Leptonic Unitarity Triangle, Geometric Neutrino Oscillation and CP Violation

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Oscillation in 2ν System

$$P_{\nu_\ell \to \nu_{\ell'}} = A \sin^2 \frac{\Delta m^2 L}{4E}; \; P_{\nu_\ell \to \nu_\ell} = 1 - A \sin^2 \frac{\Delta m^2 L}{4E}$$

L: Baseline Length
E: Neutrino Energy

Oscillation in 3ν System

$$
\begin{aligned}
P_{\nu_\ell \to \nu_{\ell'}} \quad = \quad & |U_{\ell 1} U_{\ell' 1}|^2 + |U_{\ell 2} U_{\ell' 2}|^2 + |U_{\ell 3} U_{\ell' 3}|^2 \\
& + 2 |U_{\ell 2} U_{\ell' 2} U_{\ell 1} U_{\ell' 1}| \cos\left(\frac{\Delta m^2_{21} L}{2E} - \phi_{\ell' \ell; 21}\right) \\
& + 2 |U_{\ell 3} U_{\ell' 3} U_{\ell 2} U_{\ell' 2}| \cos\left(\frac{\Delta m^2_{32} L}{2E} - \phi_{\ell' \ell; 32}\right) \\
& + 2 |U_{\ell 3} U_{\ell' 3} U_{\ell 1} U_{\ell' 1}| \cos\left(\frac{\Delta m^2_{31} L}{2E} - \phi_{\ell' \ell; 31}\right)
\end{aligned}
$$

$$
\phi_{\ell' \ell; jk} = \arg\left(U_{\ell' j} U^*_{\ell j} U_{\ell k} U^*_{\ell' k}\right)
$$

see: PDG-2018, eq.(14.37)

PMNS Mixing Matrix

- **Unitary mixing matrix U (in PDG notations) contains 4 independent parameters (θ_{12}, θ_{23}, θ_{13}, δ) ,**

$$U=\begin{pmatrix} c_s c_x & s_s c_x & s_x \\ -e^{-i\delta} s_s c_a - c_s s_a s_x & e^{-i\delta} c_s c_a - s_s s_a s_x & s_a c_x \\ e^{-i\delta} s_s s_a - c_s c_a s_x & -e^{-i\delta} c_s s_a - s_s c_a s_x & c_a c_x \end{pmatrix},$$

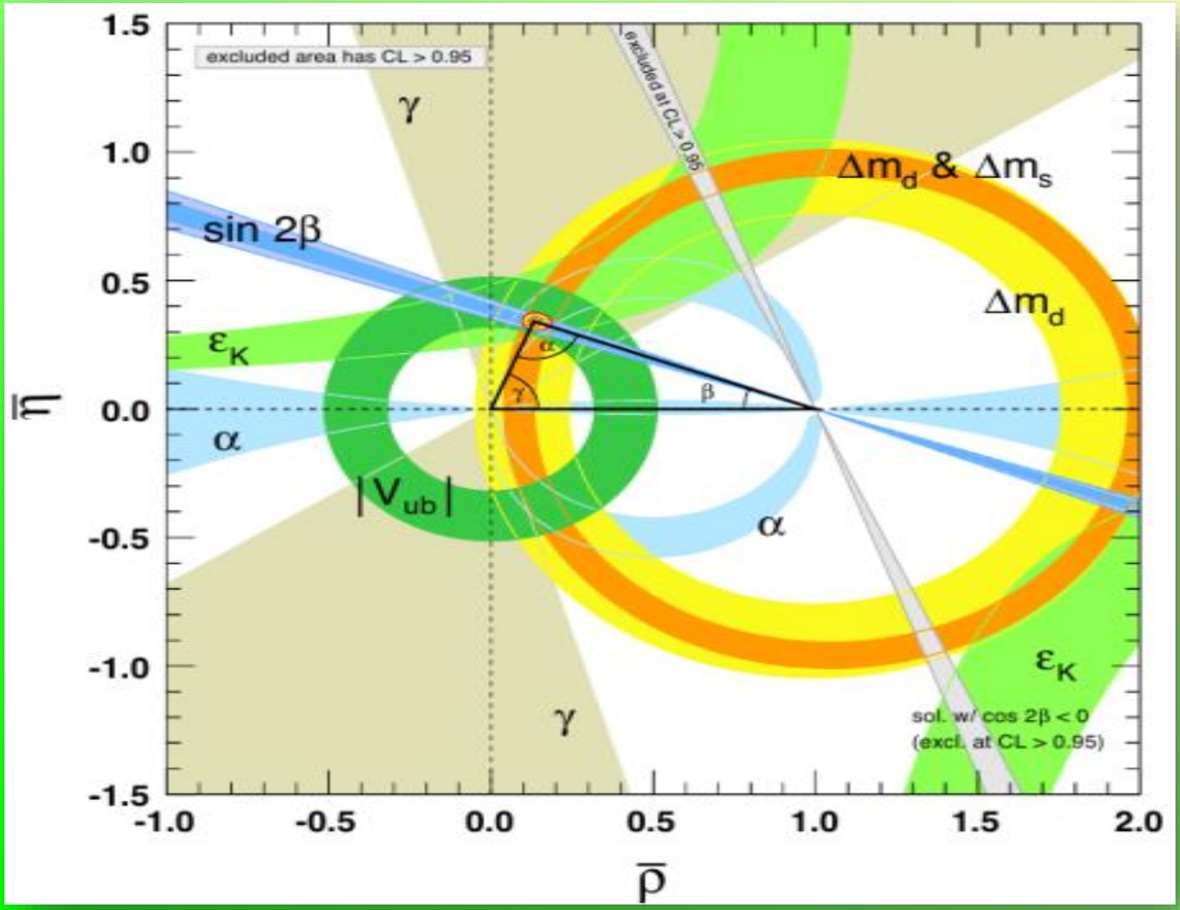
$$(\theta_s,\, \theta_a,\, \theta_x) = (\theta_{12},\, \theta_{23},\, \theta_{13})$$

