The formula for the inverse DFT is most commonly written as

$$\mathrm{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{X}[k] e^{j2\pi kn/N}$$

but, since X[k] is periodic with period N, it can be written more generally as

$$\mathrm{x}[n] = \frac{1}{N} \sum_{k=\langle N \rangle} \mathrm{X}[k] e^{j2\pi kn/N}.$$

ORTHOGONALITY AND THE HARMONIC FUNCTION

We can find the forward DFT X[k] of x[n] by a process analogous to the one used for the CTFS. To streamline the notation in the equations to follow let

$$W_N = e^{j2\pi/N} \tag{7.1}$$

Since the beginning point of the summation $\sum_{k=\langle N \rangle} \mathrm{X}[k] e^{j2\pi kn/N}$ is arbitrary let it be $k = 0$. Then, if we write $e^{j2\pi kn/N}$ for each n in $n_0 \le n < n_0 + N$, using (7.1) we can write the matrix equation

$$\underbrace{\begin{bmatrix} \mathrm{x}[n_0] \\ \mathrm{x}[n_0+1] \\ \vdots \\ \mathrm{x}[n_0+N-1] \end{bmatrix}}_{\mathbf{x}} = \frac{1}{N} \underbrace{\begin{bmatrix} W_N^0 & W_N^{n_0} & \cdots & W_N^{n_0(N-1)} \\ W_N^0 & W_N^{n_0+1} & \cdots & W_N^{(n_0+1)(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{n_0+N-1} & \cdots & W_N^{(n_0+N-1)(N-1)} \end{bmatrix}}_{\mathbf{W}} \underbrace{\begin{bmatrix} \mathrm{X}[0] \\ \mathrm{X}[1] \\ \vdots \\ \mathrm{X}[N-1] \end{bmatrix}}_{\mathbf{X}} \tag{7.2}$$

or in the compact form $N\mathbf{x} = \mathbf{WX}$. If $\mathbf{W}$ is nonsingular, we can directly find $\mathbf{X}$ as $\mathbf{X} = \mathbf{W}^{-1} N\mathbf{x}$. Equation (7.2) can also be written in the form

Construct transform matrix W

Size= number of modes

- W= [[W^r=0_p=0 W^r=0_p=1 … W^r=0_p=N]
  [W^r=1_p=0 W^r=1_p=1 … W^r=1_p=N]
  …
  [W^r=N_p=0 W^r=N_p=1 … W^r=N_p=N]]

Size= number of data points

Issues

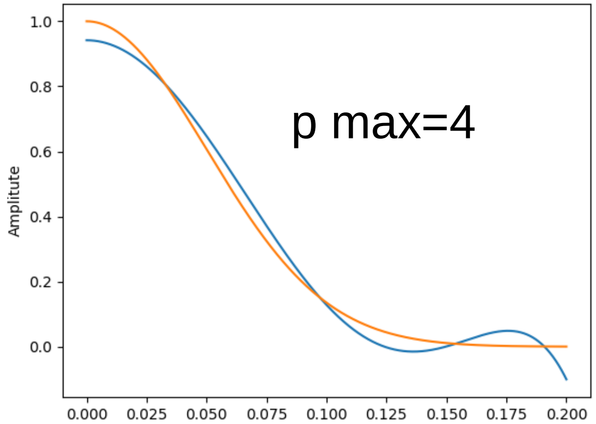
- If we contain W as a squared matrix, there are some issues when the matrix size goes larger.
- Recall the LG solution, there is a term in Laguerre polynomial is (d/dx)^p.
- Numerically if you take a large enough order derivative of something, you gotta get 0.
- Then you get a bunch of zeros in the matrix, so det(W)=0. Then W^-1 cannot be calculated.

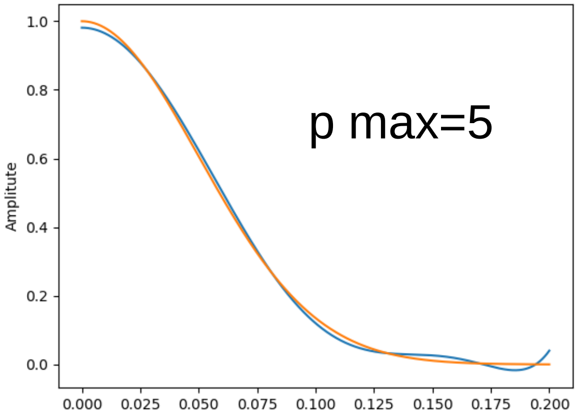
- Keep a large number of steps (data points, at least 1000), while only a small number of p. → Non-squared W

