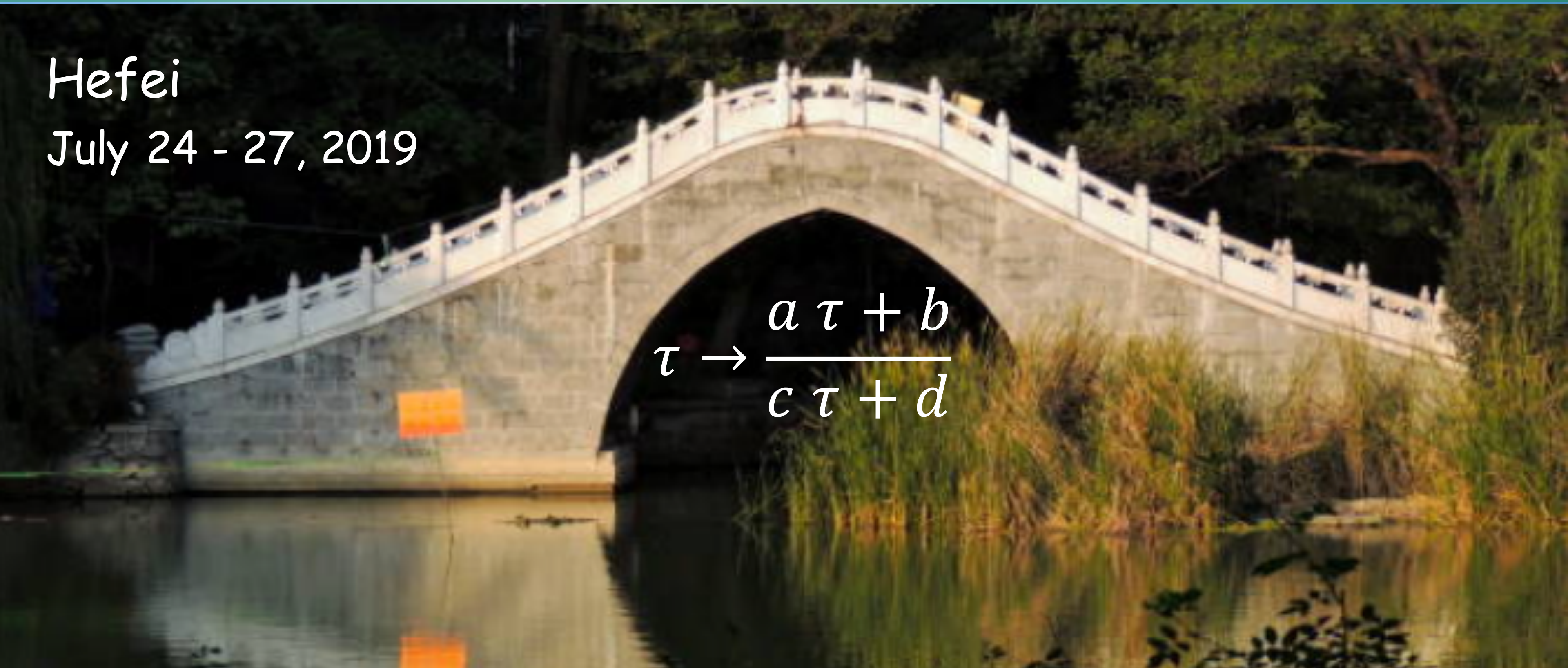
Neutrino Masses and Modular Invariance



Hefei

July 24 - 27, 2019

$$\tau \to \frac{a\,\tau + b}{c\,\tau + d}$$

Flasy 2019

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Based on:
- F.F. 1706.08749
- Juan Carlos Criado, F.F., 1807.01125
- Juan Carlos Criado, F.F, S.J.D King, in preparation

Precision Era for Neutrino Physics

	IO	NO	
$r \equiv \lvert\Delta m^2_{sol}/\Delta m^2_{atm}\rvert$	$0.0301(8)$	$0.0299(8)$	2.7%
$\sin^2\theta_{12}$	$0.303(13)$	$0.304(13)$	4.3%
$\sin^2\theta_{13}$	$0.0218(8)$	$0.0214(8)$	3.7%
$\sin^2\theta_{23}$	$0.56(3)$	$0.55(3)$	5.4%
δ/π	$1.52(14)$	$1.32(19)$	≈ 10%

[F. Capozzi, E. Lisi, A. Marrone and A. Palazzo 1804.09678]

independent global fits:
de Salas, Gariazzo, Mena, Ternes , Tortola, 1806.11051,
Gariazzo, Archidiacono, de Salas, Mena, Ternes, Tortola, 1801.04946
de Salas, Forero, Ternes, Tortola, J. W. F. Valle, 1708.01186
Esteban, Gonzalez-Garcia, Hernandez -Cabezudo, Maltoni and Schwetz 1811.05487

NO preferred over the IO

stimulating time for
for models of neutrino masses
and mixing angles.

$y_e(m_Z)$	$2.794745(16) \times 10^{-6}$
$y_\mu(m_Z)$	$5.899863(19) \times 10^{-4}$
$y_\tau(m_Z)$	$1.002950(91) \times 10^{-2}$

[Antusch and Maurer 1306.6879]

Flavour Symmetry approach

One of the few tools we have, but with several obstacles

reviews:
Ishimori, Kobayashi, Ohki, Shimizu, Okada, Tanimoto , 1003.3552;
King, Luhn, 1301.1340;
King, Merle, Morisi, Shimizu, Tanimoto, 1402.4271;
King, 1701.04413
Hagedorn, 1705.00684;
.
.
.

- high number of free parameters

lowest order Lagrangian parameters

higher dimensional operators

$$m_{ij}(\tau) = m_{ij}^{0} + m_{ij}^{1\alpha}\ \tau_\alpha + m_{ij}^{1\bar{\alpha}}\ \bar{\tau}_{\bar{\alpha}} + m_{ij}^{2\alpha\beta}\ \tau_\alpha \tau_\beta + \ldots$$

vacuum alignment in SB sector

SUSY breaking effects
RGE corrections
(Λ_{UV}, m_{SUSY} , $\tan\beta$)

This proposal [F.F. 1706.08749]

a) neutrino masses and mixings depend on a small number of fields [ideally a single complex field τ] ➡ $m_{ij}(\tau)$

b) dependence of m_{ij} on τ is holomorphic ➡ supersymmetric model

c) flavour symmetry acts non-linearly [to determine all higher dimensional operators] ➡ $\begin{cases} \tau \to F(\tau) \\ \varphi \to G(\tau,\varphi) \end{cases}$ non-linear

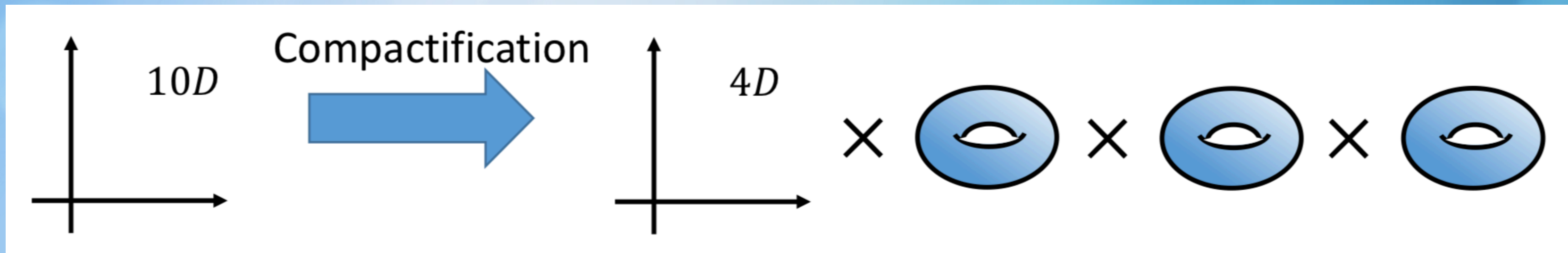
a) + b) + c) ➡ the functional form of $m_{ij}(\tau)$ is completely determined

d) the VEV τ is selected by some unknown mechanism [anarchy in vacuum selection]