Connection to Leptonic Unitarity Triangle ?

Quark Unitarity Triangle for CKM

for example, d-b triangle:

$$V_{ud}^* V_{ub} + V_{cd}^* V_{cb} + V_{td}^* V_{tb} = 0 .$$



Key to uncover CPV
in Quark Sector !!

Leptonic Unitarity Triangle (LUT)

- **Dirac LUT:**

$$U_{\ell 1}U^*_{\ell' 1} + U_{\ell 2}U^*_{\ell' 2} + U_{\ell 3}U^*_{\ell' 3} = 0\,, \quad (\ell \neq \ell')$$

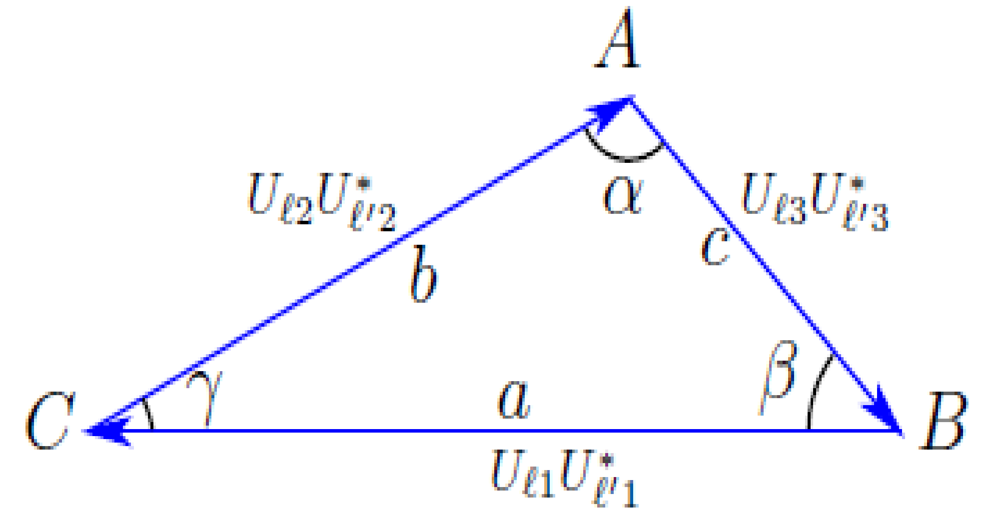
- **Majorana LUT:**

$$U^*_{ej}U_{ej'} + U^*_{\mu j}U_{\mu j'} + U^*_{\tau j}U_{\tau j'} = 0\,, \quad (j \neq j')$$

