Modular A5 Symmetry for Flavour Model Building

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Outline

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- Modular group and modular forms

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- **Modular forms of weight 2 and level 5**

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- Modular forms of higher weight
- Examples of modular A5 flavour models
- Conclusions

Modular group(s)

Modular group $\overline{\Gamma} \simeq PSL(2,\mathbb{Z})$

$$\mathscr{H} = \mathrm{Im}\,\tau > 0 \qquad \overline{\Gamma} = \left\{ \gamma : \quad \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}$$

Generators *S* and *T*: $S^2 = (ST)^3 = I$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

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Infinite normal subgroups of $SL(2,\mathbb{Z})$, $N = 2, 3, 4, \ldots$:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

Principal congruence subgroups of the modular group:

$$\overline{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\} \qquad \overline{\Gamma}(N) \equiv \Gamma(N), \quad N > 2$$

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$$\Gamma_2 \simeq S_3 \qquad \Gamma_3 \simeq A_4 \qquad \Gamma_4 \simeq S_4 \qquad \Gamma_5 \simeq A_5$$

Modular forms

Holomorphic functions transforming under $\overline{\Gamma}(N)$

$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^{k} f(\tau)\,, \quad \gamma \in \overline{\Gamma}(N)$$

Modular forms

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weight (non-negative, even)

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Modular forms of weight k and level N form a linear space of finite dimension

N\k	0	2	4	6
2	1	2	3	4
3	1	3	5	7
4	1	5	9	13
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There exists a basis in this space s.t.

$$f_i\left(\gamma\tau\right) = \left(c\tau + d\right)^{k} \rho\left(\gamma\right)_{ij} f_j(\tau)\,, \quad \gamma \in \overline{\Gamma}$$

ρ is a unitary representation of $\overline{\Gamma}_N$

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Modular forms of weight $k > 2$ can be constructed from homogeneous polynomials in the modular forms of weight 2

Modular forms of weight 2

Level *N* = 2 $\left(\Gamma_2 \simeq S_3 : \quad S^2 = (ST)^3 = T^2 = I\right)$

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2	1	**2**	3	4

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Seed functions: $\eta\left(\frac{\tau}{2}\right), \quad \eta\left(\frac{\tau+1}{2}\right), \quad \eta\left(2\tau\right)$

Kobayashi, Tanaka, Tatsuishi, 1803.10391 (following Feruglio for N = 3, 1706.08749)

$\eta(\tau) \equiv q^{1/24}\prod_{n=1}^{\infty}\left(1-q^n\right) , \quad q = e^{2\pi i\tau},$ is the Dedekind eta function

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$$\eta(\tau+1) = e^{i\pi/12}\,\eta(\tau) \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

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S (up to multiplicative factors)

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$$Y(a_1, a_2, a_3 \,|\, \tau) \equiv \sum_{i=1}^{3} a_i \frac{\mathrm{d}}{\mathrm{d}\tau} \log \eta_i(\tau) \,, \qquad \sum_{i=1}^{3} a_i = 0$$

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$$Y_{\mathbf{2}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix} \equiv c \begin{pmatrix} Y(1, 1, -2 \,|\, \tau) \\ Y(\sqrt{3}, -\sqrt{3}, 0 \,|\, \tau) \end{pmatrix}$$

S_3 doublet of weight 2 modular forms

Modular forms of weight 2

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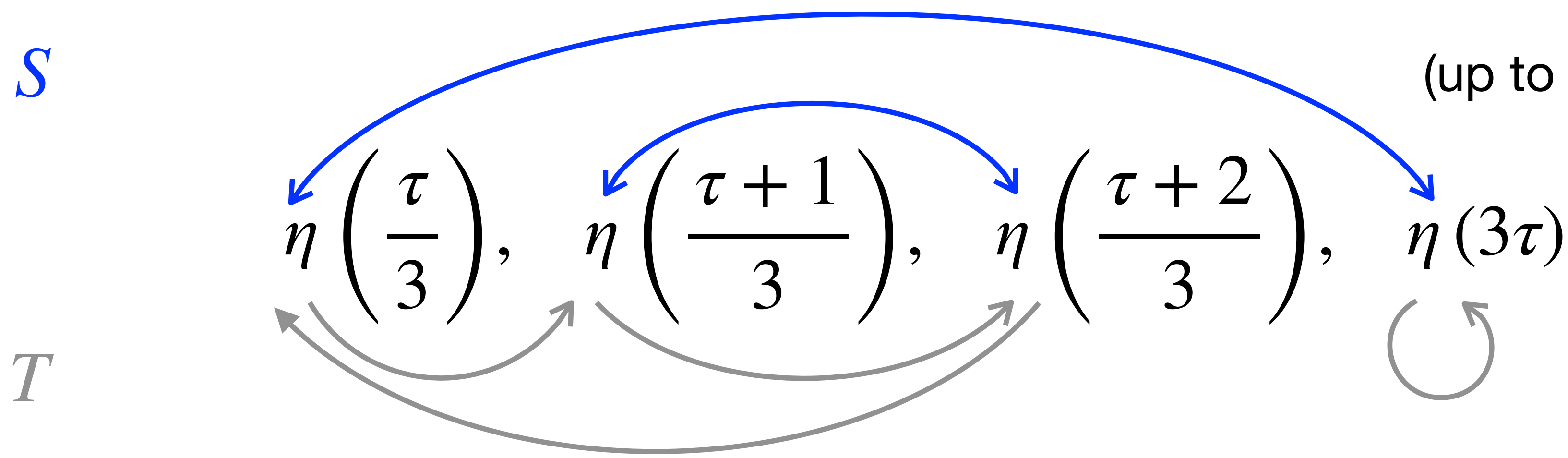
(up to multiplicative factors)