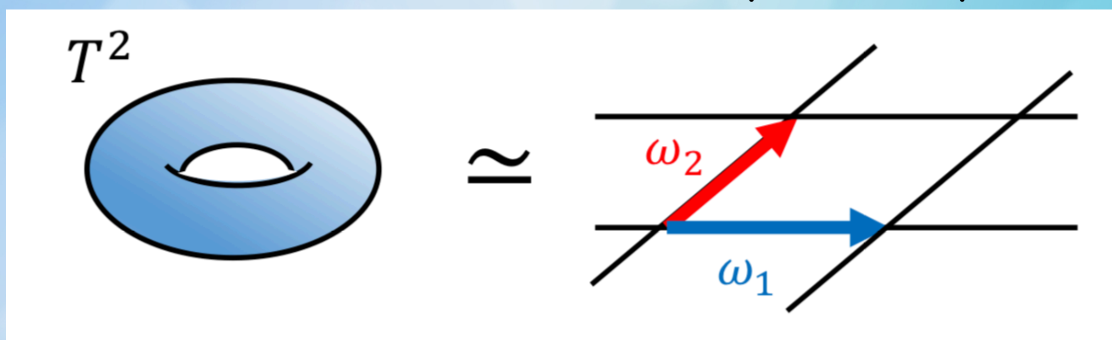
Here: a) + b) + c) from modular invariance as flavour symmetry

Modular Invariance as Flavour Symmetry

string theory in d=10 need 6 compact dimensions



simplest compactification: 3 copies of a torus T^2



completely characterized by

$$\tau = \frac{\omega_2}{\omega_1} \quad Im(\tau) > 0$$

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(g(\tau), \varphi)$$

lattice left invariant by modular transformations:

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

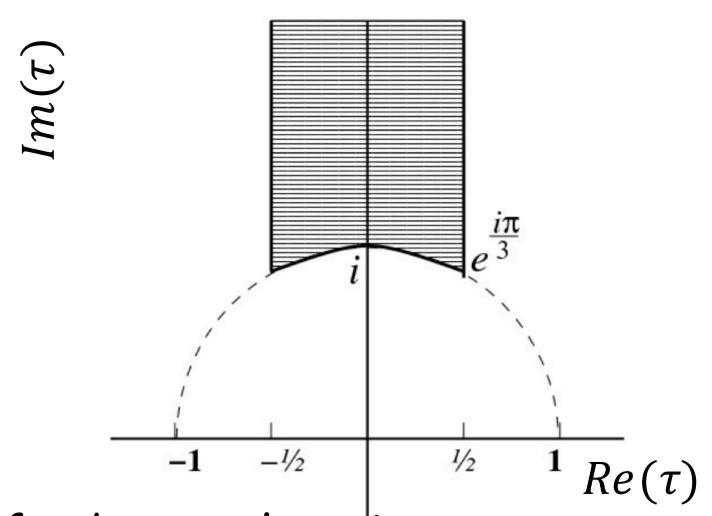
a,b,c,d integers
ad-bc=1

$\mathcal{L}_{eff}$ modular invariant

[figures from Tatsuishi's talk at Flasy 2018] they form the (discrete, infinite) modular group $\bar{\Gamma}$ generated by

$$S : \tau \to -\frac{1}{\tau} \quad , \qquad T : \tau \to \tau + 1$$

duality — discrete shift symmetry



$$S^2 = 1 \quad , \quad (ST)^3 = 1$$

- can be thought of as a gauge symmetry
- with a "gauge choice" τ can be restricted to a fundamental region

most general transformation on a set of $\mathcal{N}$=1 SUSY chiral multiplets $\varphi^{(I)}$

$$\begin{cases} \tau & \to \quad \gamma\tau \equiv \dfrac{a\tau + b}{c\tau + d} \\ \varphi^{(I)} & \to \quad (c\tau + d)^{k_I} \rho^{(I)}(\gamma) \varphi^{(I)} \end{cases}$$

e.g. $\varphi^{(I)} = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}$

[Ferrara, Lust, Shapere and Theisen, 1989]

the weight, a real number

unitary representation of the finite modular group

$$\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$$

$N = 1,2,3,\ldots$

$\mathcal{N}$=1 SUSY modular invariant theories

Yukawa interactions in $\mathcal{N}$=1 global SUSY [extension to $\mathcal{N}$=1 SUGRA straightforward]

$$S = \int d^4x d^2\theta\, w(\tau,\varphi) + h.c + kinetic\ terms$$

invariance satisfied by "minimal" Kahler potential

$$\boxed{w(\tau,\varphi) = \sum_n Y_{I_1 \dots I_n}(\tau)\varphi^{(I_1)} \dots \varphi^{(I_n)}}$$

field-dependent Yukawa couplings

invariance of $w(\Phi)$ guaranteed by an holomorphic $Y_{I_1 \dots I_n}(\tau)$ such that

$$Y_{I_1 \dots I_n}(\gamma\tau) = (c\tau + d)^{k_Y(n)} \rho(\gamma)\, Y_{I_1 \dots I_n}(\tau)$$

1. $k_Y(n) + k_{I_1} + \cdots + k_{I_n} = 0$
2. $\rho \times \rho^{I_1} \times \dots \times \rho^{I_n} \supset 1$

modular forms of level N and weight k_Y ⇒ form a linear space $\mathcal{M}_k(\Gamma_N)$ of finite dimension

Example

$\Gamma_3 \approx A_4$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \to (c\tau + d)^{-1} \rho(\gamma) \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$k_\nu = -1$

~ 3 of Γ_3

$$w(\tau, \nu) = m_0\, \nu\, Y(\tau)\, \nu + h.c.$$

modular form of level 3
$k = +2$ and $\rho \subset 3 + 1 + 1' + 1''$

$$d(\mathcal{M}_2(\Gamma_3)) = 3$$
$$\rho = 3$$

$$m(\tau) = m_0 \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

[F.F. 1706.08749]

mass matrix completely determined in terms of τ up to an overall constant

no corrections from higher order operators in the exact SUSY limit

Modular Invariance and CP

[Novichkov, Penedo, Petcov and Titov 1905.11970
see also: Baur, Nilles, Trautner and Vaudrevange,
1901.03251]

- a unique CP law consistent with the modular group [Im(τ) > 0]

$$\tau \to -\tau^*$$

[up to modulartransformations]

$\Updownarrow$

outer automorphism of $\bar{\Gamma}$

$$S \to S \quad T \to T^{-1}$$

- CP on matter multiplets

$$\varphi^{(I)} \to X_{CP}\, \varphi^{(I)*}$$

$X_{CP} = \mathbb{I}$ not restrictive if
S and T symmetric matrices
[canonical CP basis]

[in such a basis]
CP invariance $\Rightarrow$ $g_i^* = g_i$ on Lagrangian parameters g_i

- CP conserved $\leftrightarrow$ τ imaginary or at the border of the fundamental region

otherwise CP spontaneously broken by $\langle\tau\rangle$

Modular Forms

k > 0 even integer

	$d(\mathcal{M}_k(\Gamma_N))$	$k=2$	$k=4$	$k\geq 6$
$\Gamma_2 \approx S_3$	$k/2+1$	2	$1+2$	...
$\Gamma_3 \approx A_4$	$k+1$	3	$1+1'+3$	...
$\Gamma_4 \approx S_4$	$2k+1$	$2+3'$	$1+2+3+3'$	...
$\Gamma_5 \approx A_5$	$5k+1$	$3+3'+5$	$1+3+3'+4+5+5$	...