- **For studying appearance ν-oscillation, we will focus on Dirac LUT .**

Leptonic Unitarity Triangle (LUT)

- **Dirac LUT:** $$U_{\ell 1}U^*_{\ell' 1} + U_{\ell 2}U^*_{\ell' 2} + U_{\ell 3}U^*_{\ell' 3} = 0\,, \quad (\ell \neq \ell')$$
- **Geometrical Presentation:**



- **Each LUT contains only 3 independent parameters**
- **These parameters are rephasing invariant.**

$$(a, b, c) = \left(|U_{\ell 1}U^*_{\ell' 1}|, |U_{\ell 2}U^*_{\ell' 2}|, |U_{\ell 3}U^*_{\ell' 3}|\right),$$

$$(\alpha, \beta, \gamma) = \arg\left(-\frac{U_{\ell 3}U^*_{\ell' 3}}{U_{\ell 2}U^*_{\ell' 2}}, -\frac{U_{\ell 1}U^*_{\ell' 1}}{U_{\ell 3}U^*_{\ell' 3}}, -\frac{U_{\ell 2}U^*_{\ell' 2}}{U_{\ell 1}U^*_{\ell' 1}}\right).$$

Connecting LUT to Oscillation in Vacuum

- **Conventional CP Phase Shift ϕ is complicated function of 4 PMNS parameters (θ_{12}, θ_{23}, θ_{13}, δ):**

$$\boxed{\phi_{\ell'\ell;jk} = \arg(U_{\ell' j}U^*_{\ell j}U_{\ell k}U^*_{\ell' k})}$$

- **We first proved: Each phase shift ϕ equals the corresponding LUT angle (mod π):**

$$(\phi_{\ell'\ell;23}, \phi_{\ell'\ell;31}, \phi_{\ell'\ell;12}) = (\alpha, \beta, \gamma) + \pi$$

HJH & Xu, arXiv:1311.4496, PRD.2014

Connecting LUT to Oscillation in Vacuum

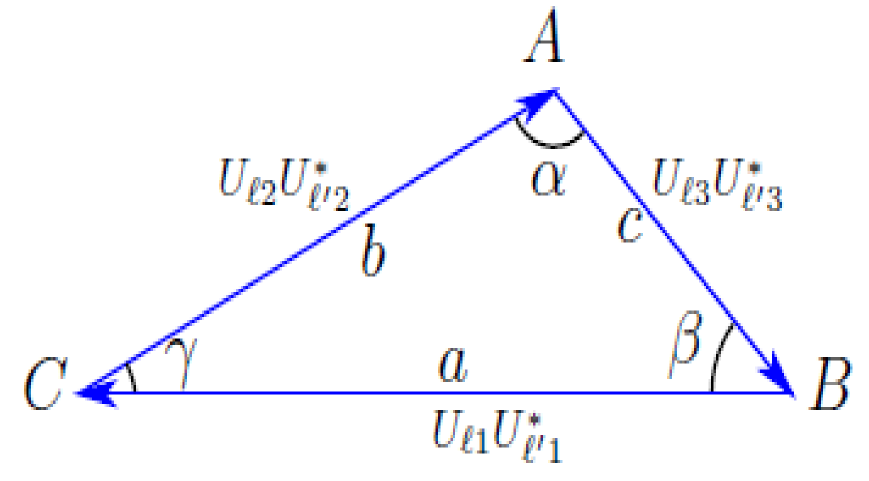
- We derive appearance formula in terms of LUT:

$$\begin{aligned} P_{\ell\to\ell'} = \; & 4ab\sin(\Delta_{12}\mp\gamma)\sin\Delta_{12} \\ & +4bc\sin(\Delta_{23}\mp\alpha)\sin\Delta_{23} \\ & +4ca\sin(\Delta_{31}\mp\beta)\sin\Delta_{31}\,, \end{aligned}$$

- We express it by only 3 Independent LUT parameters:

