Quark and Lepton Flavors in A_4 Modular invariance

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Collaborated
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Outline of my talk

1 Introduction

We have a big question since the discovery of Muon.

What is the principle to control flavors of quarks and leptons ?

Symmetry Approach: S_3 , A_4 , S_4 , A_5 …

New approach appears by using **Modular group**

- Flavor symmetry is a finite subgroup of the modular group
- Flavor symmetry acts non-linealy (Modular forms)
- Lepton/Quark masses and mixing depend on a modulus τ, which is stabilized by some unknown mechanism

2 Modular Group

What is the origin of finite groups ?

It is well known that the superstring theory
on certain compactifications lead to
non-Abelian finite groups.

Indeed, torus compactification leads to Modular symmetery,
which includes S_3, A_4, S_4, A_5 as its subgroup.

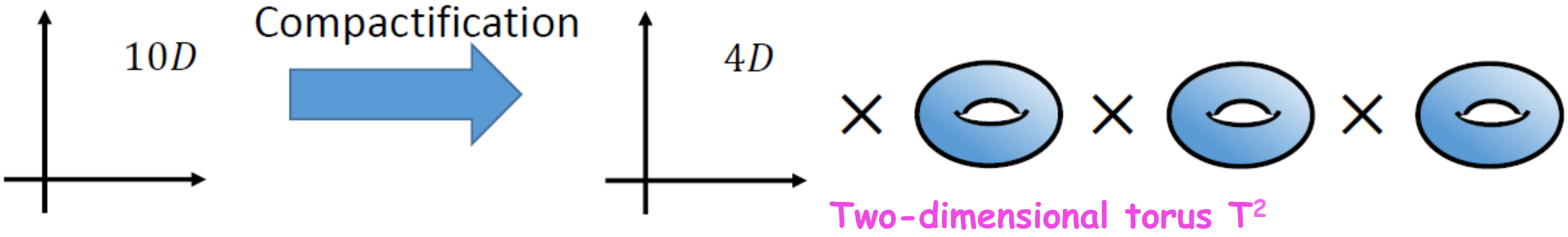
R.Toorop, F.Feruglio, C.Hagedorn, arXiv:1112.1340;
F.Feruglio, arXiv:1706.08749; A_4

T.Kobayashi, K.Tanaka, T.H.Tatsuishi, Phys.Rev.D98(2018)016004, arXiv:1803.10391; S_3
J.T.Penedo, S.T.Petcov, Nucl.Phys.B939(2019)292, arXiv:1806.11040; S_4
P.P.Novichkov, J.T.Penedo, S.T.Petcov, A.V.Titov,
JHEP 1904(2019) 174, arXiv:1812.02158; A_5
G.J.Ding, S.F.King, X.G.Liu, arXiv:1903.12588; A_5
X.~G.~Liu and G.~J.~Ding, arXiv:1907.01488 [hep-ph]; T'

Superstring theory 10D
Our universe is 4D

The extra 6D should be compactified.

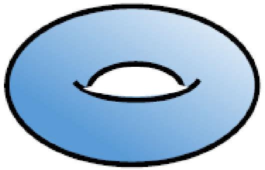
Torus compactification



We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \to \int d^4x \, \mathcal{L}_{\mathrm{eff}}$$

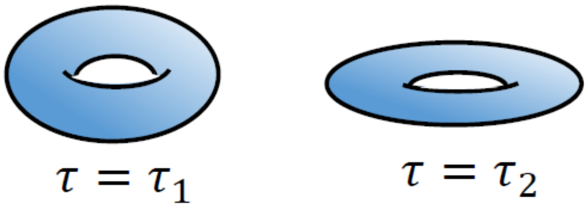
$\mathcal{L}_{\mathrm{eff}}$ depends on the structure of



- $4D$ effective theory depends on internal space

The shape of a torus T^2 is represented by a modulus τ

The different value of τ realize the different shape of T^2

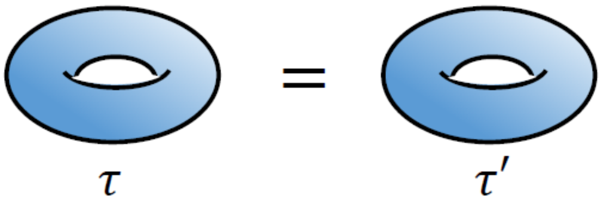


However,

there are specific transformations of τ which don't change T^2

Modular transformation

$\tau \to \tau'$



$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

ad-bc=1

a,b,c,d are integer SL(2,Z)

This transformation is generated by S and T :

$$\boxed{\begin{aligned} S &: \tau \longrightarrow -\frac{1}{\tau}, \\ T &: \tau \longrightarrow \tau + 1. \end{aligned}}$$

$$S : \tau \longrightarrow -\frac{1}{\tau},$$ duality

$$T : \tau \longrightarrow \tau + 1.$$ Dicrete shift symmetry

$$S^2 = 1, \qquad (ST)^3 = 1.$$

generate infinite discrete group

Modular group

$$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$$

Modular group has interesting subgroups

Impose congruence condition

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\}$$

called principal congruence subgroups

$\Gamma_N \equiv \Gamma / \Gamma(N)$ quotient group finite group

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$$\Gamma_2 \simeq S_3 \qquad \Gamma_3 \simeq A_4 \qquad \Gamma_4 \simeq S_4 \qquad \Gamma_5 \simeq A_5$$

We can consider effective theories with Γ_N symmetry.

$$\mathcal{L}_{\rm eff} \in \boxed{f(\tau)}\phi^{(1)}\cdots\phi^{(n)}$$

$f(\tau), \phi^{(I)}$: non-trivial rep. of Γ_N

In some cases of Γ_N, explicit forms of $f(\tau)$ have been obtained.

Famous modular function : Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{2\pi i\tau}$$

$$\eta(-1/\tau) = \boxed{\sqrt{-i\tau}}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

So called **Modular weight 1/2**

Modular transformation of chiral superfields in MSSM

$$\phi^{(I)} \to (c\tau+d)^{\boxed{-k_I}}\boxed{\rho^{(I)}(\gamma)}\phi^{(I)}$$

Modular weight

Representation matrix

3 Modular A_4 invariance as flavor symmetry

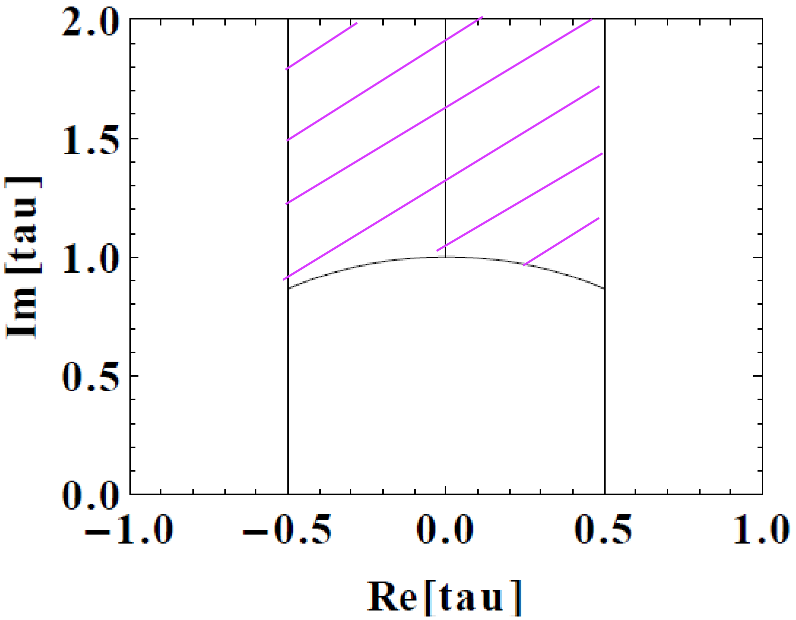
$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