Feruglio, 1706.08749

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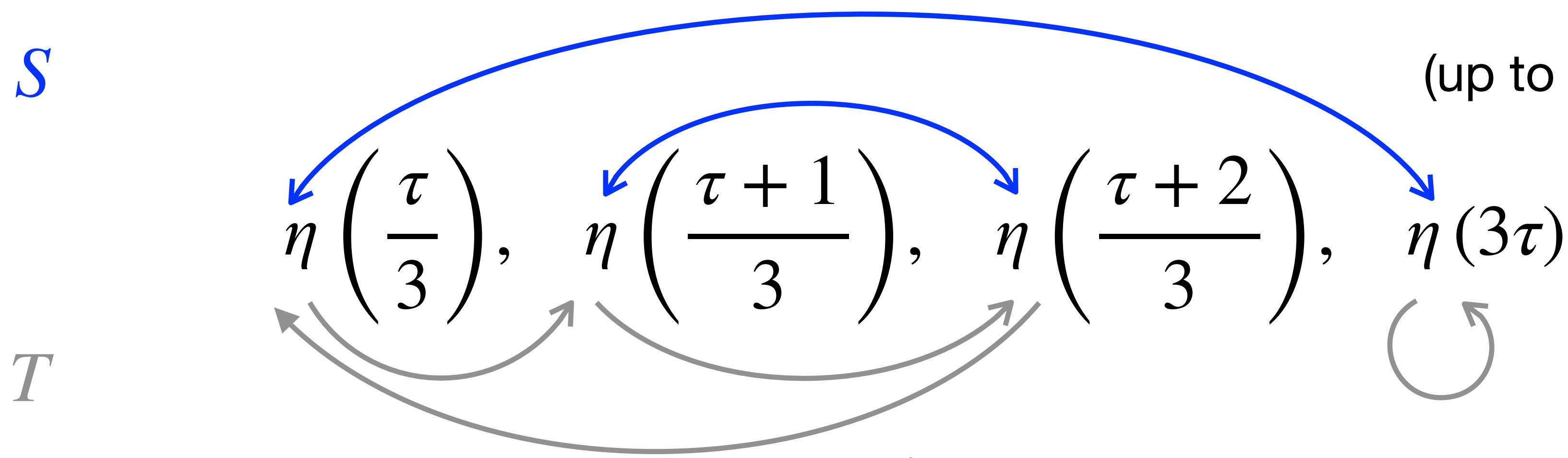

$$\eta\left(\frac{\tau}{3}\right), \quad \eta\left(\frac{\tau+1}{3}\right), \quad \eta\left(\frac{\tau+2}{3}\right), \quad \eta\,(3\tau)$$

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A_4 triplet of weight 2 modular forms

Modular forms of weight 2

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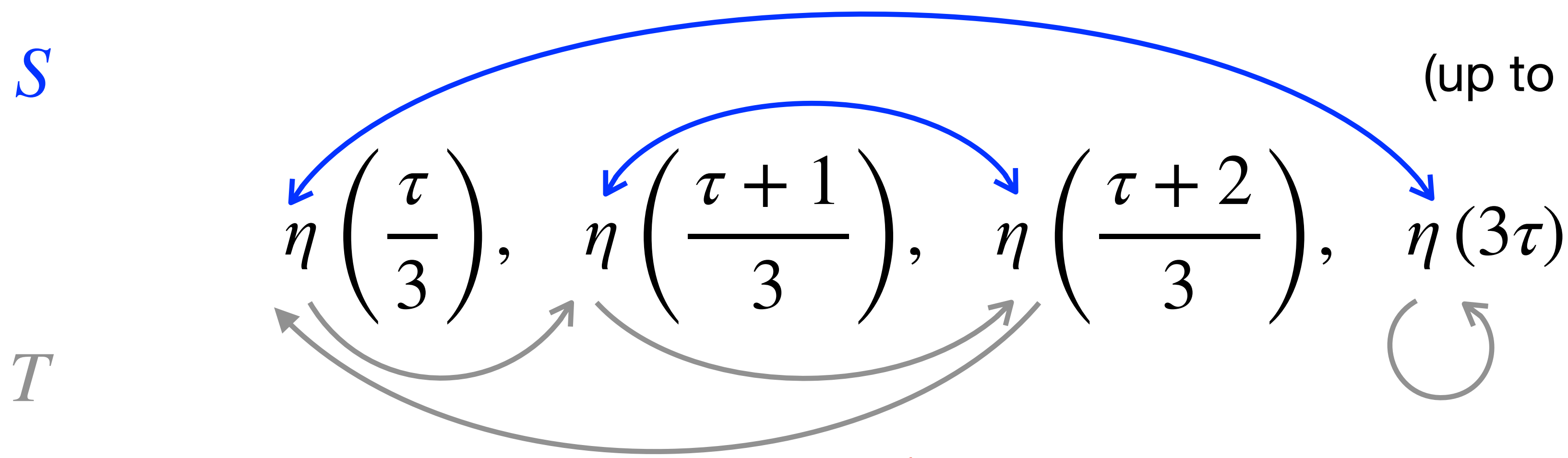

Feruglio, 1706.08749

A_4 triplet of weight 2 modular forms

Level *N* = 4 $\left(\Gamma_4 \simeq S_4 : \quad S^2 = (ST)^3 = T^4 = I\right)$

N\k	0	2	4	6
4	1	**5**	9	13

Modular forms of weight 2

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(up to multiplicative factors)

Feruglio, 1706.08749

S

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A_4 triplet of weight 2 modular forms

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N\k	0	2	4	6
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Penedo, Petcov, 1806.11040

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$$\eta\left(\tau+\frac{1}{2}\right),\quad \eta\,(4\tau),\quad \eta\left(\frac{\tau}{4}\right),\quad \eta\left(\frac{\tau+1}{4}\right),\quad \eta\left(\frac{\tau+2}{4}\right),\quad \eta\left(\frac{\tau+3}{4}\right)$$

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(up to multiplicative factors)

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S_4 doublet and triplet ($\mathbf{3}'$) of weight 2 modular forms

Modular forms of weight 2

Level *N* = 5 $\left(\Gamma_5 \simeq A_5 : \quad S^2 = (ST)^3 = T^5 = I\right)$

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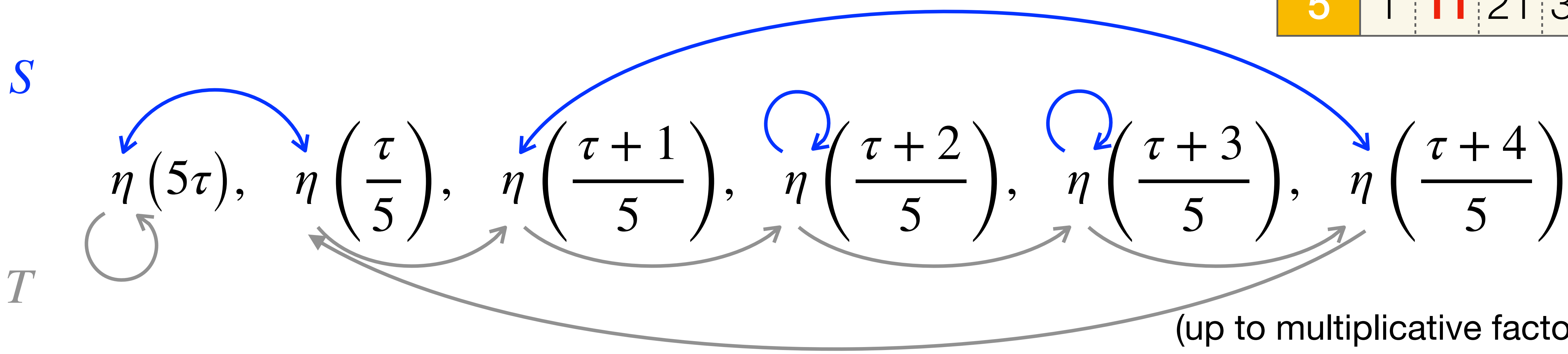