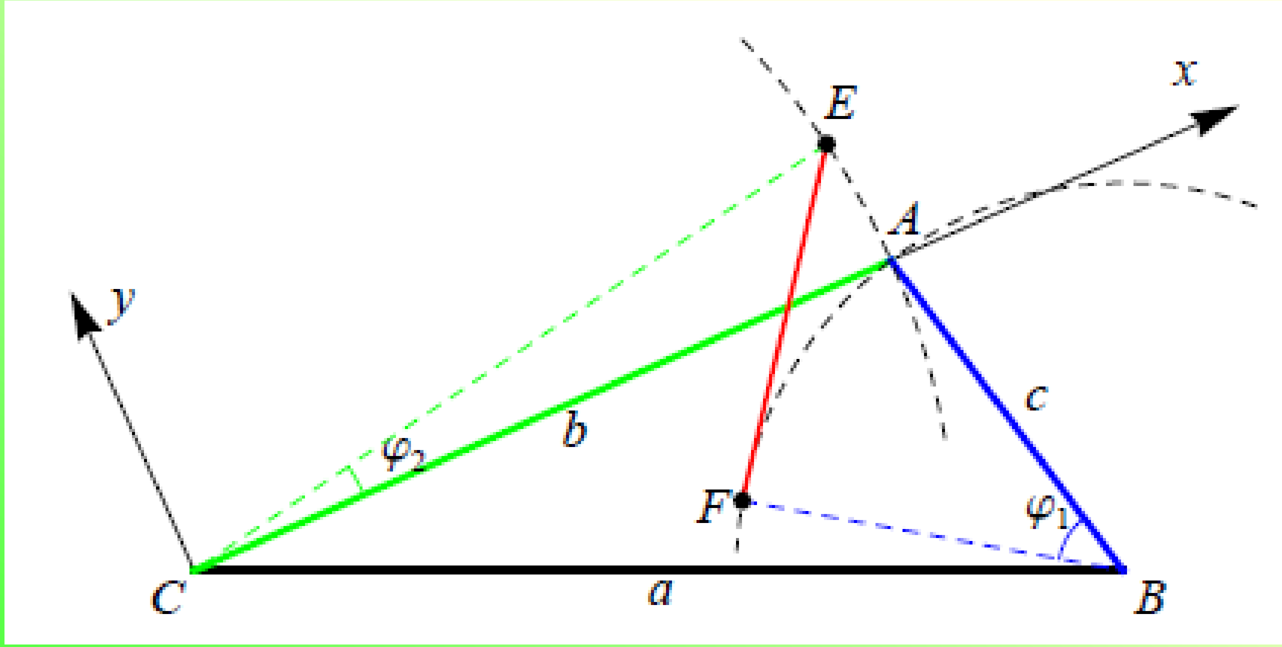
$$a = \sqrt{b^2+c^2-2bc\cos\alpha}\,,$$

$$\gamma = \arcsin\left(\frac{c}{a}\sin\alpha\right), \qquad \beta = \pi-(\alpha+\gamma)$$



HJH and Xu, 1311.4496, PRD.2014

Geometrical Picture of Vacuum Oscillation



$$\begin{aligned} P_{\ell\to\ell'} &= |U_{\ell 1}U^*_{\ell' 1}e^{\mathrm{i}2\Delta_1} + U_{\ell 2}U^*_{\ell' 2}e^{\mathrm{i}2\Delta_2} + U_{\ell 3}U^*_{\ell' 3}e^{\mathrm{i}2\Delta_3}|^2 \\ &= |a + be^{\mathrm{i}(\gamma-\pi)}e^{\mathrm{i}2\Delta_{21}} + ce^{\mathrm{i}(\pi-\beta)}e^{\mathrm{i}2\Delta_{31}}|^2, \end{aligned}$$