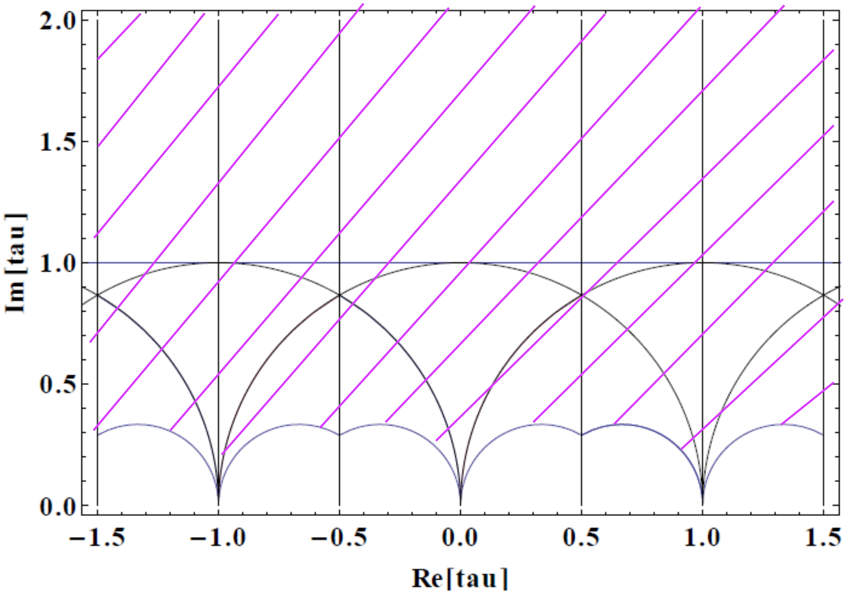
Taking $T^3=1$, we have A_4 modular group (Γ_3).

There are 3 linealy independent modular forms for weight 2 .

A_4 triplet ! weight 2 is minimal one for non-trival representation of A_4

Fundamental domain of τ on SL(2,Z) ➡ Fundamental domain of τ on Γ(3)





F. Feruglio, arXiv:1706.08749

A_4 triplet of modular function with weight 2

S transformation **T transformation**

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\ Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\ Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \end{aligned}$$

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned}$$

$q = e^{2\pi i\tau}$

$|q| \ll 1$

$Y_2^2 + 2Y_1Y_3 = 0$

Seesaw Model : 3 (L_e, L_μ, L_τ) 3 (ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$)

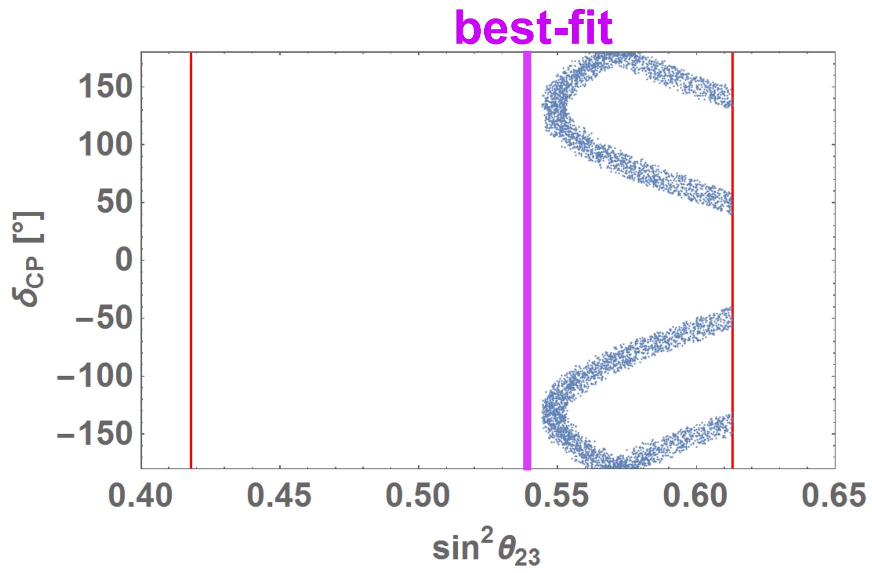
T.Kobayashi, N.Omoto,Y.Shimizu, K.Takagi, M.T, T.H.Tatsuishi, arXiv:1808.03012

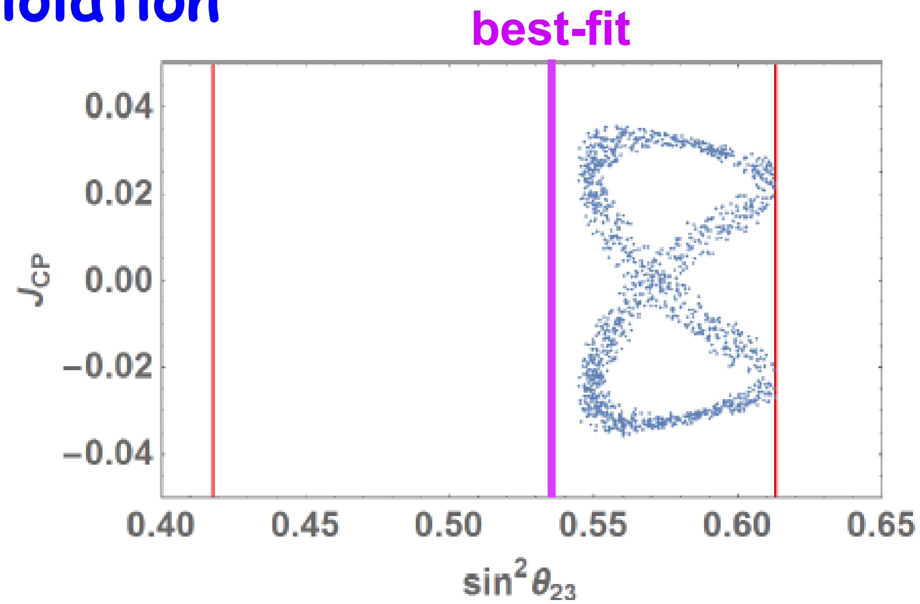
$$\mathcal{Y}_e = \begin{pmatrix} \alpha Y_1 & \alpha Y_3 & \alpha Y_2 \\ \beta Y_2 & \beta Y_1 & \beta Y_3 \\ \gamma Y_3 & \gamma Y_2 & \gamma Y_1 \end{pmatrix}$$

$$\mathcal{Y}_\nu = \begin{pmatrix} 2g_1 Y_1 & (-g_1+g_2)Y_3 & (-g_1-g_2)Y_2 \\ (-g_1-g_2)Y_3 & 2g_1 Y_2 & (-g_1+g_2)Y_1 \\ (-g_1+g_2)Y_2 & (-g_1-g_2)Y_1 & 2g_1 Y_3 \end{pmatrix}$$