(up to multiplicative factors)

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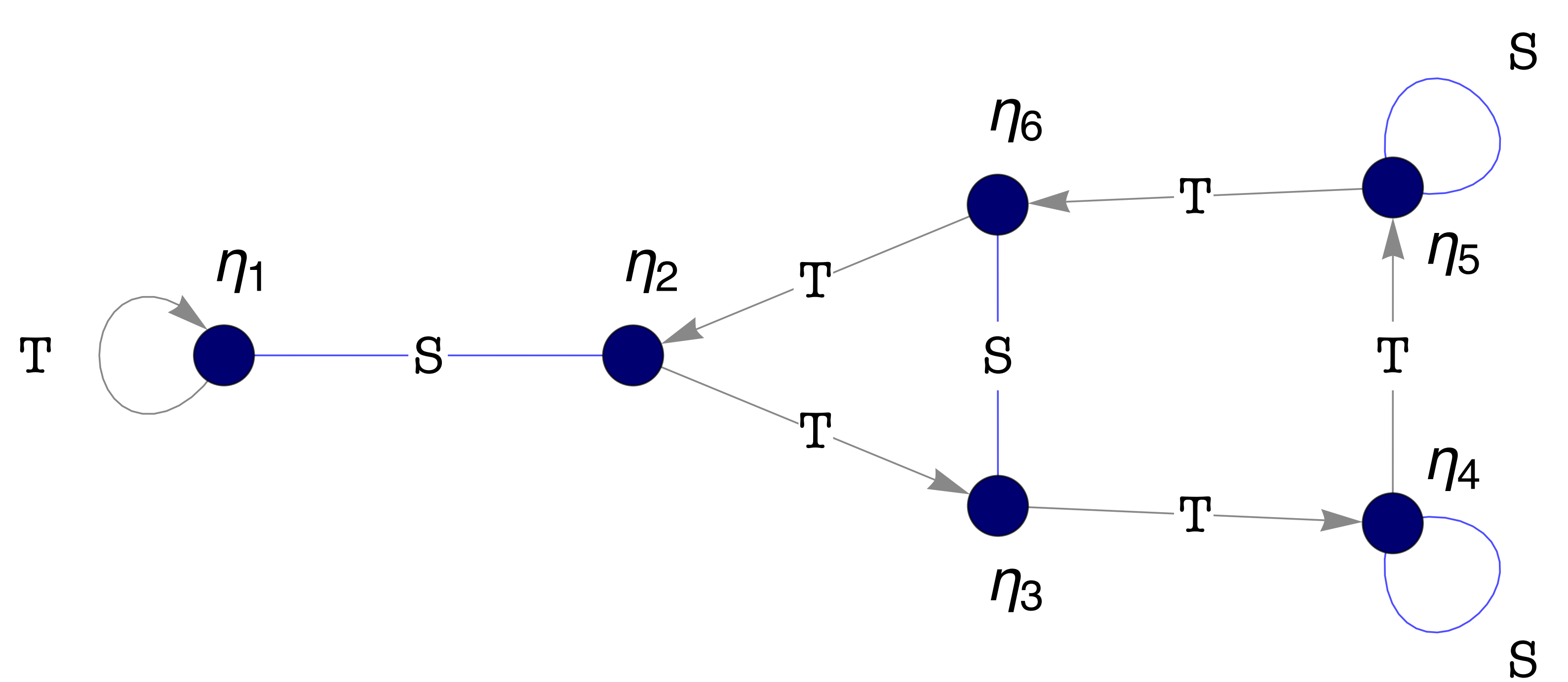

Modular forms of weight 2

$$X(a_1, \ldots, a_6 \,|\, \tau) \equiv \sum_{i=1}^{6} a_i \frac{\mathrm{d}}{\mathrm{d}\tau} \log \eta_i(\tau) \,, \qquad \sum_{i=1}^{6} a_i = 0$$

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$$X_{\mathbf{5}}(\tau) = \begin{pmatrix} X_1(\tau) \\ X_2(\tau) \\ X_3(\tau) \\ X_4(\tau) \\ X_5(\tau) \end{pmatrix} \equiv c \begin{pmatrix} -\frac{1}{\sqrt{6}} X\left(-5,1,1,1,1,1 \,|\, \tau\right) \\ X(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta \,|\, \tau) \\ X(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2 \,|\, \tau) \\ X(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3 \,|\, \tau) \\ X(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4 \,|\, \tau) \end{pmatrix} , \qquad \zeta = e^{2\pi i/5}$$

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A_5 quintet of weight 2 modular forms

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A_5 quintet of weight 2 modular forms

$$11 = \mathbf{5} + \mathbf{3} + \mathbf{3}'$$

Franc, Mason, 1503.05519

Modular forms of weight 2

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Franc, Mason, 1503.05519

How to construct the triplets?

Jacobi theta function

Jacobi theta function:

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n z\right)$$

Jacobi theta function

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Useful properties:

Kharchev, Zabrodin, 1502.04603

$$\theta_3(z+1,\tau) = \theta_3(z,\tau)\,, \quad \theta_3(z+\tau,\tau) = e^{-\pi i(2z+\tau)}\theta_3(z,\tau)$$
$$\theta_3(z+1/2,\tau) = \theta_4(z,\tau)\,, \quad \theta_3(z+\tau/2,\tau) = e^{-\pi i(z+\tau/4)}\theta_2(z,\tau)$$
$$\theta_3(z,\tau+1) = \theta_4(z,\tau)\,, \quad \theta_3(z/\tau,-1/\tau) = \sqrt{-i\tau}\,e^{\pi i z^2/\tau}\theta_3(z,\tau)$$

θ_1, θ_2 and θ_4 are auxiliary theta functions

Jacobi theta function

Jacobi theta function:

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n z\right)$$

Useful properties:

Kharchev, Zabrodin, 1502.04603

$$\theta_3(z+1,\tau) = \theta_3(z,\tau)\,, \quad \theta_3(z+\tau,\tau) = e^{-\pi i(2z+\tau)}\theta_3(z,\tau)$$

$$\theta_3(z+1/2,\tau) = \theta_4(z,\tau)\,, \quad \theta_3(z+\tau/2,\tau) = e^{-\pi i(z+\tau/4)}\theta_2(z,\tau)$$

$$\theta_3(z,\tau+1) = \theta_4(z,\tau)\,, \quad \theta_3(z/\tau,-1/\tau) = \sqrt{-i\tau}\,e^{\pi i z^2/\tau}\theta_3(z,\tau)$$