[TTT]

[F]

[PP]

[NPPT DKL]

$$\left.\begin{matrix}\Gamma_8 \supset \Delta(96) \\ \Gamma_{16} \supset \Delta(384)\end{matrix}\right\} \quad k=2 \quad \rho=3$$

[KT]

[TTT = T. Kobayashi, K. Tanaka and T. H. Tatsuishi, 1803.10391
F = F. Feruglio 1706.08749
PP = J. T. Penedo and S. T. Petcov 1806.11040
NPPT = P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov 1812.02158
DKL = G. J. Ding, S. F. King and X. G. Liu 1903.12588
KT = T. Kobayashi and S. Tamba, 1811.11384]

built in terms of
Dedekind eta function
Klein forms
Jacobi theta functions

k > 0 odd/even integer

fall in representations of homogeneous finite modular groups Γ'_N
e.g. N=3 and k=1 gives a doublet of $\Gamma'_3 = T'$
[Ding, Liu 1907.01488]

Selection of results

See talks by King, Novichkov, Penedo, Tanimoto, Titov, Zhou

- freedom in a bottom-up approach: $\Gamma_N, \rho^{(I)}, k_I$

- Majorana neutrinos

  [Dirac case explored in T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, 1808.03012]

- weight k = 2 in $Y(\tau)$ of neutrino sector (too many parameters if k > 2)

number of free parameters p:

- p = Total – 3 (charged fermion masses)

number of observables in neutrino sector = 9
(3 angles + 3 masses + 3 phases)

→ (9-p) predictions

$p \geq 3$ (always includes $Re(\tau), Im(\tau)$ and one overall scale)
Here $p > 5$ not considered

Example 1: $m_\nu(\tau)$ and $m_e(\tau)$

[P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, 1905.11970]

Γ_4

no additional flavons beyond τ

CP spontaneously broken by $\langle\tau\rangle$

1 extra parameters from Dirac neutrino mass

→ $p = 4$

$\mathrm{Re}\,\tau$	± 0.09922
$\mathrm{Im}\,\tau$	1.016
g'/g	-0.02093
$v_u^2\, g^2/\Lambda$ [eV]	0.0135

r	0.02981
δm^2 $[10^{-5}$ eV$^2]$	7.326
$\|\Delta m^2\|$ $[10^{-3}$ eV$^2]$	2.457
$\sin^2\theta_{12}$	0.305
$\sin^2\theta_{13}$	0.02136
$\sin^2\theta_{23}$	0.4862

Ordering	NO
m_1 [eV]	0.01211
m_2 [eV]	0.01483
m_3 [eV]	0.05139
$\sum_i m_i$ [eV]	0.07833
$\|\langle m\rangle\|$ [eV]	0.01201
δ/π	± 1.641
α_{21}/π	± 0.3464
α_{31}/π	± 1.254

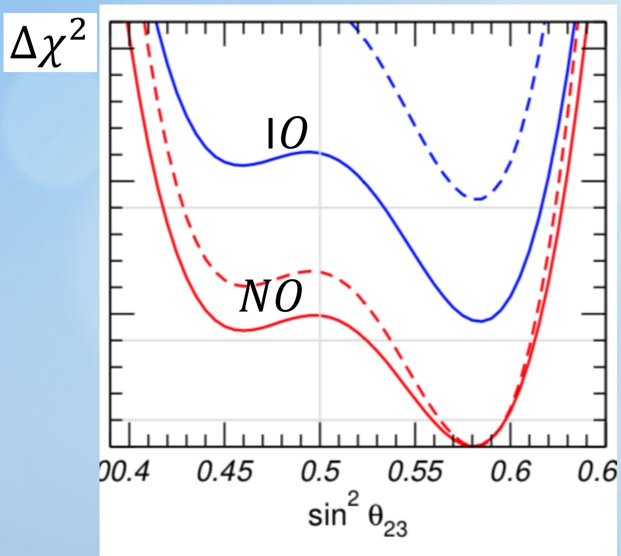
$N\sigma$	1.012

Example 2: $m_\nu(\tau)$ and $m_e(\varphi)$ [Juan Carlos Criado, F.F., 1807.01125]

Γ_3

charged lepton sector depending on additional flavon
$\varphi = (1,0,\varphi_3)$ $\varphi_3 \ll 1$ real
CP spontaneously broken by $\langle\tau\rangle$

$\Rightarrow$ $p = 4$



T. Schwetz @ Neutrino Telescopes, Venice, 19 March 2019

Seesaw -> NO	best value
$r \equiv \lvert\Delta m^2_{sol}/\Delta m^2_{atm}\rvert$	$0.0299(12)$
m_3/m_2	$3.68(5)$
$\sin^2\theta_{12}$	$0.306(11)$
$\sin^2\theta_{13}$	$0.0211(12)$
$\sin^2\theta_{23}$	$0.459(5)$
δ/π	$1.438(8)$
α_{21}/π	$1.704(5)$
α_{31}/π	$1.201(16)$

all dimensionless neutrino data determined in terms of 3 vacuum parameters

$Re(\tau), Im(\tau), \varphi_3$

τ	$-0.2005 + i\ 1.0578$
φ_3	0.117

$\Leftarrow$ 1^{st} octant

$N\sigma = 2.5 \div 3$

$m_1 = 1.09(3)\times 10^{-2}\text{eV}$, $m_2 = 1.39(2)\times 10^{-2}\text{eV}$, $m_3 = 5.11(4)\times 10^{-2}\text{eV}$

$|m_{ee}| = 1.04(2)\times 10^{-2}\text{eV}$

Other examples $m_\nu(\tau)$ and $m_e(\varphi)$ [Criado, F, S.J.D. King, in preparation]

Γ_4

$$\left[\begin{pmatrix} 0 & Y_1 & -Y_2 \\ Y_1 & -Y_2 & 0 \\ -Y_2 & 0 & Y_1 \end{pmatrix} + \xi \begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix}\right]$$

Type I seesaw
CP-invariant

1 extra parameters in ν^c mass

$p = 5$

	best value
$\Delta m^2_{21} \cdot 10^5\ \mathrm{eV}^{-2}$	7.39
$\Delta m^2_{3\ell} \cdot 10^3\ \mathrm{eV}^{-2}$	2.53
$\sin^2\theta_{12}$	0.317
$\sin^2\theta_{13}$	0.0224
$\sin^2\theta_{23}$	0.580
δ/π	1.25

	best value
$m_1 \cdot 10^2\ \mathrm{eV}^{-1}$	4.26
$m_2 \cdot 10^2\ \mathrm{eV}^{-1}$	4.35
$m_3 \cdot 10^2\ \mathrm{eV}^{-1}$	6.59
α_{21}/π	0.11
α_{31}/π	0.30
$m_{ee} \cdot 10^2\ \mathrm{eV}^{-1}$	4.25