➢ **For L = 0: Δ = 0, P = 0 → It reduces to LUT:**

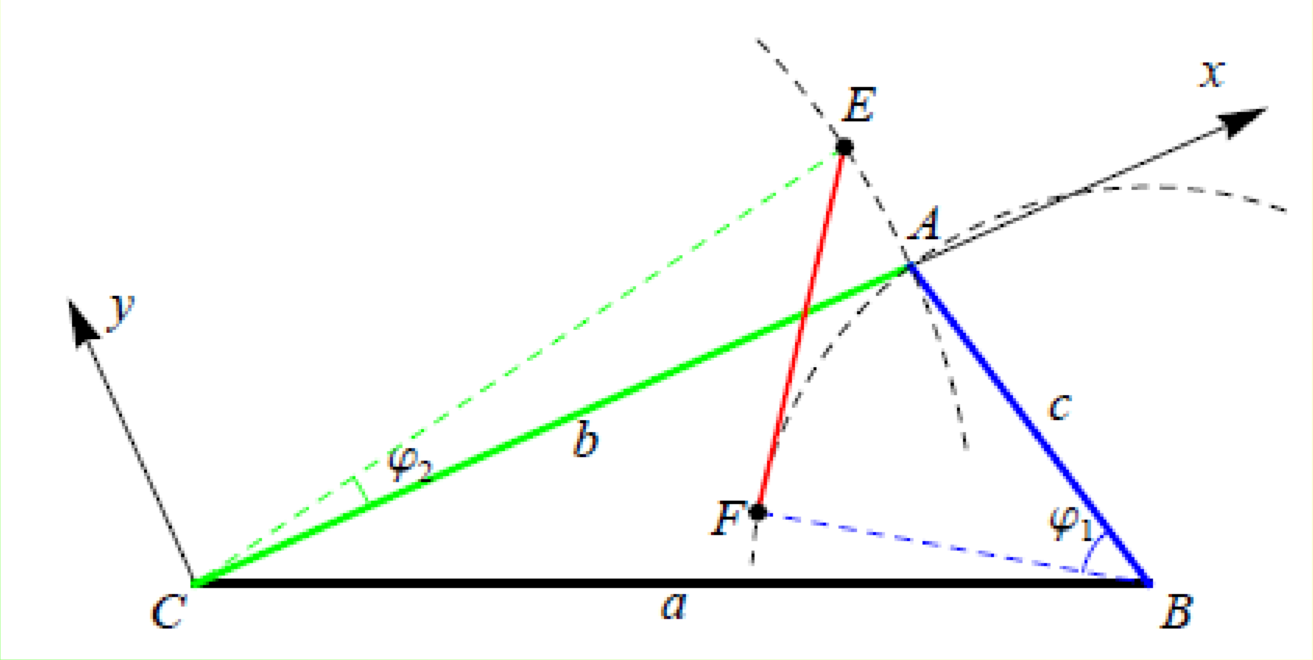
$$a + b\,e^{\mathrm{i}(\gamma-\pi)} + c\,e^{\mathrm{i}(\pi-\beta)} = 0.$$

HJH and Xu,
arXiv:1606.04054, PRD.2017

Geometrical Picture of Vacuum Oscillation

L/E ≠ 0 → P ≠ 0:

$$|EF|^2 = P_{\ell\to\ell'}$$



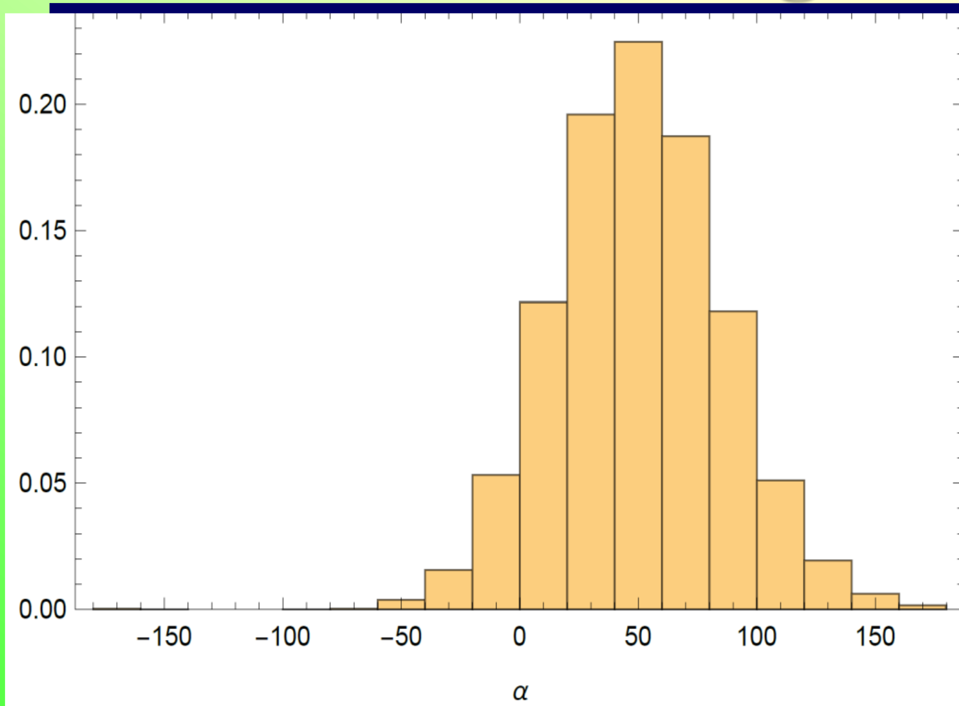
$$P_{\ell\to\ell'} = 4c^2\sin^2\Delta$$
$$-8bc\sin\Delta\sin\epsilon\Delta\cos[(1-\epsilon)\Delta+\alpha]$$
$$+4b^2\sin^2\epsilon\Delta\,,$$