θ_1, θ_2 and θ_4 are auxiliary theta functions

Alternative construction invoking Klein forms has been worked out in
Ding, King, Liu, 1903.12588

12 seed functions

$$\alpha_{1,-1}(\tau) \equiv \theta_3\left(\frac{\tau+1}{2}, 5\tau\right) \qquad \alpha_{2,-1}(\tau) \equiv e^{2\pi i\tau/5}\,\theta_3\left(\frac{3\tau+1}{2}, 5\tau\right)$$

$$\alpha_{1,0}(\tau) \equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau}{5}\right) \qquad \alpha_{2,0}(\tau) \equiv \theta_3\left(\frac{\tau+7}{10}, \frac{\tau}{5}\right)$$

$$\alpha_{1,1}(\tau) \equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+1}{5}\right) \qquad \alpha_{2,1}(\tau) \equiv \theta_3\left(\frac{\tau+8}{10}, \frac{\tau+1}{5}\right)$$

$$\alpha_{1,2}(\tau) \equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+2}{5}\right) \qquad \alpha_{2,2}(\tau) \equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau+2}{5}\right)$$

$$\alpha_{1,3}(\tau) \equiv \theta_3\left(\frac{\tau+2}{10}, \frac{\tau+3}{5}\right) \qquad \alpha_{2,3}(\tau) \equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+3}{5}\right)$$

$$\alpha_{1,4}(\tau) \equiv \theta_3\left(\frac{\tau+3}{10}, \frac{\tau+4}{5}\right) \qquad \alpha_{2,4}(\tau) \equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+4}{5}\right)$$

12 seed functions

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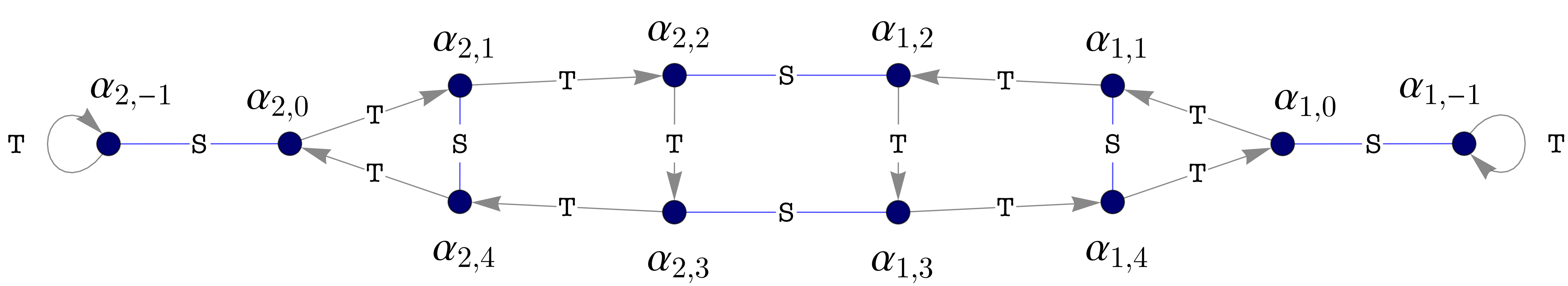
$$\alpha_{1,4}(\tau) \equiv \theta_3\left(\frac{\tau+3}{10}, \frac{\tau+4}{5}\right) \qquad \alpha_{2,4}(\tau) \equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+4}{5}\right)$$



Modular forms of weight 2

$$Y(c_{1,-1}, \ldots, c_{1,4}; c_{2,-1}, \ldots, c_{2,4} \,|\, \tau) \equiv \sum_{i,j} c_{i,j} \frac{\mathrm{d}}{\mathrm{d}\tau} \log \alpha_{i,j}(\tau)\,, \qquad \sum_{i,j} c_{i,j} = 0$$

Modular forms of weight 2

$$Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}\,|\,\tau)\equiv\sum_{i,j}c_{i,j}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(\tau)\,,\qquad \sum_{i,j}c_{i,j}=0$$

$$Y_{\mathbf{5}}(\tau)=\begin{pmatrix}Y_1(\tau)\\Y_2(\tau)\\Y_3(\tau)\\Y_4(\tau)\\Y_5(\tau)\end{pmatrix}\equiv\begin{pmatrix}-\frac{1}{\sqrt{6}}Y\left(-5,1,1,1,1,1;-5,1,1,1,1,1\,|\,\tau\right)\\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,|\,\tau)\\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,|\,\tau)\\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,|\,\tau)\\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,|\,\tau)\end{pmatrix}$$

Modular forms of weight 2

$$Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}\,|\,\tau) \equiv \sum_{i,j} c_{i,j}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(\tau)\,, \qquad \sum_{i,j} c_{i,j} = 0$$

$$Y_{\mathbf{5}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{\sqrt{6}}Y\left(-5,1,1,1,1,1;-5,1,1,1,1,1\,|\,\tau\right) \\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,|\,\tau) \\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,|\,\tau) \\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,|\,\tau) \\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,|\,\tau) \end{pmatrix}$$

$$Y_{\mathbf{3}}(\tau) = \begin{pmatrix} Y_6(\tau) \\ Y_7(\tau) \\ Y_8(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}Y\left(-\sqrt{5},-1,-1,-1,-1,-1;\sqrt{5},1,1,1,1,1\,|\,\tau\right) \\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,-1,-\zeta^4,-\zeta^3,-\zeta^2,-\zeta\,|\,\tau) \\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,-1,-\zeta,-\zeta^2,-\zeta^3,-\zeta^4\,|\,\tau) \end{pmatrix}$$

Modular forms of weight 2

$$Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}\,|\,\tau) \equiv \sum_{i,j} c_{i,j}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(\tau)\,, \qquad \sum_{i,j} c_{i,j}=0$$