$$\tau = 2.506 + i\,0.591$$
$$\xi = -2.595$$
$$\varphi_3 = i\,0.108$$

$\chi^2_{min} = 0.34$

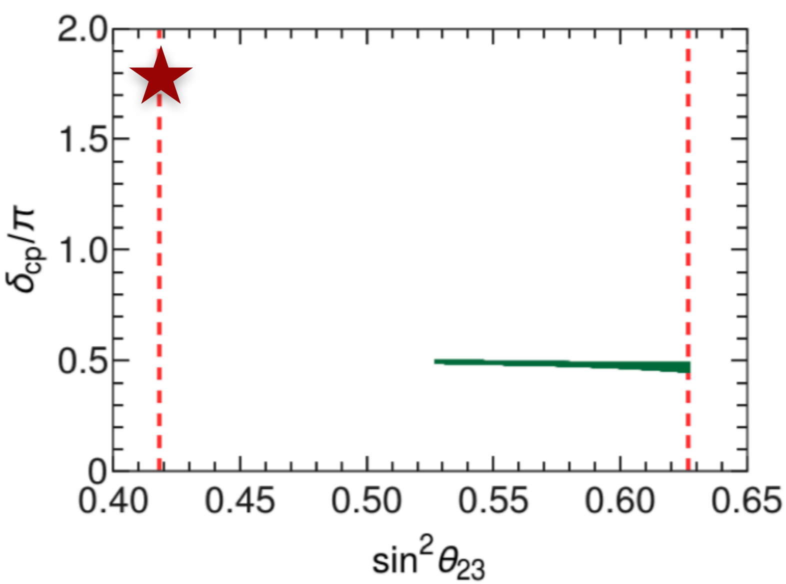
Γ_5	y_e diagonal			
Weinberg	$p = 3$ not viable			
Seesaw	$p = 5$ NO			
	★			
m_1	$83\ meV$	$49\ meV$	$26\ meV$	$26\ meV$
m_2	$83\ meV$	$50\ meV$	$27\ meV$	$27\ meV$
m_3	$97\ meV$	$70\ meV$	$57\ meV$	$57\ meV$
$\lvert m_{ee} \rvert$	$46\ meV$	$50\ meV$	$24\ meV$	$23\ meV$
δ_{CP}	$\pm 1.7\ \pi$	$\pm 1.5\ \pi$	$\pm 1.2\ \pi$	$\pm 1.0\ \pi$

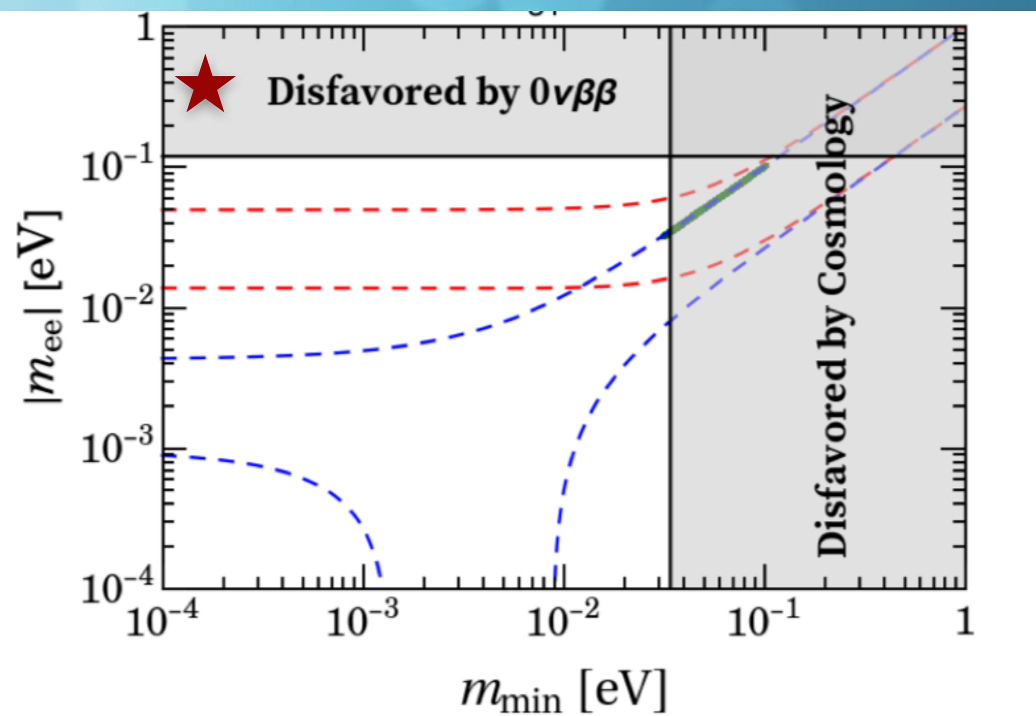
no known solutions with $p \leq 5$ and $y_e(\tau)$

2 extra parameters from Dirac neutrino mass

more solutions with IO

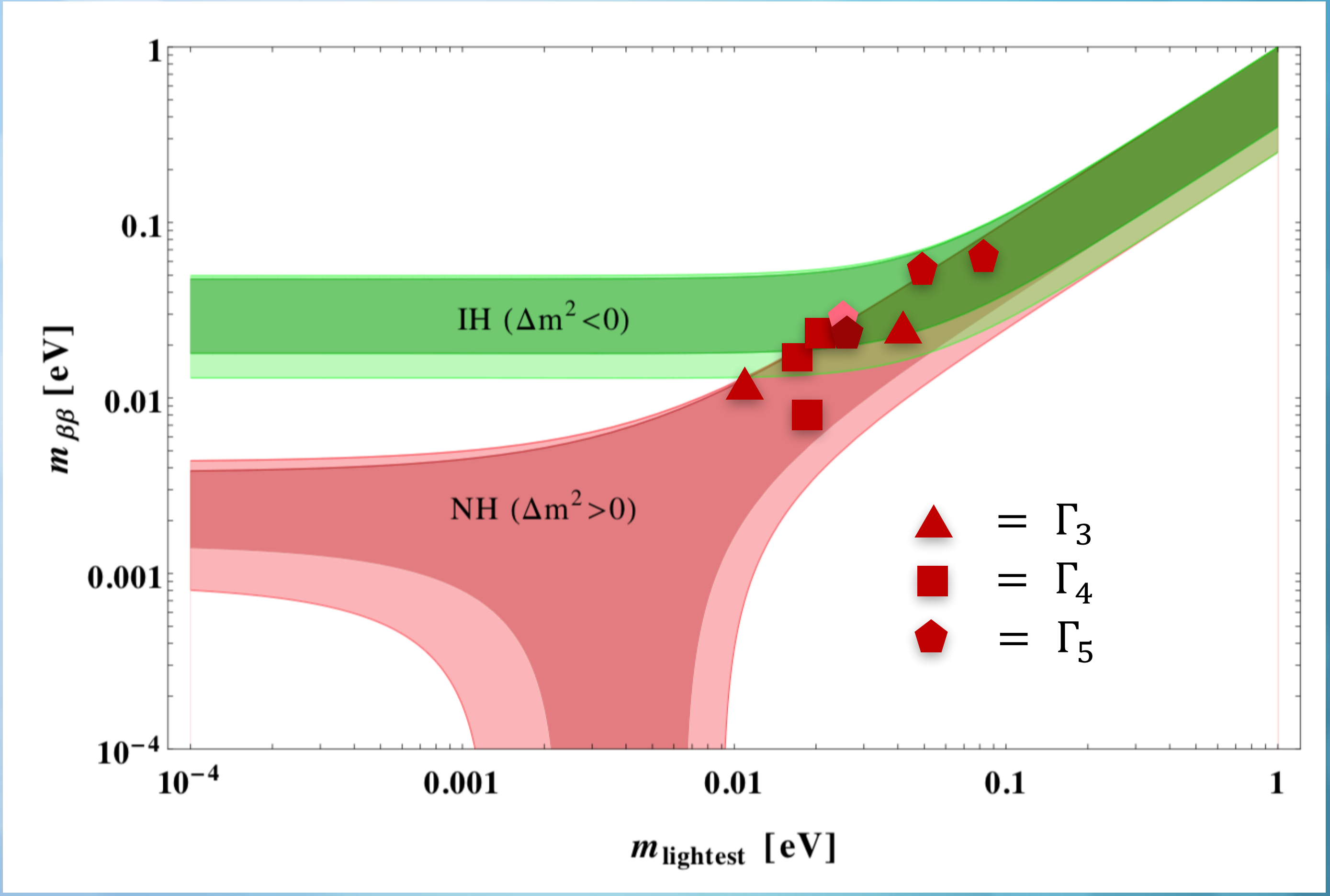
[P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov 1812.02158
G. J. Ding, S. F. King and X. G. Liu 1903.12588]