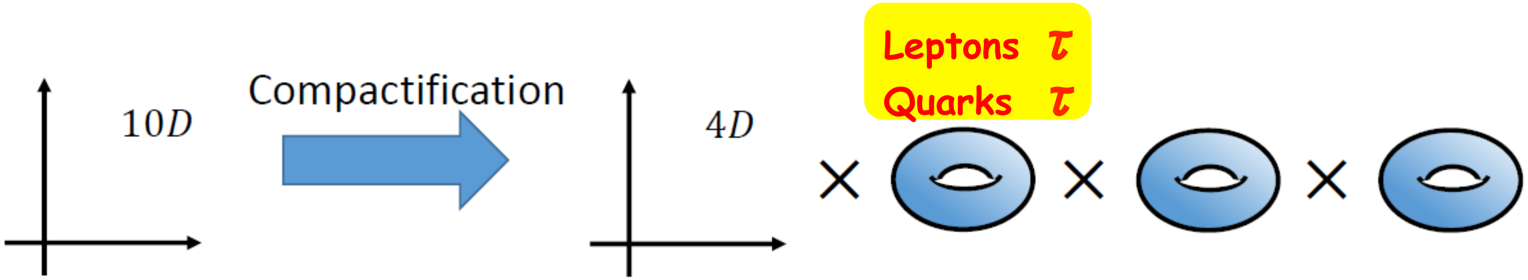
$$\mathcal{M}_R = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \Lambda$$

CP violation





Consider both quarks and leptons in modular symmetry



H.Okada, M.T, arXiv:1905.13421 [hep-ph] A_4

T.Kobayashi, Y.Shimizu, K.Takagi, M.T, T.H.Tatsuishi, arXiv:1906.10341 [hep-ph] S_3

Simple A_4 model: left-handed doublet 3, right-handed singlets 1, 1”, 1’

Quarks

$$w_u = \alpha_u u^c H_u \mathrm{Y}_3^{(2)} Q + \beta_u c^c H_u \mathrm{Y}_3^{(2)} Q + \gamma_u t^c H_u \mathrm{Y}_3^{(2)} Q$$

$$w_d = \alpha_d d^c H_d \mathrm{Y}_3^{(2)} Q + \beta_d s^c H_d \mathrm{Y}_3^{(2)} Q + \gamma_d b^c H_d \mathrm{Y}_3^{(2)} Q$$

$$M_q = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

for both up- and down-quarks

Prarameters α_q, β_q, γ_q are responsible for quark mass hierarchy.
After removing parameters α_q, β_q, γ_q by inputting quark masses,
one can examine CKM matrix elements by scanning modulus τ .

This simple mass matrix cannot reproduce observed three CKM mixing angles by fixing one complex parameter τ ! We need extra free parameters !

Let us consider Modular forms with higher weights k=4, 6 …

# of modular forms is k+1

Weight 2
3 Modular forms

$$\mathbf{Y}_3{}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1}'} \oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1}''} \oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}}$$

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}\,, \qquad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}''\,, \qquad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}'\,, \qquad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}\,.$$

Modular forms with higher weights are constructed by the tensor product of modular forms of weight 2

J.T.Penedo, S.T.Petcov, Nucl.Phys.B939(2019)292

Weight 4
5 Modular forms

$$\mathbf{Y}_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2Y_3\,, \quad \mathbf{Y}_{\mathbf{1}'}^{(4)} = Y_3^2 + 2Y_1Y_2\,, \quad \mathbf{Y}_{\mathbf{1}''}^{(4)} = Y_2^2 + 2Y_1Y_3 = 0\,,$$

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}$$

Weight 6
7 Modular forms

$$\mathbf{Y}_{\mathbf{1}}^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3\,,$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_1^2Y_2 + 2Y_2^2Y_3 \\ Y_1^2Y_3 + 2Y_3^2Y_2 \end{pmatrix}, \qquad \mathbf{Y}_{\mathbf{3}'}^{(6)} \equiv \begin{pmatrix} Y_1^{\prime(6)} \\ Y_2^{\prime(6)} \\ Y_3^{\prime(6)} \end{pmatrix} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^2Y_1 + 2Y_1^2Y_2 \\ Y_3^2Y_2 + 2Y_2^2Y_1 \end{pmatrix}$$

4 Modular A_4 invariance in Quarks and Leptons

Quark Sector

We have two A_4 triplets in weight 6 modular forms.

	Q	(q_1^c, q_2^c, q_3^c)	H_q	$\mathbf{Y}_{\mathbf{3}}^{(6)}$, $\mathbf{Y}_{\mathbf{3'}}^{(6)}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1}, \mathbf{1''}, \mathbf{1'})$	$\mathbf{1}$	$\mathbf{3}$, $\mathbf{3'}$
$-k_I$	-2	$(-4, -4, -4)$	0	$k=6$

$Q=(Q_1,Q_2,Q_3)$ Left-handed doublets

Freedom of Interchange 1,2,3

$(q_1{}^c, q_2{}^c, q_3{}^c)$ Right-handed singlets

Freedom of Interchange 1,2,3

no effect on masses and mixing

Modular invariant 6 couplings

$$w_q = \alpha_q q_1^c H_q \mathbf{Y}_{\mathbf{3}}^{(6)} Q + \alpha'_q q_1^c H_q \mathbf{Y}_{\mathbf{3'}}^{(6)} Q + \beta_q q_2^c H_q \mathbf{Y}_{\mathbf{3}}^{(6)} Q + \beta'_q q_2^c H_q \mathbf{Y}_{\mathbf{3'}}^{(6)} Q + \gamma_q q_3^c H_q \mathbf{Y}_{\mathbf{3}}^{(6)} Q + \gamma'_q q_3^c H_q \mathbf{Y}_{\mathbf{3'}}^{(6)} Q$$

$$M_q = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \left[\begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix} + \begin{pmatrix} g_{q1} & 0 & 0 \\ 0 & g_{q2} & 0 \\ 0 & 0 & g_{q3} \end{pmatrix} \begin{pmatrix} Y_1'^{(6)} & Y_3'^{(6)} & Y_2'^{(6)} \\ Y_2'^{(6)} & Y_1'^{(6)} & Y_3'^{(6)} \\ Y_3'^{(6)} & Y_2'^{(6)} & Y_1'^{(6)} \end{pmatrix} \right]$$

Extra 3 complex parameters appear in addition to α_q, β_q, γ_q !