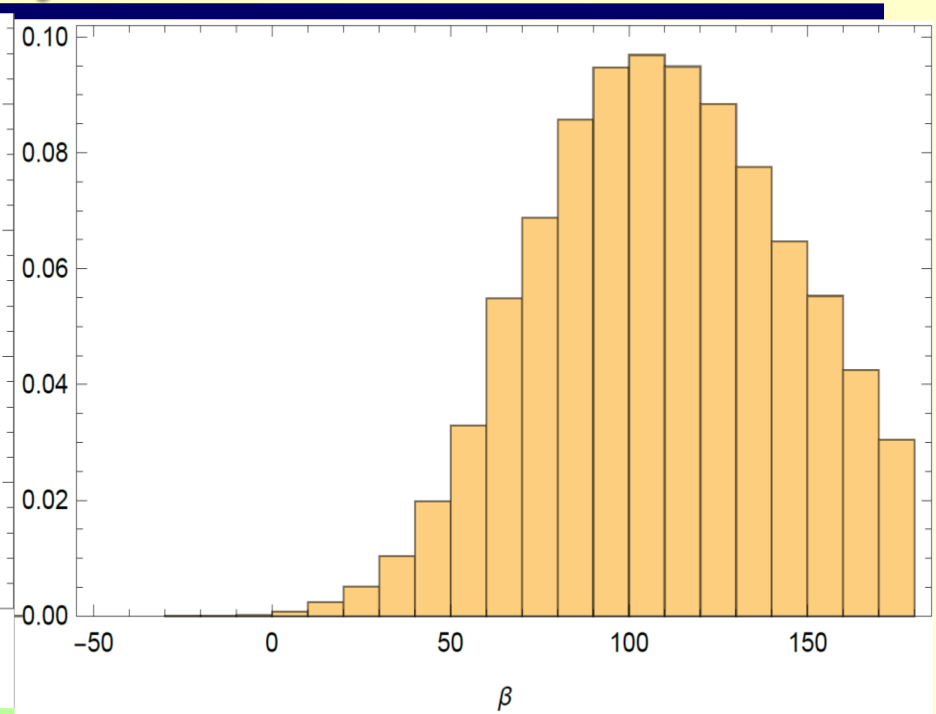
CPV

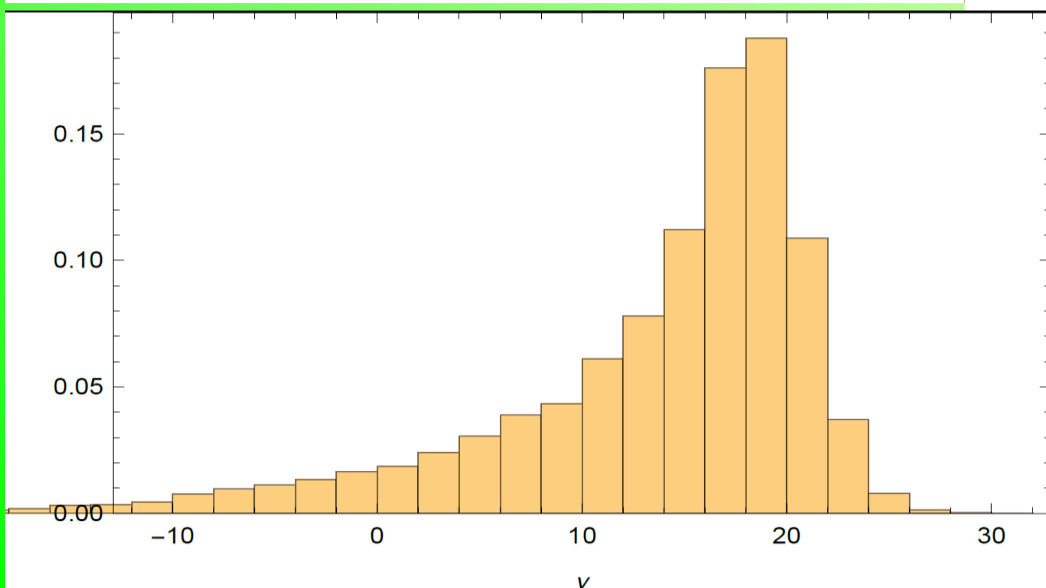
$$\Delta \equiv \Delta_{31} = \frac{\Delta m^2_{31}L}{4E}, \qquad \epsilon \equiv \frac{\Delta_{21}}{\Delta_{31}} = \frac{\Delta m^2_{21}}{\Delta m^2_{31}}\,.$$