Some trend

- most of the solutions with NO prefer a nearly degenerate spectrum $m_1 > 10\, meV$ and $|m_{ee}|$ on the high side of allowed range



[$0\nu\beta\beta$ region from: S. Dell'Oro, S. Marcocci and F. Vissani, 1404.2616]

Some trend

- most of the solutions with NO prefer a nearly degenerate spectrum $m_1 > \ 10\, meV$ and $|m_{ee}|$ on the high side of allowed range

- higher N -> more solutions

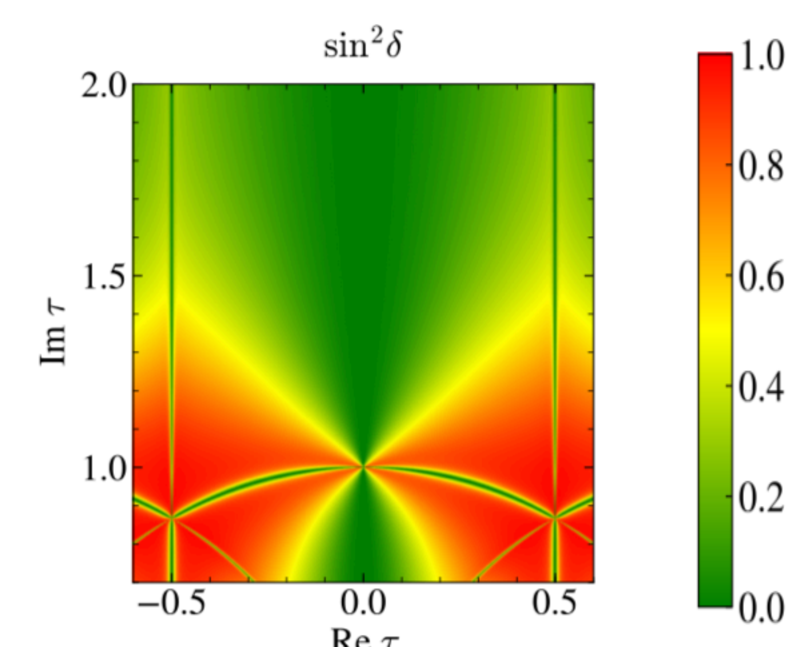
- any preferred value of τ ?

- extrema of $V(\tau)$ at the border of the fundamental region and along the $Im(\tau)$ axis ?

[Cvetic, Font, Ibanez, Lust and Quevedo, Nucl.Phys.B 361 (1991) 194]

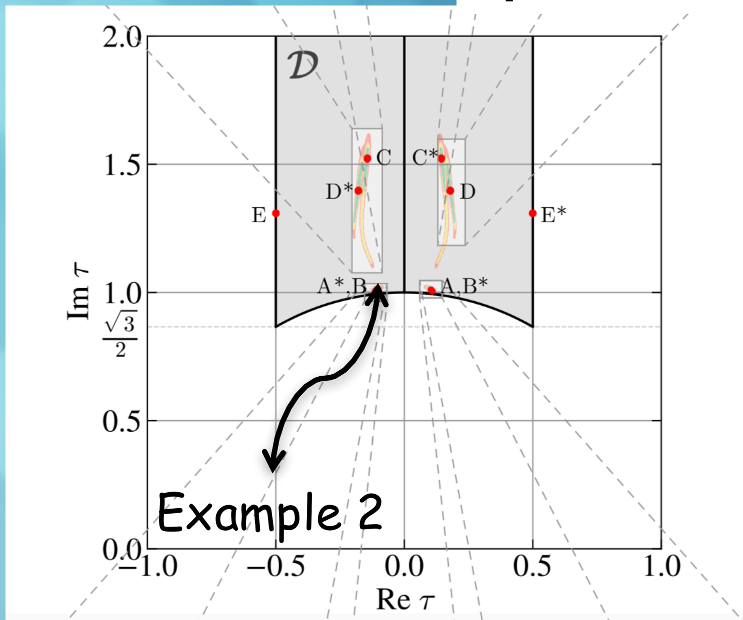
[P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, 1811.04933 and 1812.02158]

CP is conserved here



[Novichkov, Penedo, Petcov and Titov 1905.11970]

Γ_4 solutions



Corrections from SUSY breaking

- unknown breaking mechanism. Here:

F-component of a chiral supermultiplet, gauge and modular invariant

$$X = \vartheta^2 F$$

messenger scale M

SUSY-breaking scale

$$m_{SUSY} = \frac{F}{M}$$

- most general correction term to lepton masses and mixing angles

$$\delta\mathcal{S} = \frac{1}{M^2}\int d^4x d^2\theta d^2\bar{\theta}\; X^\dagger\; f(\Phi,\bar{\Phi}) + h.c.$$

$f(\Phi,\bar{\Phi})$ has dimension 3, determined by gauge invariance and lepton number conservation (treating Λ as spurion with L=+2)

$\Rightarrow$ $$\delta\mathcal{W}/\mathcal{W} \approx \delta\mathcal{Y}/\mathcal{Y} \approx \frac{m_{SUSY}}{M}$$

tiny, if sufficient gap between m_{SUSY} and M

10^{-10} for $m_{SUSY}=10^8$ GeV, $M = 10^{18}$ GeV

Conclusions

- neutrino data set a new standard in model building
  - accurate predictions
  - falsifiable models

- modular invariance as flavour symmetry can determine the functional dependence of Yukawa couplings on a modulus field

$\mathcal{Y}(\tau)$

- no/less flavons,
- less parameters
- no corrections from higher dimensional operators
- stability against SUSY breaking corrections

- can be implemented in a bottom-up approach:

absolute masses, m_{ee} and phases are predicted

- new directions:

+ role of modular symmetry in quark sector [Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi and Uchida, 1812.11072 Okada and Tanimoto, 1812.09677]

+ in GUTs [de Anda, King and Perdomo, 1812.05620]

+ more moduli [de Medeiros Varzielas, King and Zhou, 1906.02208]

Backup Slides

Modular forms of level 3 [1706.08749]

dimension of linear space $\mathcal{M}_k\left(\Gamma(3)\right)$ is (k+1) , k > 0 even integer

- 3 linearly independent modular forms of level 3 and minimal weight $k_I = 2$

$$
\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}-\frac{27\eta'(3\tau)}{\eta(3\tau)}\right]\\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]\\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]\quad .
\end{aligned}
$$

can be expressed in terms of the Dedekind eta function

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q\equiv e^{i2\pi\tau}$$

- they transform in a triplet 3 of Γ_3

$$Y(-1/\tau)=\tau^2\,\rho(S)Y(\tau) \qquad Y(\tau+1)=\rho(T)Y(\tau)$$

$$\rho(S)=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix} \qquad \rho(T)=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix} \qquad \omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