Viable Model for Quarks

	Q	(d^c, s^c, b^c)	(u^c, c^c, t^c)	$H_{u,d}$	$\mathbf{Y}_{\mathbf{3}}^{(2)}$	$\mathbf{Y}_{\mathbf{3}}^{(6)},\ \mathbf{Y}_{\mathbf{3'}}^{(6)}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})$	$(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}\ ,\quad \mathbf{3'}$
$-k_I$	-2	$(0,\ 0,\ 0)$	$(-4,\ -4,\ -4)$	0	$k=2$	$k=6$

$$M_d = \begin{pmatrix} \alpha_d & 0 & 0 \\ 0 & \beta_d & 0 \\ 0 & 0 & \gamma_d \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL} ,$$

Weight 2 modular forms

$$M_u = \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \left[\begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix} + \begin{pmatrix} g_{u1} & 0 & 0 \\ 0 & g_{u2} & 0 \\ 0 & 0 & g_{u3} \end{pmatrix} \begin{pmatrix} Y_1'^{(6)} & Y_3'^{(6)} & Y_2'^{(6)} \\ Y_2'^{(6)} & Y_1'^{(6)} & Y_3'^{(6)} \\ Y_3'^{(6)} & Y_2'^{(6)} & Y_1'^{(6)} \end{pmatrix} \right]_{RL}$$

Weight 6 modular forms

After removing parameters α_q, β_q, γ_q by inputting quark masses, we have 3 complex parameters in addition to τ (8 real parameters).

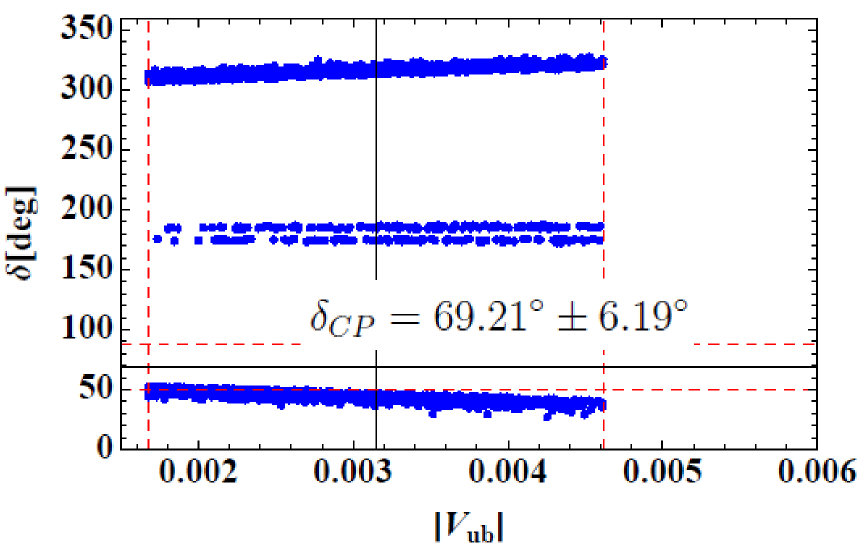
We set $g_{u1}=g_{u2}=g_{u3}$. 4 real parameters ⇔ 4 CKM elements

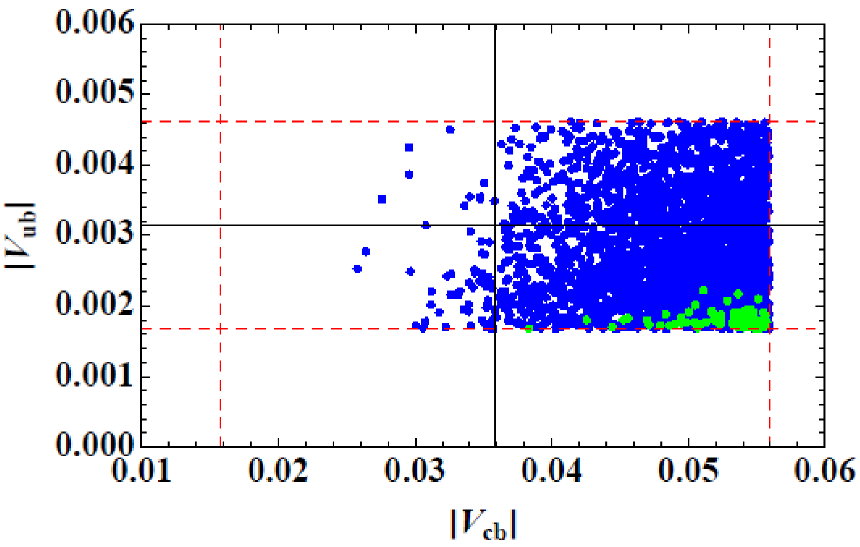
Input : quark masses and three mixing angles at GUT scale

Effect of RGE depends on tan β , M_{SUSY} , threshold effect ….

(tan β =5, M_{SUSY}=1 TeV)

Output : CP violating phase δ_{CP} (PDG)





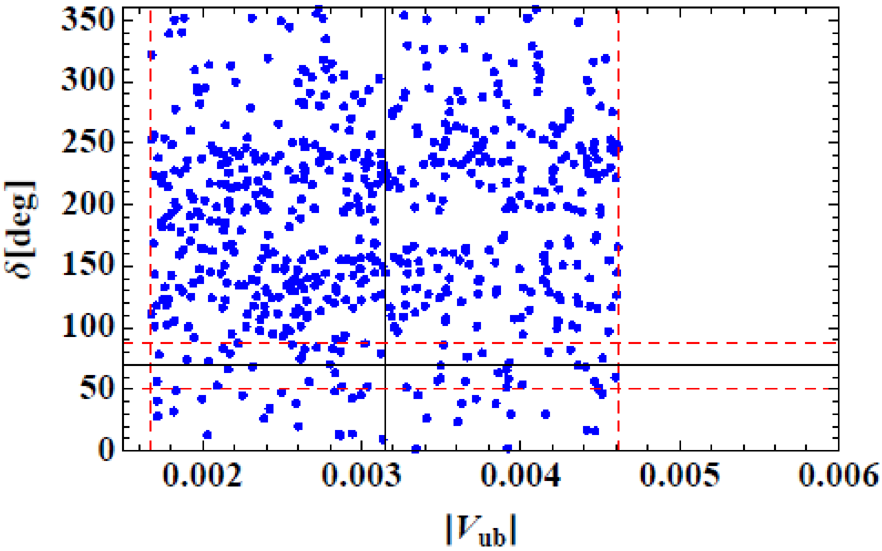
$$|V_d| = \begin{pmatrix} 0.5537 & 0.6135 & 0.5631 \\ 0.8110 & 0.2439 & 0.5317 \\ 0.1889 & 0.7511 & 0.6326 \end{pmatrix} , \qquad |V_u| = \begin{pmatrix} 0.4857 & 0.6859 & 0.5419 \\ 0.8198 & 0.2382 & 0.5208 \\ 0.3034 & 0.6876 & 0.6596 \end{pmatrix}$$