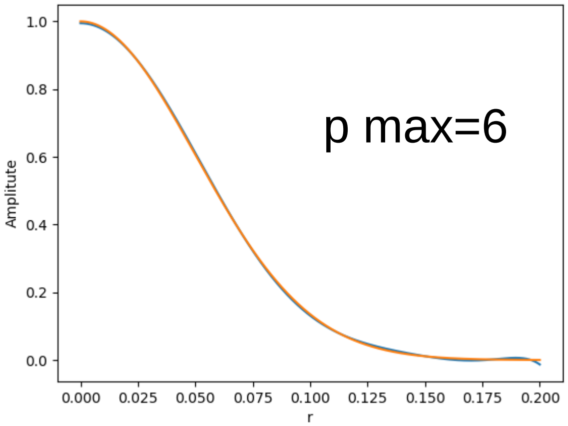
- To compute left inverse need to compute the pseudo-inverse of a matrix, W+, by singular-value decomposition (SVD), essentially the leastsq.
- If the the length is too large (length>10), SVD does not converge, so the pseudo-inverse will not form.
- If the length is too small (legnth<0.1), the pseudo-inverse will have a large numerical error.
- Numerical error: calculate WW+. In principle, WW+ should equal to unit matrix, the elements in cross line are 1, and others are 0. When the numerically error appears, the elements in the cross line decrease and others increase.

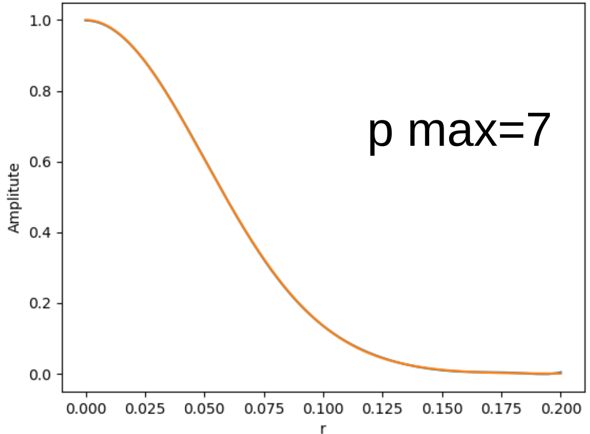
Signal Reconstruction

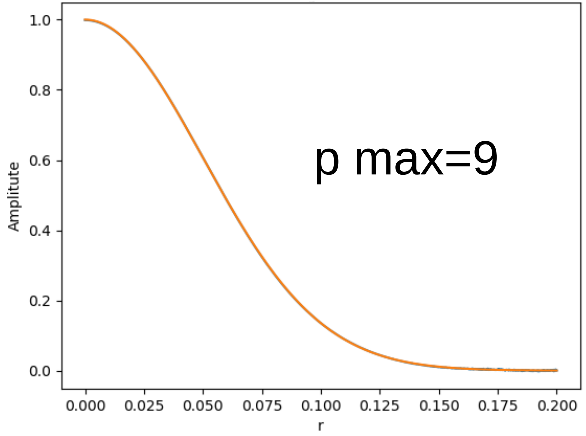
- The orange one is the original test signal, the blue one is the reconstructed signal.