- they generate the whole ring $\mathcal{M}(\Gamma(3))$

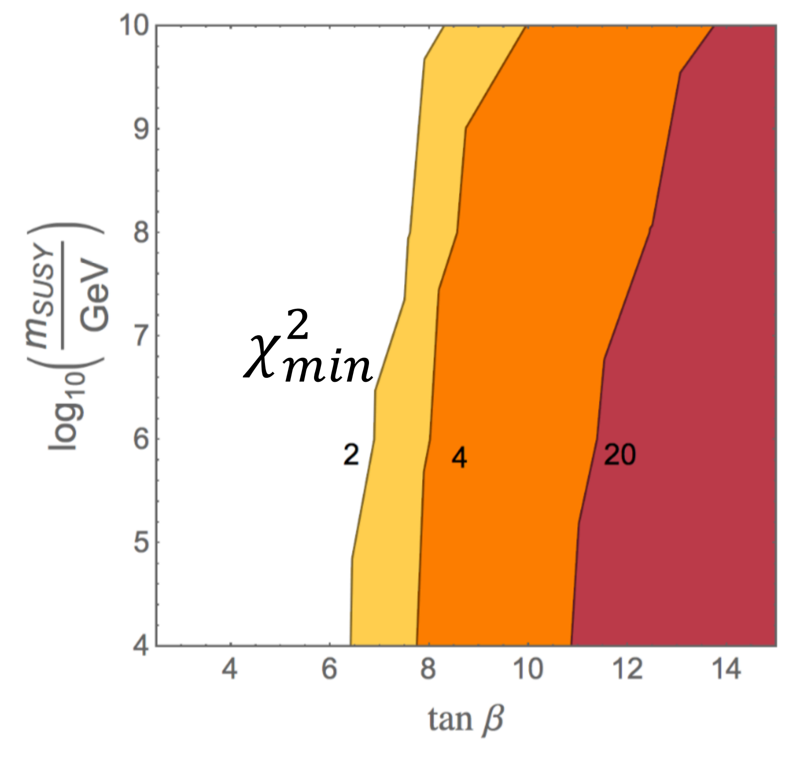
any modular form of level 3 and weight 2k can be written as an homogeneous polynomial in Y_i of degree k

Corrections from RGE

Model 1 (IO)
- r and $sin^2\vartheta_{12}$ mostly affected, at large $tan\beta$

$\Lambda = 10^{15}$ GeV

m_{SUSY}	Quantity	$\tan\beta = 2.5$	$\tan\beta = 10$	$\tan\beta = 15$
10^4 GeV	r	0.0302	0.0292	0.0288
	$\sin^2\theta_{12}$	0.304	0.345	0.418
	χ^2_{min}	0.4	12.2	82.0
10^8 GeV	r	0.0302	0.0294	0.0286
	$\sin^2\theta_{12}$	0.303	0.335	0.389
	χ^2_{min}	0.4	7.0	47.7



Model 2 (NO)
negligible corrections for $tan\beta$ up to 25 and m_{SUSY} as low as 10^4 GeV

$\mathcal{N}$=1 SUSY modular invariant theories

- known since late 1980s

S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B **225** (1989) 363.

S. Ferrara, .D. Lust and S. Theisen, Phys. Lett. B **233** (1989) 147.

focus on Yukawa interactions and $\mathcal{N}$=1 global SUSY

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta\ w(\Phi) + h.c.$$

$\Phi = (\tau, \varphi)$

Kahler potential, kinetic terms

superpotential, holomorphic function of Φ
Yukawa interactions

S invariant if
$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi) + f(\overline{\Phi}) \end{cases}$$

- invariance of the Kahler potential easy to achieve. For example:

$$K(\Phi, \bar{\Phi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

minimal K

- extension to $\mathcal{N}$=1 SUGRA straightforward: ask invariance of $G=K+\log|w|^2$

Few facts about (level-N) Modular Forms

- transformation property under the modular group

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

unitary representation of the finite modular group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$

- q-expansion

$$f(\tau + N) = f(\tau) \Rightarrow f(\tau) = \sum_{i=0}^{\infty} a_n q_N^n \qquad q_N = e^{\frac{i2\pi\tau}{N}}$$

- $k < 0 \Rightarrow f(\tau) = 0$

  $k = 0 \Rightarrow f(\tau) =$ constant

  $k > 0$ (even integer) $\Rightarrow f(\tau) \in \mathcal{M}_k(\Gamma(N))$ finite-dimensional linear space

- ring of modular forms generated by few elements

$$\mathcal{M}(\Gamma(N)) = \bigoplus_{k=0}^{\infty} \mathcal{M}_{2k}(\Gamma(N))$$

an explicit example in a moment

Level N modular forms

N	g	$d_{2k}(\Gamma(N))$	μ_N	Γ_N
2	0	$k+1$	6	S_3
3	0	$2k+1$	12	A_4
4	0	$4k+1$	24	S_4
5	0	$10k+1$	60	A_5
6	1	$12k$	72	
7	3	$28k-2$	168	

Table 1: Some properties of modular forms: g is the genus of the space $\mathcal{H}/\Gamma(N)$ after compactification, $d_{2k}(\Gamma(N))$ the dimension of the linear space $\mathcal{M}_{2k}(\Gamma(N))$, μ_N is the dimension of the quotient group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$, which, for $N \leq 5$, is isomorphic to a permutation group.

Modular forms of level 3 [1706.08749]

dimension of linear space $\mathcal{M}_k\left(\Gamma(3)\right)$ is (k+1) , k > 0 even integer

- 3 linearly independent modular forms of level 3 and minimal weight $k_I = 2$

$$
\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}-\frac{27\eta'(3\tau)}{\eta(3\tau)}\right]\\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right]\\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)}+\omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)}+\omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] .
\end{aligned}
$$

can be expressed in terms of the Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q \equiv e^{i2\pi\tau}$$

- they transform in a triplet 3 of Γ_3

$$Y(-1/\tau) = \tau^2\,\rho(S)Y(\tau) \qquad Y(\tau+1) = \rho(T)Y(\tau)$$

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \omega = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

- they generate the whole ring $\mathcal{M}(\Gamma(3))$

any modular form of level 3 and weight 2k can be written as an homogeneous polynomial in Y_i of degree k

Fundamental domain of Γ(3)

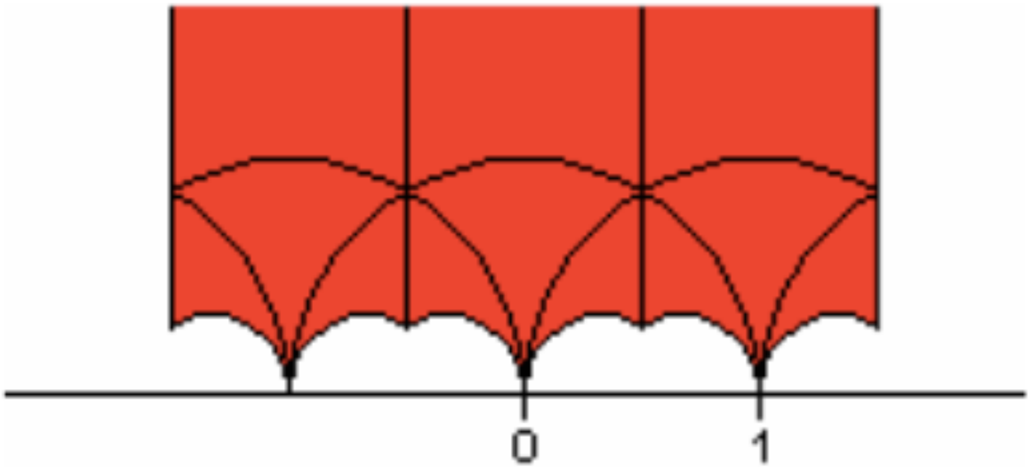


Figure 1: Fundamental domain for $\Gamma(3)$.