Hierarchical CKM elements are obtained by cancellation between V_d and V_u.

$$M_d = \begin{pmatrix} \alpha_d & 0 & 0 \\ 0 & \beta_d & 0 \\ 0 & 0 & \gamma_d \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL} ,$$

$$M_u = \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \left[\begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix} + \begin{pmatrix} g_{u1} & 0 & 0 \\ 0 & g_{u2} & 0 \\ 0 & 0 & g_{u3} \end{pmatrix} \begin{pmatrix} Y_1^{\prime(6)} & Y_3^{\prime(6)} & Y_2^{\prime(6)} \\ Y_2^{\prime(6)} & Y_1^{\prime(6)} & Y_3^{\prime(6)} \\ Y_3^{\prime(6)} & Y_2^{\prime(6)} & Y_1^{\prime(6)} \end{pmatrix} \right]_{RL}$$

If we set g_{u1} , g_{u2} , g_{u3} are different ones, two extra phases appear !



[0-2π] is allowed for δ

Alternative assignment of weight for quarks

Alternative Quark Mass Matrix appears !

	Q	(q_1^c, q_2^c, q_3^c)	H_q	$\mathbf{Y_3^{(k)}}$
$SU(2)$	**2**	**1**	**2**	**1**
A_4	**3**	**(1, 1", 1')**	**1**	**3**
$-k_I$	-2	$(-4,\ -2,\ 0)$	0	$k = 2, 4, 6$

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1'^{(6)} & Y_3^{(6)} + g_q Y_3'^{(6)} & Y_2^{(6)} + g_q Y_2'^{(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

After removing parameters α_q, β_q, γ_q by inputting quark masses, we have 1 complex parameters in addition to τ **(4 real parameters)**.

4 real parameters ⇔ 4 CKM elements

Lepton Sector

Common modulus τ for both quarks and leptons

	L	(e^c, μ^c, τ^c)	H_u	H_d	$\mathbf{Y_r}^{(2)}$, $\mathbf{Y_r}^{(4)}$
$SU(2)$	**2**	**1**	**2**	**2**	**1**
A_4	**3**	(**1**, **1**", **1**')	**1**	**1**	**3**, $\{\mathbf{3}, \mathbf{1}, \mathbf{1}'\}$
$-k_I$	-2	$(0,\ 0,\ 0)$	0	0	2, 4

$$M_E = \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_\mu & 0 \\ 0 & 0 & \gamma_\tau \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L \mathbf{Y_r^{(k)}})_\mathbf{1}$$

Weinberg operator by using weight 4 modular forms

$$M_\nu = \frac{v_u^2}{\Lambda}\left[\begin{pmatrix} 2Y_1^{(4)} & -Y_3^{(4)} & -Y_2^{(4)} \\ -Y_3^{(4)} & 2Y_2^{(4)} & -Y_1^{(4)} \\ -Y_2^{(4)} & -Y_1^{(4)} & 2Y_3^{(4)} \end{pmatrix} + g_{\nu 1} Y_\mathbf{1}^{(4)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_{\nu 2} Y_{\mathbf{1}'}^{(4)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]_{LL}$$

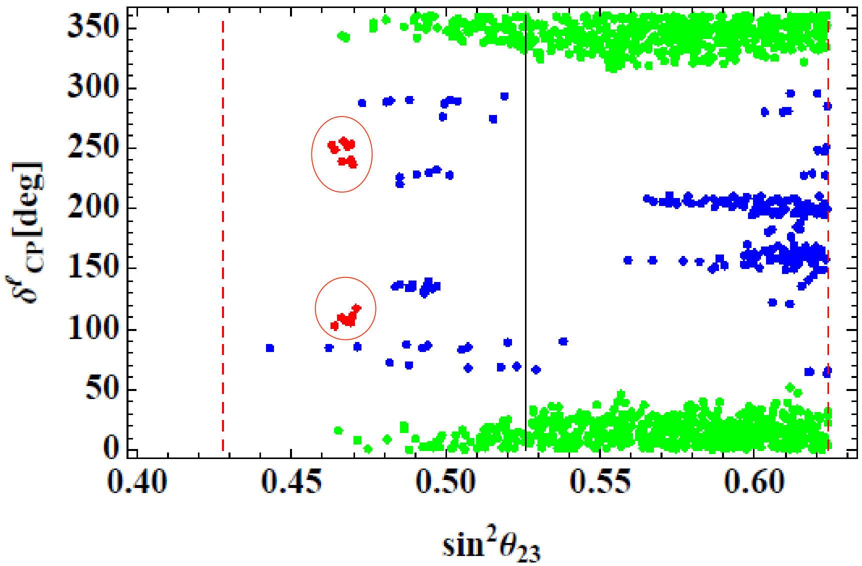
After removing parameters α_e, β_μ, γ_τ by inputting charged lepton masses, we have 2 complex parameters (4 real parameters if τ is fixed).

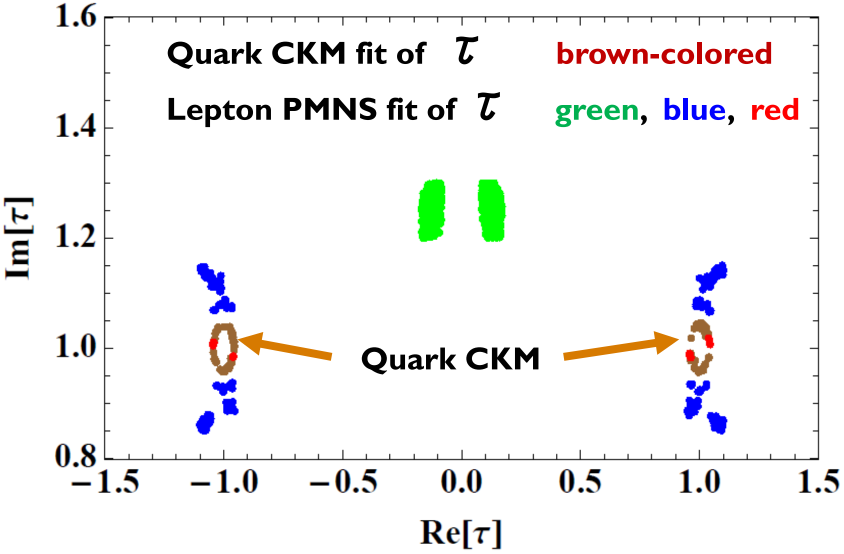
Input of 4 observed values: θ_{12}, θ_{23}, θ_{13}, $\Delta m^2_{sol} / \Delta m^2_{atm}$