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$$Y_{\mathbf{3}}(\tau)=\begin{pmatrix}Y_6(\tau)\\Y_7(\tau)\\Y_8(\tau)\end{pmatrix}\equiv\begin{pmatrix}\frac{1}{\sqrt{2}}Y\left(-\sqrt{5},-1,-1,-1,-1,-1;\sqrt{5},1,1,1,1,1\,|\,\tau\right)\\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,-1,-\zeta^4,-\zeta^3,-\zeta^2,-\zeta\,|\,\tau)\\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,-1,-\zeta,-\zeta^2,-\zeta^3,-\zeta^4\,|\,\tau)\end{pmatrix}$$

$$Y_{\mathbf{3'}}(\tau)=\begin{pmatrix}Y_9(\tau)\\Y_{10}(\tau)\\Y_{11}(\tau)\end{pmatrix}\equiv\begin{pmatrix}\frac{1}{\sqrt{2}}Y\left(\sqrt{5},-1,-1,-1,-1,-1;-\sqrt{5},1,1,1,1,1\,|\,\tau\right)\\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,-1,-\zeta^3,-\zeta,-\zeta^4,-\zeta^2\,|\,\tau)\\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,-1,-\zeta^2,-\zeta^4,-\zeta,-\zeta^3\,|\,\tau)\end{pmatrix}$$

Modular forms of weight 2

$$Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}\,|\,\tau) \equiv \sum_{i,j} c_{i,j}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(\tau)\,, \qquad \sum_{i,j} c_{i,j} = 0$$

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$$Y_{\mathbf{3'}}(\tau) = \begin{pmatrix} Y_9(\tau) \\ Y_{10}(\tau) \\ Y_{11}(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}Y\left(\sqrt{5},-1,-1,-1,-1,-1;-\sqrt{5},1,1,1,1,1\,|\,\tau\right) \\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,-1,-\zeta^3,-\zeta,-\zeta^4,-\zeta^2\,|\,\tau) \\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,-1,-\zeta^2,-\zeta^4,-\zeta,-\zeta^3\,|\,\tau) \end{pmatrix}$$

A_5 quintet and triplets of weight 2 modular forms

Modular forms of higher weight

Weight 4: $Y_i Y_j$ 66 combinations - 45 constraints

= 21 independent combinations

N\k	0	2	4	6
5	1	11	**21**	31

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$$Y_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5$$

$$Y_{\mathbf{3}}^{(4)} = \begin{pmatrix} -2Y_1Y_6 + \sqrt{3}\,Y_5Y_7 + \sqrt{3}\,Y_2Y_8 \\ \sqrt{3}\,Y_2Y_6 + Y_1Y_7 - \sqrt{6}\,Y_3Y_8 \\ \sqrt{3}\,Y_5Y_6 - \sqrt{6}\,Y_4Y_7 + Y_1Y_8 \end{pmatrix}$$

$$Y_{\mathbf{3'}}^{(4)} = \begin{pmatrix} \sqrt{3}\,Y_1Y_6 + Y_5Y_7 + Y_2Y_8 \\ Y_3Y_6 - \sqrt{2}\,Y_2Y_7 - \sqrt{2}\,Y_4Y_8 \\ Y_4Y_6 - \sqrt{2}\,Y_3Y_7 - \sqrt{2}\,Y_5Y_8 \end{pmatrix}$$

$$Y_{\mathbf{4}}^{(4)} = \begin{pmatrix} 2Y_4^2 + \sqrt{6}\,Y_1Y_2 - Y_3Y_5 \\ 2Y_2^2 + \sqrt{6}\,Y_1Y_3 - Y_4Y_5 \\ 2Y_5^2 - Y_2Y_3 + \sqrt{6}\,Y_1Y_4 \\ 2Y_3^2 - Y_2Y_4 + \sqrt{6}\,Y_1Y_5 \end{pmatrix}$$

$$Y_{\mathbf{5},1}^{(4)} = \begin{pmatrix} \sqrt{2}\,Y_1^2 + \sqrt{2}\,Y_3Y_4 - 2\sqrt{2}\,Y_2Y_5 \\ \sqrt{3}\,Y_4^2 - 2\sqrt{2}\,Y_1Y_2 \\ \sqrt{2}\,Y_1Y_3 + 2\sqrt{3}\,Y_4Y_5 \\ 2\sqrt{3}\,Y_2Y_3 + \sqrt{2}\,Y_1Y_4 \\ \sqrt{3}\,Y_3^2 - 2\sqrt{2}\,Y_1Y_5 \end{pmatrix}$$

$$Y_{\mathbf{5},2}^{(4)} = \begin{pmatrix} \sqrt{3}\,Y_5Y_7 - \sqrt{3}\,Y_2Y_8 \\ -Y_2Y_6 - \sqrt{3}\,Y_1Y_7 - \sqrt{2}\,Y_3Y_8 \\ -2Y_3Y_6 - \sqrt{2}\,Y_2Y_7 \\ 2Y_4Y_6 + \sqrt{2}\,Y_5Y_8 \\ Y_5Y_6 + \sqrt{2}\,Y_4Y_7 + \sqrt{3}\,Y_1Y_8 \end{pmatrix}$$

Modular forms of higher weight

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N\k	0	2	4	6
5	1	11	**21**	31

$$Y^{(4)}_{\mathbf{1}} = Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5$$

$$Y^{(4)}_{\mathbf{3}} = \begin{pmatrix} -2Y_1Y_6 + \sqrt{3}\,Y_5Y_7 + \sqrt{3}\,Y_2Y_8 \\ \sqrt{3}\,Y_2Y_6 + Y_1Y_7 - \sqrt{6}\,Y_3Y_8 \\ \sqrt{3}\,Y_5Y_6 - \sqrt{6}\,Y_4Y_7 + Y_1Y_8 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{3'}} = \begin{pmatrix} \sqrt{3}\,Y_1Y_6 + Y_5Y_7 + Y_2Y_8 \\ Y_3Y_6 - \sqrt{2}\,Y_2Y_7 - \sqrt{2}\,Y_4Y_8 \\ Y_4Y_6 - \sqrt{2}\,Y_3Y_7 - \sqrt{2}\,Y_5Y_8 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{4}} = \begin{pmatrix} 2Y_4^2 + \sqrt{6}\,Y_1Y_2 - Y_3Y_5 \\ 2Y_2^2 + \sqrt{6}\,Y_1Y_3 - Y_4Y_5 \\ 2Y_5^2 - Y_2Y_3 + \sqrt{6}\,Y_1Y_4 \\ 2Y_3^2 - Y_2Y_4 + \sqrt{6}\,Y_1Y_5 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{5},1} = \begin{pmatrix} \sqrt{2}\,Y_1^2 + \sqrt{2}\,Y_3Y_4 - 2\sqrt{2}\,Y_2Y_5 \\ \sqrt{3}\,Y_4^2 - 2\sqrt{2}\,Y_1Y_2 \\ \sqrt{2}\,Y_1Y_3 + 2\sqrt{3}\,Y_4Y_5 \\ 2\sqrt{3}\,Y_2Y_3 + \sqrt{2}\,Y_1Y_4 \\ \sqrt{3}\,Y_3^2 - 2\sqrt{2}\,Y_1Y_5 \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{5},2} = \begin{pmatrix} \sqrt{3}\,Y_5Y_7 - \sqrt{3}\,Y_2Y_8 \\ -Y_2Y_6 - \sqrt{3}\,Y_1Y_7 - \sqrt{2}\,Y_3Y_8 \\ -2Y_3Y_6 - \sqrt{2}\,Y_2Y_7 \\ 2Y_4Y_6 + \sqrt{2}\,Y_5Y_8 \\ Y_5Y_6 + \sqrt{2}\,Y_4Y_7 + \sqrt{3}\,Y_1Y_8 \end{pmatrix}$$