HJH and Xu, arXiv:1606.04054, PRD2017

e-μ LUT Angles by Current Data







HJH and Xu, PRD2017
And in preparation.

Phase Shift Effect via LUT

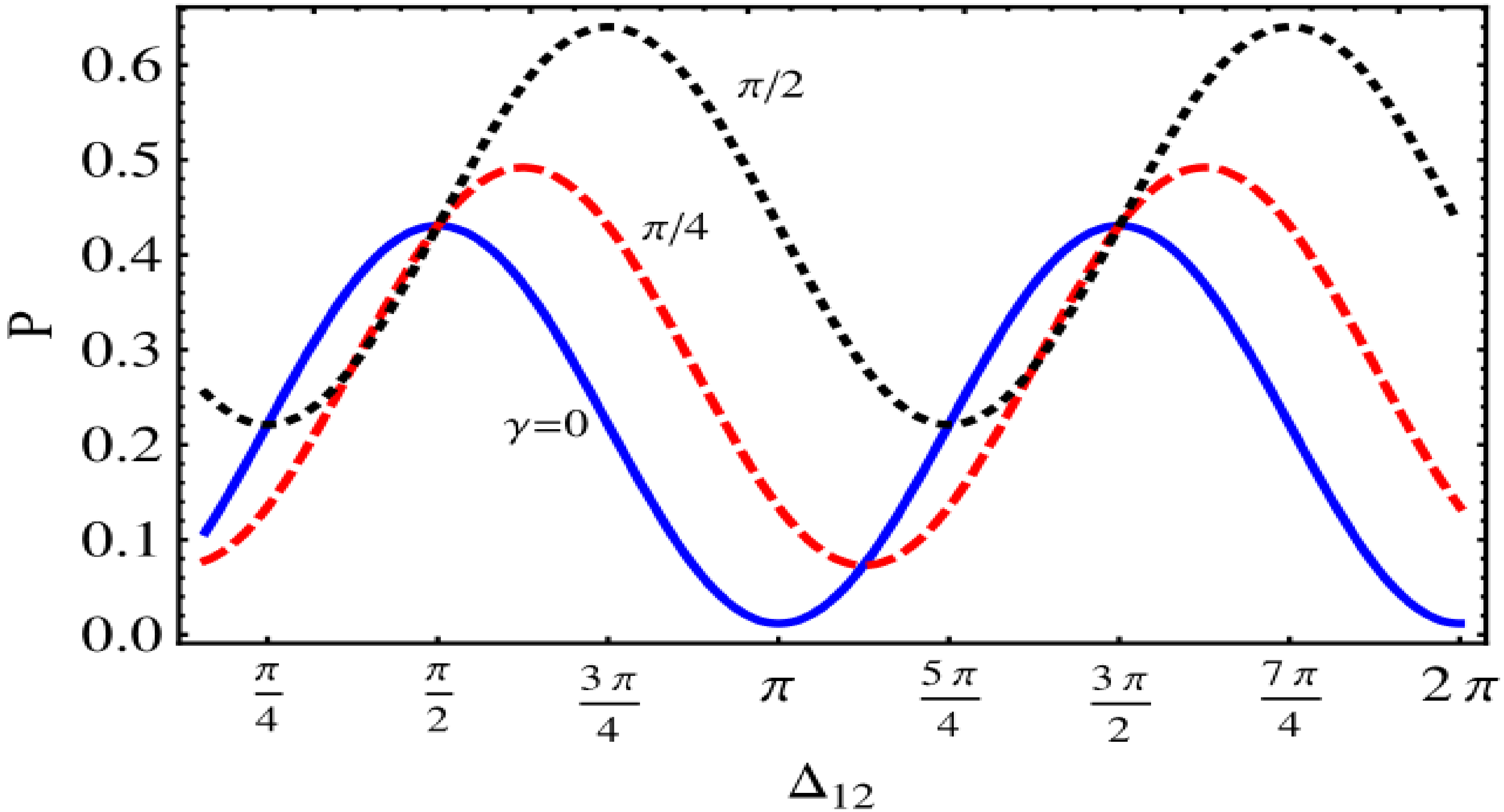


FIG. 3. Phase-shift effects of γ on neutrino oscillation probability $P[\bar{\nu}_\ell \rightarrow \bar{\nu}_{\ell'}]$. For illustration, we plot three curves for $\gamma = 0$ (blue solid), $\frac{\pi}{4}$ (red dashed), and $\frac{\pi}{2}$ (black dotted).

HJH & Xu, 1311.4496, PRD.2014

Some Key Points

- LUT Angles have direct Physical Meaning:
  serve as CPV Phase-Shift in neutrino oscillations.

- Neutrino Oscillation Probability has a fully Geometrical Formulation via LUT: $|EF|^2 = P_{\ell \to \ell'}$
  in terms of (b, c, α).
  ⟶ They are Rephasing Invariant.
  $\alpha \neq 0 \rightarrow$ CP Violation .
  ⟶ ν Probability depends on only 3 independent LUT parameters (rather than 4 in PMNS matrix).

- Allow to directly measure LUT parameters from a given appearance channel.

Connecting LUT to ν Oscillation in Matter

Matter Effects

➢ **Evolution Equation:**

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}L}|\nu(L)\rangle = H|\nu(L)\rangle,$$

➢ **Effective Hamiltonian:**

$$H = H_0 + H_i$$

$$H_0 = \frac{1}{2E}U\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}U^\dagger$$

$$H_0 \gg H_i$$

$$H_i = \sqrt{2}G_F N_e \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix},$$