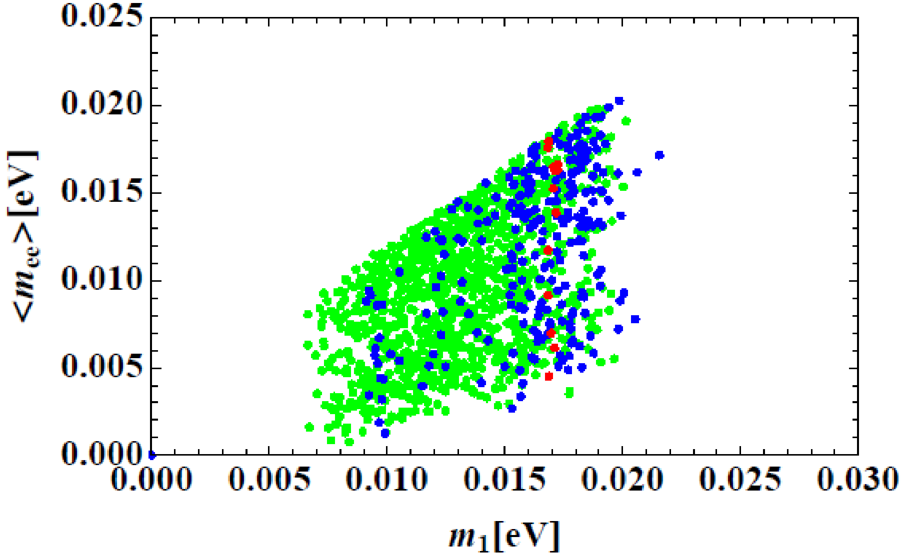
output : δ_{CP}, $\langle m_{ee} \rangle$, Σm_i

Normal hierarchy of neutrino masses

● **Common τ for quarks and leptons**







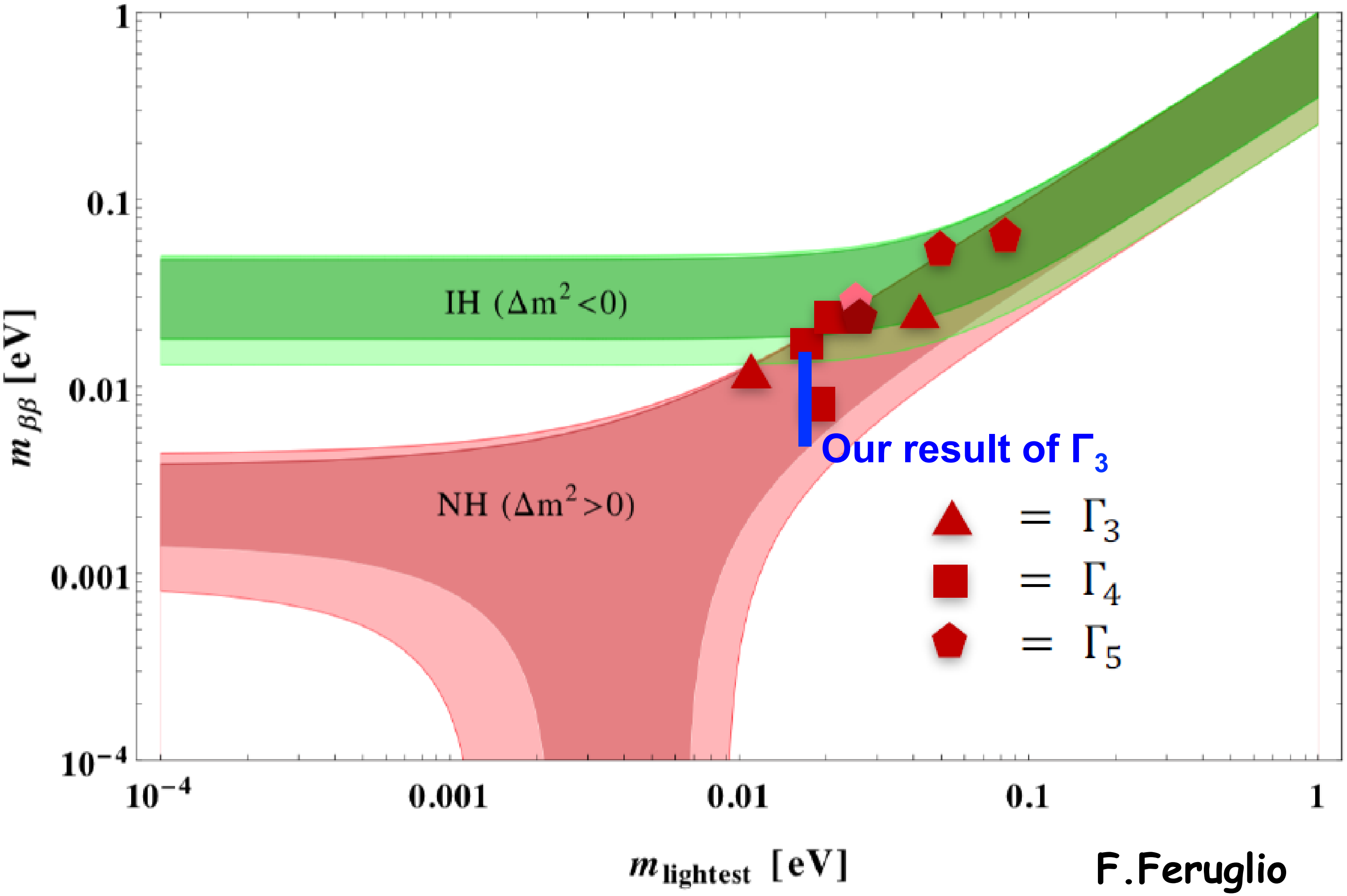
$$\sin^2\theta_{23} = 0.46\text{–}0.47$$

$$\delta^\ell_{CP} = 100^\circ\text{–}120^\circ \text{ and } 240^\circ\text{–}260^\circ$$

$$\langle m_{ee}\rangle = 4\text{–}18\text{meV}$$

$$\sum m_i \simeq 90\text{meV}$$

Cosmological bound 120meV


1
0.1
0.01
0.001
10^{-4}
$m_{\beta\beta}$ [eV]
IH ($\Delta m^2<0$)
NH ($\Delta m^2>0$)
10^{-4}
0.001
0.01
0.1
1
m_{lightest} [eV]
Our result of Γ_3
▲ = Γ_3
■ = Γ_4
⬟ = Γ_5

F.Feruglio

5 Summary

A_4 modular symmetry for quarks and leptons gives a viable model by using weight 2,4 and 6 modular forms

- Modulus τ is common in both quarks and leptons
- Quark Mass matrices is consistent with observed CKM matrix
- Lepton mass matrix is consistent with 3 observed mixing angles

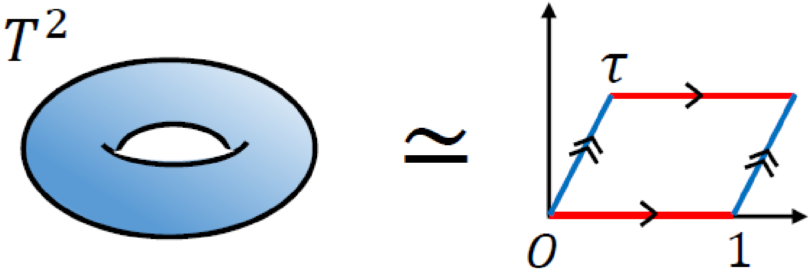
  NH is favored. (IH is not allowed)