Modular form multiplets of weights 6, 8 and 10 have been derived in
Novichkov, Penedo, Petcov, AT, 1812.02158

Examples of modular A5 models

Weinberg operator

$$W \supset \frac{g}{\Lambda} \left(L \, H_u \, L \, H_u \, Y \right)_{\mathbf{1}}$$

Diagonal charged leptons assumed (more on that later)

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Weight	$k_Y/2$	0	k_Y
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	L	H_u	Y
Weight	$k_Y/2$	0	k_Y
A_5	**3** (**3'**)	**1**	$\mathbf{r}_Y$

$$k_Y = 0: \qquad \mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} = \mathbf{1} \oplus \mathbf{3}^{(\prime)} \oplus \mathbf{5} \qquad (LL)_{\mathbf{1}} = L_1L_1 + L_2L_3 + L_3L_2$$

$$M_\nu = \frac{2v_u^2 g}{\Lambda}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \Rightarrow$$ degenerate neutrino masses

Examples of modular A5 models

$k_Y = 2: \qquad \left(\mathbf{1} \oplus \cancel{\mathbf{3}}^{(\prime)} \oplus \mathbf{5}\right) \otimes \mathbf{r}_Y \;\Rightarrow\; \mathbf{r}_Y = \mathbf{5} \qquad (LLY_{\mathbf{5}})_{\mathbf{1}} = \ldots$

$$M_\nu^{\mathbf{3}^{(\prime)}}(\tau) = \frac{v_u^2 g}{\Lambda} \begin{pmatrix} 2Y_1(\tau) & -\sqrt{3}Y_{5(4)}(\tau) & -\sqrt{3}Y_{2(3)}(\tau) \\ -\sqrt{3}Y_{5(4)}(\tau) & \sqrt{6}Y_{4(2)}(\tau) & -Y_1(\tau) \\ -\sqrt{3}Y_{2(3)}(\tau) & -Y_1(\tau) & \sqrt{6}Y_{3(5)}(\tau) \end{pmatrix}$$

Examples of modular A5 models

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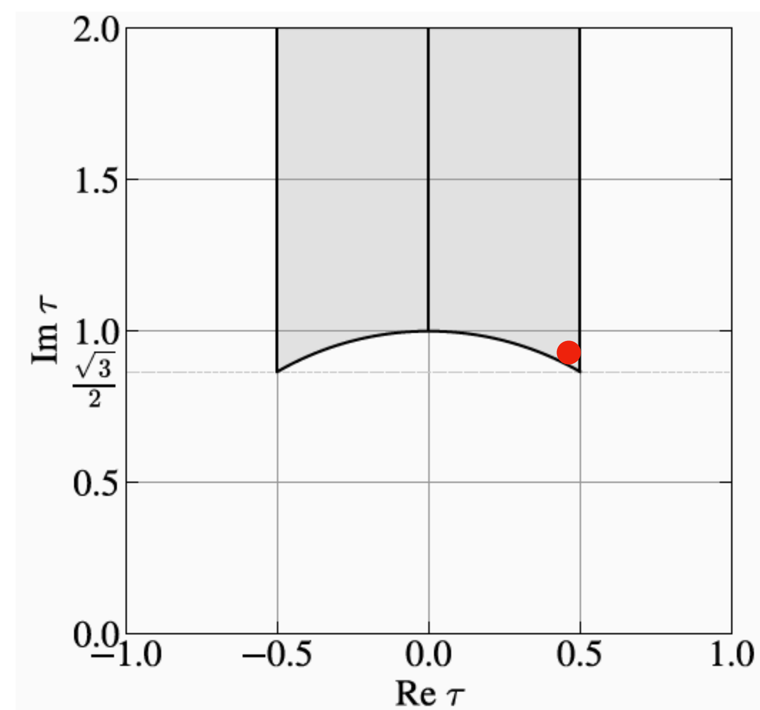
$$\mathbf{r}_L = \mathbf{3}' \qquad \langle\tau\rangle = 0.48 + 0.873\, i$$

$m_1 = 0.020$ eV, $\quad m_2 = 0.022$ eV, $\quad m_3 = 0.054$ eV,

$\sin^2\theta_{12} = 0.325, \quad \sin^2\theta_{13} = 0.166, \quad \sin^2\theta_{23} = 0.421,$

$\delta = 1.50\,\pi, \quad \alpha_{21} = 1.90\,\pi, \quad \alpha_{31} = 1.95\,\pi$



Examples of modular A5 models

$$k_Y = 4: \qquad \left(\mathbf{1} \oplus \cancel{\mathbf{3}}^{(\prime)} \oplus \mathbf{5}\right) \otimes \mathbf{r}_Y \;\Rightarrow\; \mathbf{r}_Y = \mathbf{1},\, \mathbf{5}$$

$$W \supset \frac{1}{\Lambda}\left\{ g_1 \left(L H_u L H_u Y^{(4)}_{\mathbf{1}}\right)_{\mathbf{1}} + g_2 \left(L H_u L H_u Y^{(4)}_{\mathbf{5},1}\right)_{\mathbf{1}} + g_3 \left(L H_u L H_u Y^{(4)}_{\mathbf{5},2}\right)_{\mathbf{1}} \right\}$$

Examples of modular A5 models

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$$M_\nu^{\mathbf{3}'} = \frac{2v_u^2 g_1}{\Lambda}\left[(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5)\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right.$$

$$+\frac{g_2}{g_1}\begin{pmatrix} Y_1^2 + Y_3Y_4 - 2Y_2Y_5 & -\frac{3}{\sqrt{2}}Y_2Y_3 - \frac{\sqrt{3}}{2}Y_1Y_4 & -\frac{\sqrt{3}}{2}Y_1Y_3 - \frac{3}{\sqrt{2}}Y_4Y_5 \\ * & \frac{3}{2}Y_4^2 - \sqrt{6}Y_1Y_2 & Y_2Y_5 - \frac{1}{2}(Y_1^2 + Y_3Y_4) \\ * & * & \frac{3}{2}Y_3^2 - \sqrt{6}Y_1Y_5 \end{pmatrix}$$

$$\left.+\frac{g_3}{g_1}\begin{pmatrix} Y_5Y_7 - Y_2Y_8 & -Y_4Y_6 - \frac{1}{\sqrt{2}}Y_5Y_8 & Y_3Y_6 + \frac{1}{\sqrt{2}}Y_2Y_7 \\ * & -\frac{1}{\sqrt{2}}Y_2Y_6 - \sqrt{\frac{3}{2}}Y_1Y_7 - Y_3Y_8 & \frac{1}{2}(Y_2Y_8 - Y_5Y_7) \\ * & * & \frac{1}{\sqrt{2}}Y_5Y_6 + Y_4Y_7 + \sqrt{\frac{3}{2}}Y_1Y_8 \end{pmatrix}\right]$$