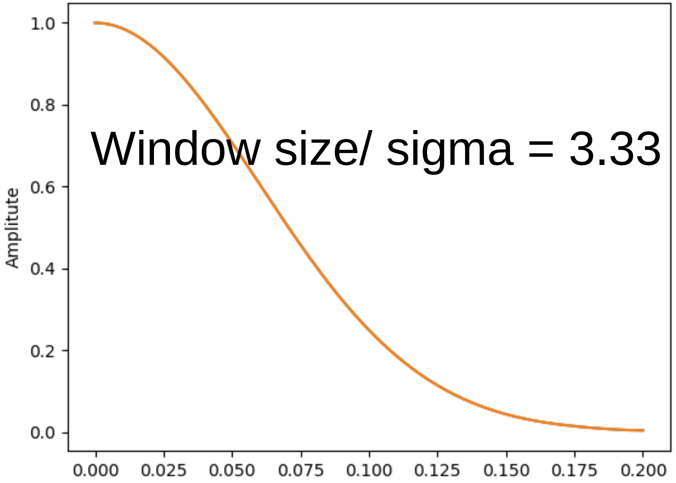


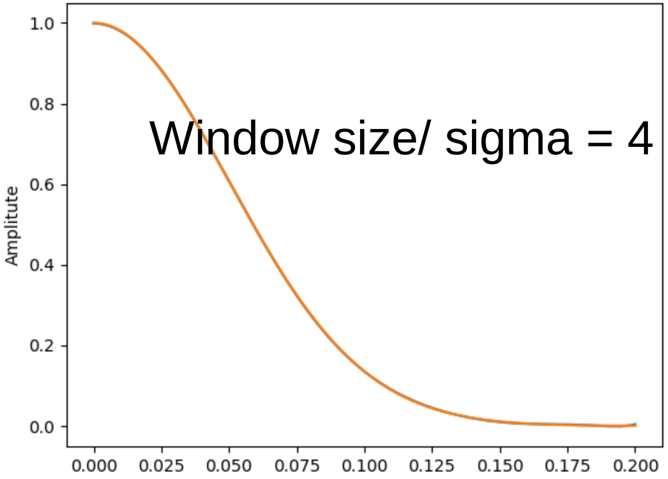
Including more than 7 modes gets very well reconstructed results

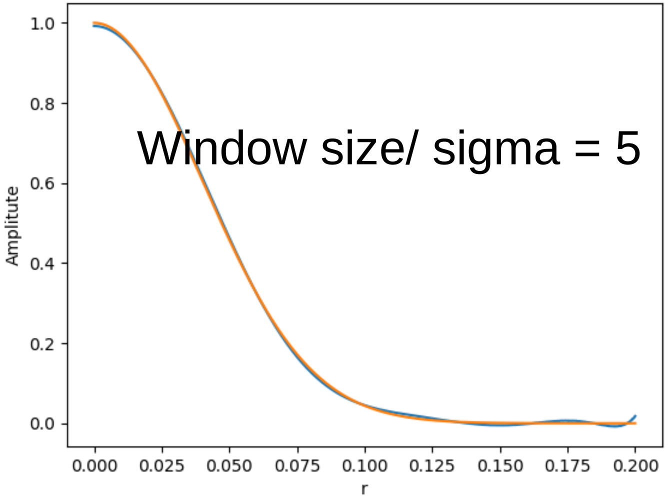
Window size/ signal width Ratio

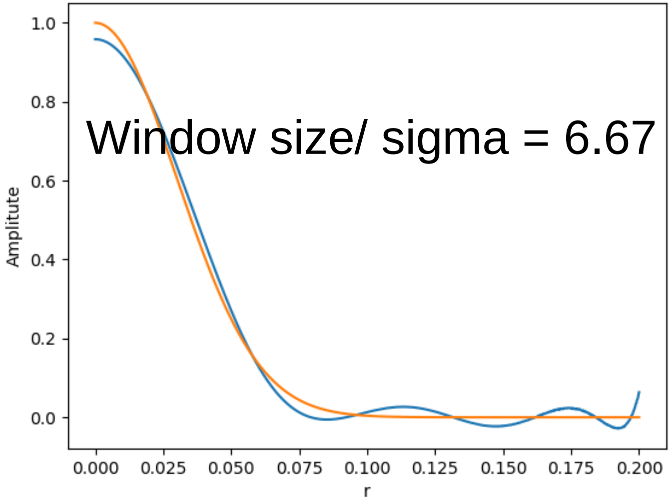
- If the width of the signal is too small compare to the window size, (when window size/ sigma >5) the numerical error dramatically increases.
- Because when the window size is too large, there will be many zeors. When doing W+ dot x[r], essentially many information in W+ are throwing away, due to the numerous zeros in x[r]. This is why the coefficients error increase, which causes the reconstructed signal error increases.

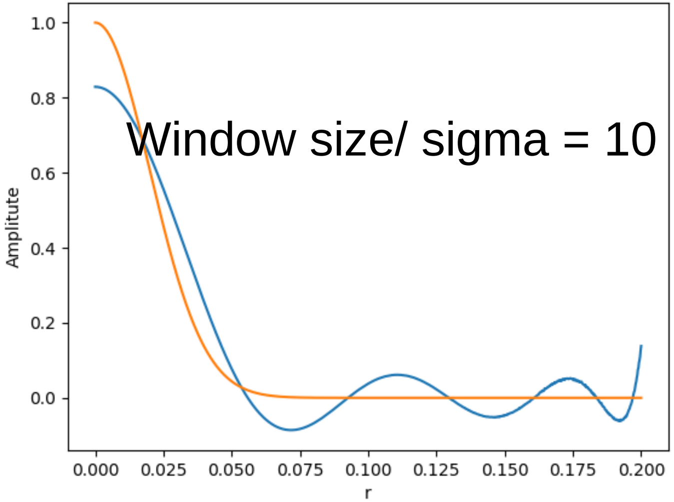
Window size/ signal width Ratio











Increasing window size/ signal width Ration increases the numerical error

r length

- Same w/s ratio, when r length is smaller, the error larger.
- Like the explanation earlier, smaller r length increases the pseudo-inverse error. → increases the reconstruction error