  By imposing common τ for quarks and leptons,
  θ_{23} and δ_{CP} are predicted distinctly !

$$\sin^2\theta_{23} = 0.46\text{–}0.47 \qquad \delta^{\ell}_{CP} = 100^\circ\text{–}120^\circ \ \text{and}\ 240^\circ\text{–}260^\circ$$

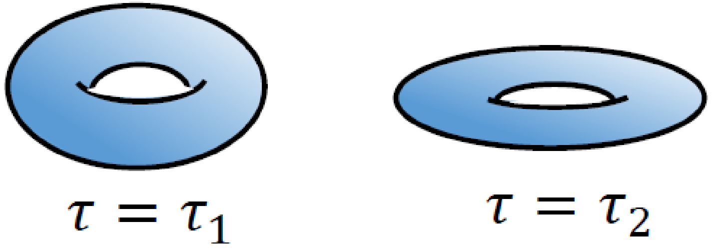
Towards Quark-Lepton unification with modular invariance !

$2D$ torus (T^2) is equivalent to parallelogram with identification of confronted sides.



Two-dimensional torus T^2 is obtained as $T^2 = \mathbb{R}^2 / \Lambda$
Λ is two-dimensional lattice

The shape of torus is represented by a modulus $\tau \in \mathbb{C}$.



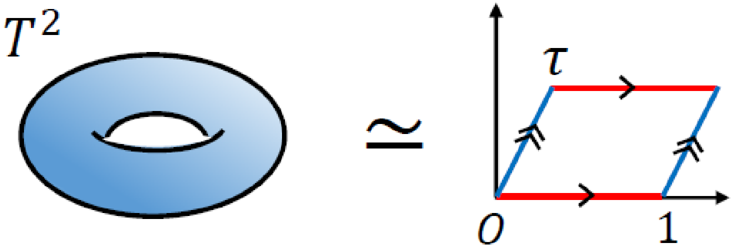
The different value of τ realize the different shape of T^2

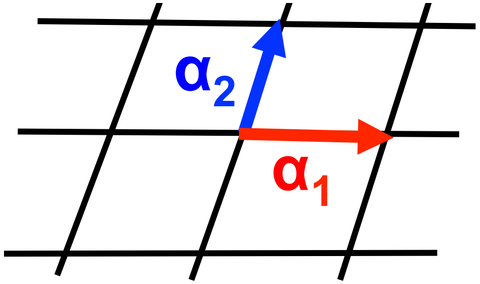
$\mathcal{L}_{\text{eff}}$ depends on τ.   e.g.) $\mathcal{L}_{\text{eff}} \supset Y(\tau)_{ij} \phi \overline{\psi_i} \psi_j + \cdots$

- $4D$ effective theory depends on a modulus τ

Modular transformation

The shape of a torus $T^2 \simeq$ The shape of a lattice on $\mathbb{C}$-plane





$(x,y) \sim (x,y) + n_1\alpha_1 + n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

Λ is two-dimensional lattice,
which is spanned by two lattice vectors

$\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\,\tau$

$\tau = \alpha_2 / \alpha_1$ is a modulus parameter (complex).

The same lattice is spanned by other bases under the transformation

$$\boxed{\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}}$$

ad-bc=1
a,b,c,d are integer SL(2,Z)

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

$\tau = \alpha_2 / \alpha_1$

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

Modular transformation

Modular transf. does not change the lattice (torus)

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

The modular transformation is generated by S and T .

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}$$

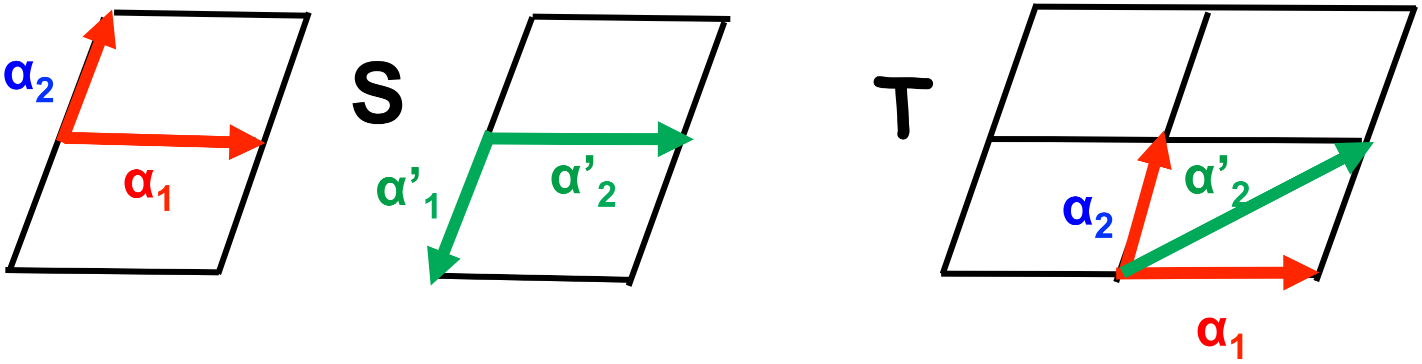
duality

$$T : \tau \longrightarrow \tau + 1$$

Dicrete shift symmetry

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$$



$$\tau = \alpha_2 / \alpha_1$$

Kinetic Term

Kinetic term of the modulus τ
$$\frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation
$$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

⇒
$$\frac{\left|\partial_\mu \tau'\right|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$
Modular invariant

How to find A_4 triplet modular functions.

Prepare 4 Dedekind eta-functions as Modular functions

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$\begin{aligned}
\eta(3\tau) &\to \sqrt{\frac{-i\tau}{3}}\,\eta(\tau/3),\\
\eta(\tau/3) &\to \sqrt{-i3\tau}\,\eta(3\tau),\\
\eta((\tau+1)/3) &\to e^{-i\pi/12}\sqrt{-i\tau}\,\eta((\tau+2)/3),\\
\eta((\tau+2)/3) &\to e^{i\pi/12}\sqrt{-i\tau}\,\eta((\tau+1)/3).
\end{aligned}$$

S : $\tau \longrightarrow -1/\tau$

$$\begin{aligned}
\eta(3\tau) &\to e^{i\pi/4}\eta(3\tau),\\
\eta(\tau/3) &\to \eta((\tau+1)/3),\\
\eta((\tau+1)/3) &\to \eta((\tau+2)/3),\\
\eta((\tau+2)/3) &\to e^{i\pi/12}\eta(\tau/3),
\end{aligned}$$

T : $\tau \longrightarrow \tau + 1$

Comment : Two special sets of τ

T(τ → τ +1) preserved : $<\tau>= i\,\infty$ ($q=0$) $(Y_1, Y_2, Y_3)=(1, 0, 0)$

S(τ → -1/ τ) preserved : $<\tau>=i$ ($q=e^{-2\pi}$) $(Y_1, Y_2, Y_3)=Y_1(i)\,(1, 1-\sqrt{3}, -2+\sqrt{3})$