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$$k_Y = 4: \qquad (\mathbf{1} \oplus \cancel{\mathbf{3}}^{(\prime)} \oplus \mathbf{5}) \otimes \mathbf{r}_Y \;\Rightarrow\; \mathbf{r}_Y = \mathbf{1},\, \mathbf{5}$$

$$W \supset \frac{1}{\Lambda}\left\{ g_1 \left(L H_u L H_u Y^{(4)}_{\mathbf{1}} \right)_{\mathbf{1}} + g_2 \left(L H_u L H_u Y^{(4)}_{\mathbf{5},1} \right)_{\mathbf{1}} + g_3 \left(L H_u L H_u Y^{(4)}_{\mathbf{5},2} \right)_{\mathbf{1}} \right\}$$

$$M_\nu^{\mathbf{3}'} = \frac{2 v_u^2 g_1}{\Lambda} \left[(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right.$$

7 real parameters

$$+ \frac{g_2}{g_1} \begin{pmatrix} Y_1^2 + Y_3Y_4 - 2Y_2Y_5 & -\frac{3}{\sqrt{2}}Y_2Y_3 - \frac{\sqrt{3}}{2}Y_1Y_4 & -\frac{\sqrt{3}}{2}Y_1Y_3 - \frac{3}{\sqrt{2}}Y_4Y_5 \\ * & \frac{3}{2}Y_4^2 - \sqrt{6}Y_1Y_2 & Y_2Y_5 - \frac{1}{2}(Y_1^2 + Y_3Y_4) \\ * & * & \frac{3}{2}Y_3^2 - \sqrt{6}Y_1Y_5 \end{pmatrix}$$

$$\left. + \frac{g_3}{g_1} \begin{pmatrix} Y_5Y_7 - Y_2Y_8 & -Y_4Y_6 - \frac{1}{\sqrt{2}}Y_5Y_8 & Y_3Y_6 + \frac{1}{\sqrt{2}}Y_2Y_7 \\ * & -\frac{1}{\sqrt{2}}Y_2Y_6 - \sqrt{\frac{3}{2}}Y_1Y_7 - Y_3Y_8 & \frac{1}{2}(Y_2Y_8 - Y_5Y_7) \\ * & * & \frac{1}{\sqrt{2}}Y_5Y_6 + Y_4Y_7 + \sqrt{\frac{3}{2}}Y_1Y_8 \end{pmatrix} \right]$$

Examples of modular A5 models

Let $\langle\tau\rangle = i \quad \Rightarrow \quad \mathbb{Z}_2^S$ residual symmetry preserved

5 real parameters: $2v_u^2\, g_1/\Lambda$, $\mathrm{Re}\left(g_2/g_1\right)$, $\mathrm{Im}\left(g_2/g_1\right)$, $\mathrm{Re}\left(g_3/g_1\right)$ and $\mathrm{Im}\left(g_3/g_1\right)$

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$2v_u^2\, g_1/\Lambda \simeq 0.0051$ eV, $\quad g_2/g_1 = -\,0.2205 - 0.1576\, i, \quad g_3/g_1 = 0.0246 - 0.0421\, i$

$$m_1 = 0.042 \text{ eV}, \quad m_2 = 0.043 \text{ eV}, \quad m_3 = 0.065 \text{ eV},$$
$$\textstyle\sum_i m_i = 0.149 \text{ eV}, \quad |\langle m \rangle| = 0.042 \text{ eV},$$
$$\sin^2\theta_{12} = 0.282, \quad \sin^2\theta_{13} = 0.0214, \quad \sin^2\theta_{23} = 0.550,$$
$$\delta = 1.32\,\pi, \quad \alpha_{21} = 1.98\,\pi, \quad \alpha_{31} = 0.93\,\pi$$

$$N\sigma = 1.7$$

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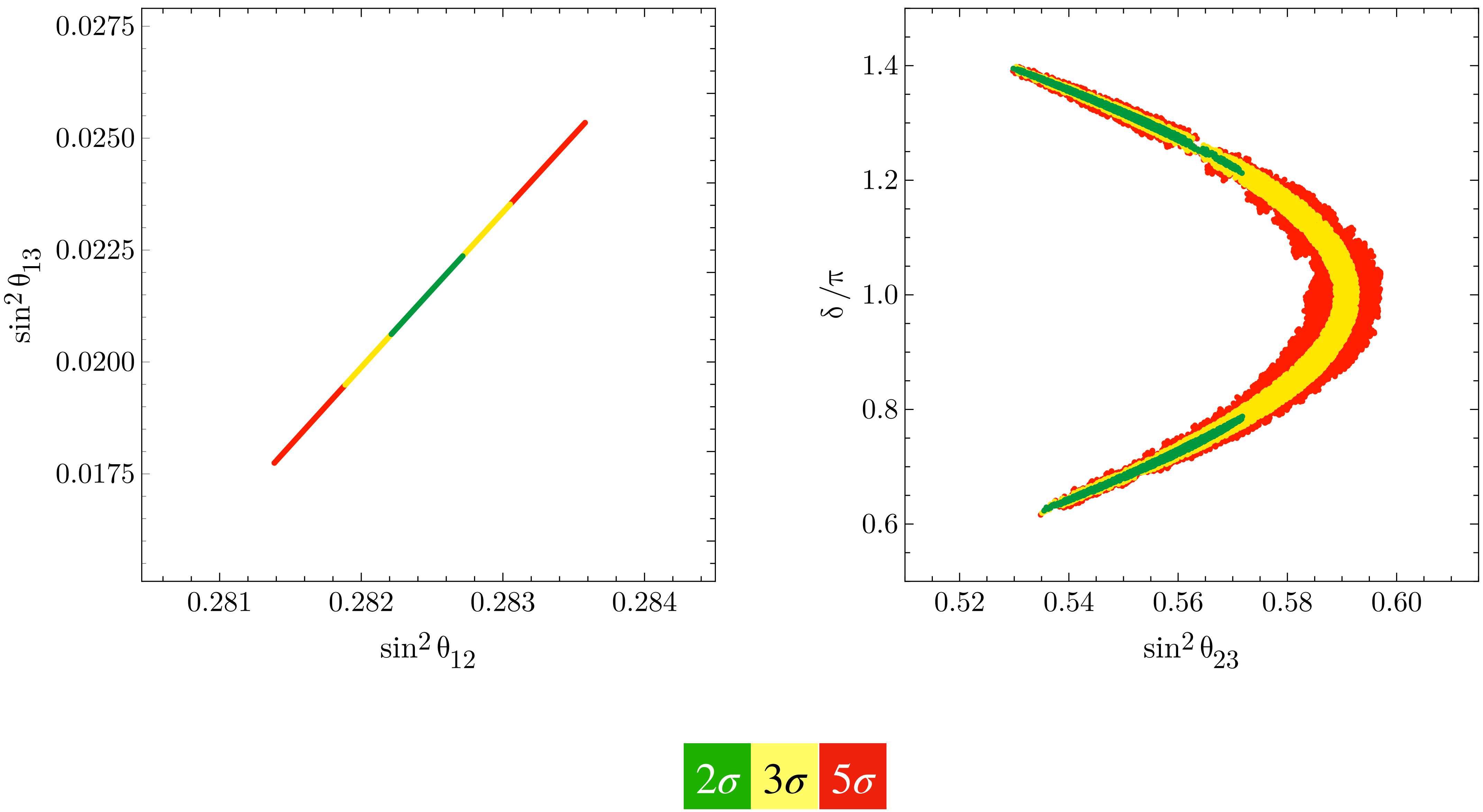