Q-expansion

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \ldots \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \quad . \end{aligned}$$

$$Y_2^2 + 2Y_1Y_3 = 0$$

some VEVs

$$(Y_1, Y_2, Y_3)|_{\tau=i\infty} = (1, 0, 0)$$

$$(Y_1, Y_2, Y_3)|_{\tau=i} = Y_1(i)(1, 1-\sqrt{3}, -2+\sqrt{3})$$

$$Y(-1/\tau)|_{\tau=i} = -\rho(S)Y(\tau)|_{\tau=i}$$

Ring of level-3 modular forms

$$Y_2^2 + 2Y_1Y_3 = 0$$

As discussed explicitly in Appendix D, the constraint ($\boxed{30}$) is essential to recover the correct dimension of the linear space $\mathcal{M}_{2k}(\Gamma(3))$. On the one side from table 1 we see that this space has dimension $2k+1$. On the other hand the number of independent homogeneous polynomial $Y_{i_1}Y_{i_2}\cdots Y_{i_k}$ of degree k that we can form with Y_i is $(k+1)(k+2)/2$. These polynomials are modular forms of weight $2k$ and, to match the correct dimension, $k(k-1)/2$ among them should vanish. Indeed this happens as a consequence of eq. ($\boxed{30}$). Therefore the ring $\mathcal{M}(\Gamma(3))$ is generated by the modular forms $Y_i(\tau)$ $(i = 1, 2, 3)$.

Models 1 and 2 are based on Γ_3

Why Γ_3 ? Γ_3 is isomorphic to A_4, smallest group of the Γ_N series possessing a 3-dimensional irreducible representation

[Ma, Rajasekaran, 0106291
Babu, Ma, Valle 0206292]

[recent extensions to Γ_2 and Γ_4 in Kobayashi, Tanaka, Tatsuishi, 1803.10391;
Penedo, Petcov 1806.11040]

	(E_1^c, E_2^c, E_3^c)	N^c	L	H_d	H_u	φ
$SU(2)_L \times U(1)_Y$	$(1, +1)$	$(1, 0)$	$(2, -1/2)$	$(2, -1/2)$	$(2, +1/2)$	$(1, 0)$
$\Gamma_3 \equiv A_4$	$(1, 1'', 1')$	3	3	1	1	3
k_I	$(k_{E_1}, k_{E_2}, k_{E_3})$	k_N	k_L	k_d	k_u	k_φ

Table 1: Chiral supermultiplets, transformation properties and weights. Model 1 has no gauge singlets N^c.

	k_{E_i}	k_N	k_L	k_d	k_u	k_φ
`Model 1`	-2	$-$	-1	0	0	$+3$
`Model 2`	-4	-1	$+1$	0	0	$+3$

Table 2: Weights of chiral multiplets. Model 1 has no gauge singlets N^c.

modular invariance broken by

τ

real

$\varphi = (1, 0, \varphi_3)$

if we go minimal

	L	H_u	Y
$SU(2)\times U(1)$	$(2,-1/2)$	$(2,+1/2)$	$(1,0)$
$\Gamma_3 \equiv A_4$	3	1	3
k_I	+1	0	+2

the operator

$$w_\nu = \frac{1}{\Lambda}(H_u H_u \, LL \, Y)_1$$

is completely specified up to an overall constant

we get

$$m_\nu = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \frac{v_u^2}{\Lambda}$$

a familiar matrix but now Y_i are determined by the choice of τ

by scanning τ VEVs the best agreement is obtained for $\tau = 0.0111 + 0.9946i$

	$\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}$	$\sin^2\vartheta_{12}$	$\sin^2\vartheta_{13}$	$\sin^2\vartheta_{23}$	$\frac{\delta_{CP}}{\pi}$	$\frac{\alpha_{21}}{\pi}$	$\frac{\alpha_{31}}{\pi}$
Exp	0.0292	0.297	0.0215	0.5	1.4	–	–
1σ	0.0008	0.017	0.0007	0.1	0.2	–	–
prediction	0.0292	0.295	0.0447	0.651	1.55	0.22	1.80

many σ away

2-parameter fit to 5 physical quantities

8 dimensionless physical quantities independent on any coupling constant!

Variants

neutrino masses from see-saw mechanism

$$w_\nu = g\,(N^c H_u L)_1 + \Lambda (N^c N^c Y)_1$$

assignement

	L	N^c	H_u	Y
$SU(2)\times U(1)$	$(2,-1/2)$	$(1,0)$	$(2,+1/2)$	$(1,0)$
$\Gamma_3 \equiv A_4$	3	3	1	3
k_I	k_L	$+1$	k_u	$+2$