Another eigenvector of S

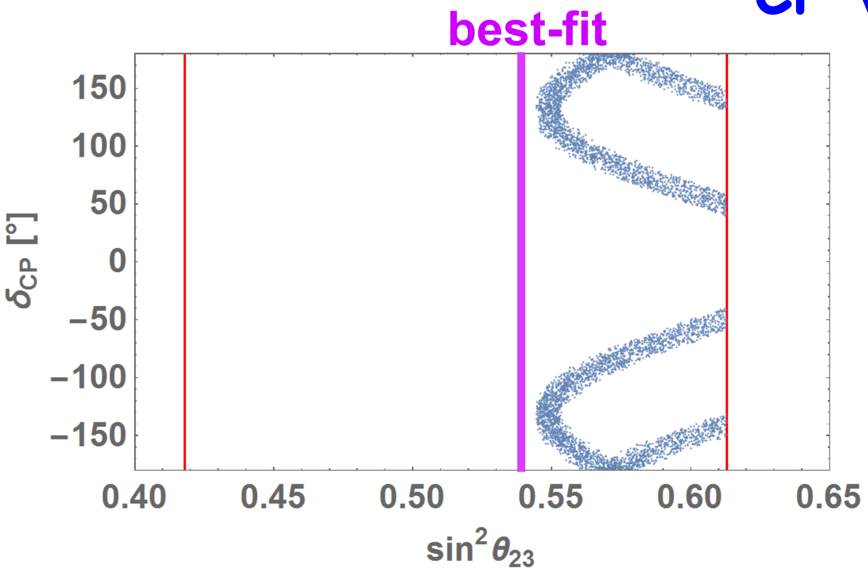
Eigenvector of S (1,1,1) cannot be realized

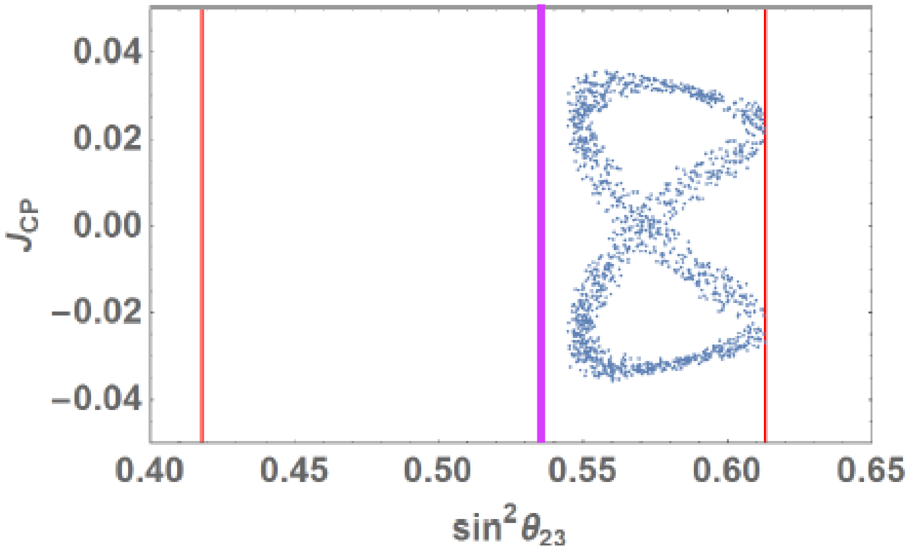
$$
\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots , \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}
$$

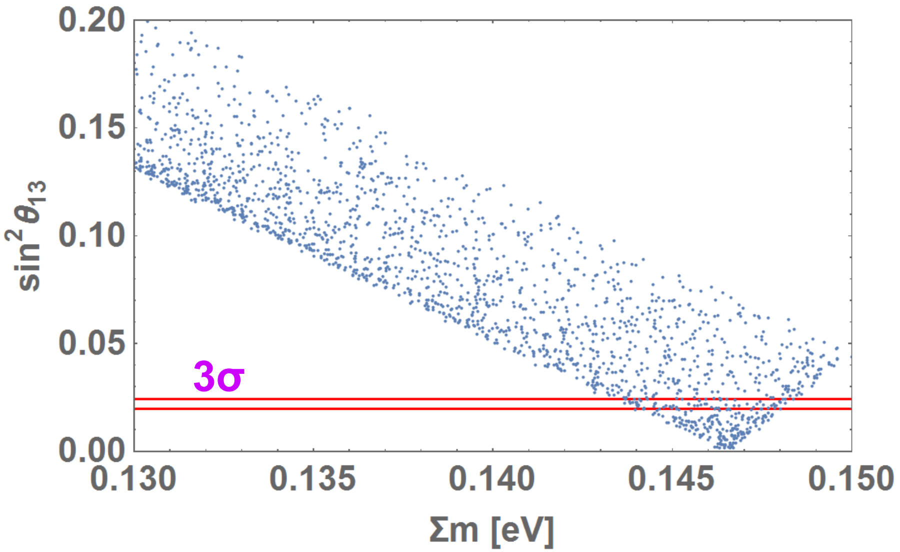
$$q = e^{2\pi i \tau}$$

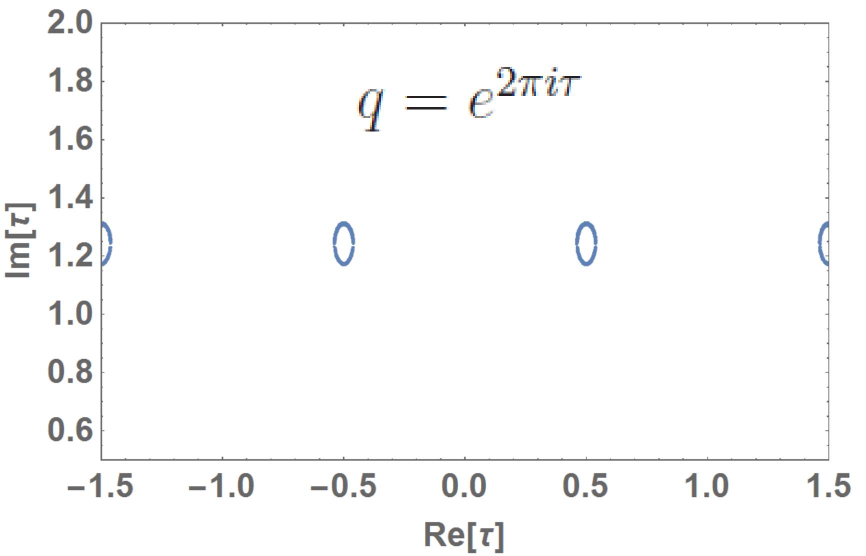
$$Y_2^2 + 2Y_1Y_3 = 0$$

CP violation









Planck 2018 results <0.12 eV@ΛCDM model ?