Examples of modular A5 models

Diagonal charged leptons

- Second modulus: $\langle \tau_l \rangle = i\infty \quad \Rightarrow \quad \mathbb{Z}_5^T$ residual symmetry

$$W \supset \alpha_1 \left(E^c L H_d Y^{(4)}_{\mathbf{1}} \right)_{\mathbf{1}} + \alpha_2 \left(E^c L H_d Y^{(4)}_{\mathbf{3}^{(\prime)}} \right)_{\mathbf{1}} + \alpha_3 \left(E^c L H_d Y^{(4)}_{\mathbf{5},1} \right)_{\mathbf{1}}$$

	L	E^c	H_d	Y
Weight	2	2	0	4
A_5	**3** (**3'**)	**3** (**3'**)	**1**	$\mathbf{r}_Y$

$$M_e M_e^\dagger = v_d^2\, \alpha_1^2 \mathrm{diag}\left(\left| 1 + 2\,\frac{\alpha_3}{\alpha_1} \right|^2, \left| 1 - \frac{\alpha_2}{\alpha_1} - \frac{\alpha_3}{\alpha_1} \right|^2, \left| 1 + \frac{\alpha_2}{\alpha_1} - \frac{\alpha_3}{\alpha_1} \right|^2 \right)$$

$$v_d\, \alpha_1 \simeq 660 \text{ MeV}, \quad \alpha_2/\alpha_1 = 1.34 \quad \text{and} \quad 1 + 2\alpha_3/\alpha_1 = -7.7 \times 10^{-4}$$

to fit the charged lepton masses

Examples of modular A5 models

Diagonal charged leptons

- Flavons: $\langle \varphi_i \rangle \quad \Rightarrow \quad \mathbb{Z}_5^T$ residual symmetry

$$W \supset \alpha_1 \left(E^c L H_d \varphi_{\mathbf{1}}\right)_{\mathbf{1}} + \alpha_2 \left(E^c L H_d \varphi_{\mathbf{3}^{(\prime)}}\right)_{\mathbf{1}} + \alpha_3 \left(E^c L H_d \varphi_{\mathbf{5}}\right)_{\mathbf{1}}$$

$-k_{E^c} - k_L > 0 \Rightarrow$ couplings to modular forms are forbidden

$$k_\varphi = k_{E^c} + k_L < 0$$

$$\langle \varphi_{\mathbf{1}} \rangle = v_1, \quad \langle \varphi_{\mathbf{3}^{(\prime)}} \rangle = \left(v_2, 0, 0\right)^T \quad \text{and} \quad \langle \varphi_{\mathbf{5}} \rangle = \left(v_3, 0, 0, 0, 0\right)^T$$

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$$Y_{\mathbf{3}}^{(4)}\Big|_{\langle\tau\rangle = i\infty} = \frac{4\pi^2}{\sqrt{15}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_{\mathbf{3}'}^{(4)}\Big|_{\langle\tau\rangle = i\infty} = -\frac{2\pi^2}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_{\mathbf{5},1}^{(4)}\Big|_{\langle\tau\rangle = i\infty} = -\frac{2\sqrt{2}\pi^2}{3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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- Example of viable modular A5 flavour model
- For more modular A5 models see Ding, King, Liu, 1903.12588

Backup

Generators of A5

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \zeta = e^{2\pi i/5}$$

$\mathbf{1}:$ $\quad \rho(S) = 1 \qquad \rho(T) = 1$

$$\mathbf{3}: \quad \rho(S) = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\varphi & 1/\varphi \\ -\sqrt{2} & 1/\varphi & -\varphi \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$$

$$\mathbf{3}': \quad \rho(S) = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1/\varphi & \varphi \\ \sqrt{2} & \varphi & -1/\varphi \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}$$

$$\mathbf{4}: \quad \rho(S) = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1/\varphi & \varphi & -1 \\ 1/\varphi & -1 & 1 & \varphi \\ \varphi & 1 & -1 & 1/\varphi \\ -1 & \varphi & 1/\varphi & 1 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & \zeta^4 \end{pmatrix}$$

$$\mathbf{5}: \quad \rho(S) = \frac{1}{5}\begin{pmatrix} -1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & 1/\varphi^2 & -2\varphi & 2/\varphi & \varphi^2 \\ \sqrt{6} & -2\varphi & \varphi^2 & 1/\varphi^2 & 2/\varphi \\ \sqrt{6} & 2/\varphi & 1/\varphi^2 & \varphi^2 & -2\varphi \\ \sqrt{6} & \varphi^2 & 2/\varphi & -2\varphi & 1/\varphi^2 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^4 \end{pmatrix}$$

Some properties of modular forms

Feruglio, 1706.08749

N	g	$d_{2k}(\Gamma(N))$	μ_N	Γ_N
2	0	$k+1$	6	S_3
3	0	$2k+1$	12	A_4
4	0	$4k+1$	24	S_4
5	0	$10k+1$	60	A_5
6	1	$12k$	72	
7	3	$28k-2$	168	

$k(\text{this presentation}) \equiv 2\,k(\text{this table})$