$1+k_L+k_u=0$

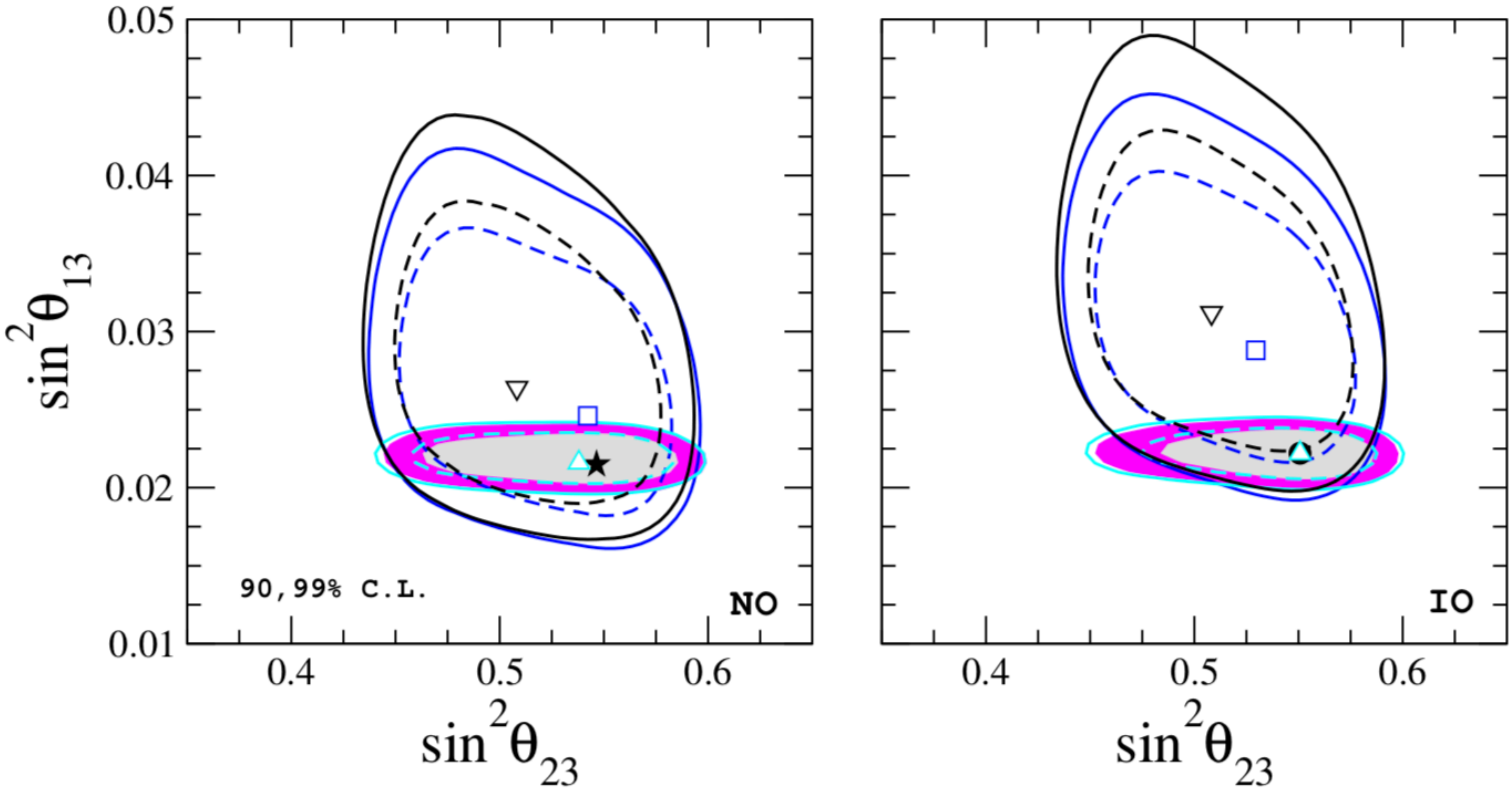
we get the best agreement at

$$\tau = -0.195 + 1.0636i$$

	$\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}$	$\sin^2\vartheta_{12}$	$\sin^2\vartheta_{13}$	$\sin^2\vartheta_{23}$	$\frac{\delta_{CP}}{\pi}$	$\frac{\alpha_{21}}{\pi}$	$\frac{\alpha_{31}}{\pi}$
Exp	0.0292	0.297	0.0215	0.5	1.4	–	–
1σ	0.0008	0.017	0.0007	0.1	0.2	–	–
$prediction$	0.0280	0.291	0.0486	0.331	1.47	1.83	1.26

Normal mass ordering is predicted

$$m_1 = 1.096\times 10^{-2}\ eV \qquad m_2 = 1.387\times 10^{-2}\ eV \qquad m_3 = 5.231\times 10^{-2}\ eV$$



Status of neutrino oscillations 2018: 3
σ hint for normal mass ordering and improved CP sensitivity
P.F. de Salas (Valencia U., IFIC), D.V. Forero (Campinas State U. & Virginia Tech.), C.A. Ternes, M. Tortola, J.W.F. Valle (Valencia U., IFIC). Aug 3, 2017. 8 pp.
Published in **Phys.Lett. B782 (2018) 633-640**

Charged Lepton Sector

$$\mathcal{Y}_e = \begin{pmatrix} a\ \varphi_1 & a\ \varphi_3 & a\ \varphi_2 \\ b\ \varphi_2 & b\ \varphi_1 & b\ \varphi_3 \\ c\ \varphi_3 & c\ \varphi_2 & c\ \varphi_1 \end{pmatrix}$$

$$U_e = \begin{pmatrix} 1 & \varphi_3 & 0 \\ 0 & -\varphi_3 & 1 \\ -\varphi_3 & 1 & \varphi_3 \end{pmatrix} + \ldots$$

where dots stand for terms of order φ_3^2, $(m_e^2/m_\mu^2)\varphi_3$ and $(m_\mu^2/m_\tau^2)\varphi_3$.

Fit to Model 1

	best value	pull
$r \equiv \lvert \Delta m^2_{sol}/\Delta m^2_{atm} \rvert$	$0.0302(11)$	$+0.13$
m_3/m_2	$0.0150(5)$	$-$
$\sin^2\theta_{12}$	$0.304(17)$	$+0.08$
$\sin^2\theta_{13}$	$0.0217(8)$	-0.13
$\sin^2\theta_{23}$	$0.577(4)$	$+0.67$
δ/π	$1.529(3)$	$+0.07$
α_{21}/π	$0.135(6)$	$-$
α_{31}/π	$1.728(18)$	$-$

Inverted mass Ordering

- no SUSY breaking effects
- no RGE corrections

best fit parameters

τ	$0.0117 + i\ 0.9948$
φ_3	-0.086

close to the self-dual critical point

$\chi^2_{min} = 0.4$

8 dimensionless physical quantities independent on any coupling constant!

$$m_1 = 4.90(3)\times 10^{-2}\mathrm{eV} \quad , \quad m_2 = 4.98(2)\times 10^{-2}\mathrm{eV} \quad , \quad m_3 = 7.5(3)\times 10^{-4}\mathrm{eV}$$

$$\lvert m_{ee}\rvert = 4.73(4)\times 10^{-2}\mathrm{eV}$$

by reproducing individually Δm^2_{sol} and Δm^2_{atm}

Fit to Yukawa couplings

Model 1

$a\cos\beta$	2.806923×10^{-6}
$b\cos\beta$	9.992488×10^{-3}
$c\cos\beta$	5.899778×10^{-4}

$y_e(m_Z)$	2.794745×10^{-6}	0.0
$y_\mu(m_Z)$	5.899864×10^{-4}	+0.05
$y_\tau(m_Z)$	1.002950×10^{-2}	0.0

Model 2

$a\cos\beta$	2.809569×10^{-6}
$b\cos\beta$	9.961316×10^{-3}
$c\cos\beta$	5.899455×10^{-4}

$y_e(m_Z)$	2.794745×10^{-6}	0.0
$y_\mu(m_Z)$	5.899863×10^{-4}	0.0
$y_\tau(m_Z)$	1.002950×10^{-2}	0.0

1σ parameter space

Intervals where $\chi^2 \leq \chi^2_{\min} + 1$:

	IO	NO
$\mathrm{Re}(\tau)$	$[0.0113, 0.0120]$	$[-0.2023, -0.1987]$
$\mathrm{Im}(\tau)$	$[0.9944, 0.9951]$	$[1.0522, 1.0633]$
$\mathrm{Re}(\varphi_3)$	$[-0.090, -0.082]$	$[0.113, 0.121]$

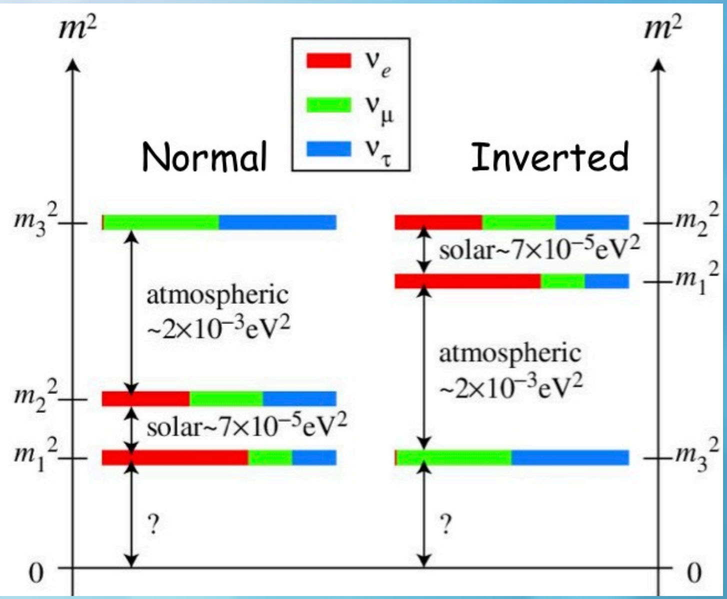
relevant parameters

$$m_1 < m_2 \; [\Delta m_{ij}^2 \equiv m_i^2 - m_j^2]$$

$$\Delta m_{21}^2 < \left|\Delta m_{32}^2\right|, \left|\Delta m_{31}^2\right|$$

i.e. 1 and 2 are, by definition, the closest levels

two possibilities: NO and IO



Mixing matrix U_{PMNS} (Pontecorvo,Maki,Nakagawa,Sakata)

$0 \le \vartheta_{ij} \le \pi/2$

$0 \le \delta < 2\pi$

$$L_{CC} = -\frac{g}{\sqrt{2}} W_\mu^- \,\overline{e}_L \gamma^\mu U_{PMNS} \nu_L$$

Majorana phases

standard parametrization

$$U_{PMNS} = \begin{array}{c} \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ -c_{12}s_{13}c_{23}e^{i\delta} + s_{12}s_{23} & -s_{12}s_{13}c_{23}e^{i\delta} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix} \end{array} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}$$