$$N_e = (Z/A)\,\rho\, N_A\,,$$

Effective Unitarity Triangle (ELUT)

- **To apply our LUT formulation, we need to define effective mixing matrix $U_m = U + \delta U$:**

$$H = \frac{1}{2E} U_m \begin{pmatrix} \tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2 \end{pmatrix} U_m^\dagger .$$

- **With U_m , we define Effective Unitarity Triangle:**

$$(b, c, \alpha) \Rightarrow (b_m, c_m, \alpha_m)$$

Effective Unitarity Triangle (ELUT)

- **In vacuum,**

$$P_{\ell\to\ell'} = 4c^2\sin^2\Delta -8bc\sin\Delta\sin\epsilon\Delta\cos[(1-\epsilon)\Delta+\alpha] +4b^2\sin^2\epsilon\Delta\,,$$

$$(b, c, \alpha)$$

$$\Downarrow$$

- **In matter,**

$$P_{\ell\to\ell'} = 4c_m^2\sin^2\Delta_m -8b_m c_m\sin\Delta_m\sin(\epsilon_m\Delta_m)\cos[(1-\epsilon_m)\Delta_m+\alpha_m] +4b_m^2\sin^2(\epsilon_m\Delta_m)\,,$$

$$(b_m, c_m, \alpha_m)$$

HJH & Xu, arXiv:1606.04054, PRD.2017

How to Compute ELUT ?

$$(b, c, \alpha) \Rightarrow (b_m, c_m, \alpha_m).$$

- **Key Point:** $\epsilon \equiv \Delta m_{21}^2/\Delta m_{31}^2 \approx 0.03 \ll 1$

- **Under perturbative expansion, we derive solutions:**

$$c_m \simeq \frac{c}{1-n_E}, \quad b_m \simeq \frac{\epsilon b}{n_E}, \quad \alpha_m \simeq \alpha \pm \pi,$$

$$\epsilon_m \simeq \frac{-n_E}{1-n_E}, \quad \Delta_m \simeq (1-n_E)\Delta,$$

$$n_E = 2\sqrt{2}\, G_F N_e E/\Delta m_{31}^2 .$$

$$\Delta \equiv \Delta_{31} = \frac{\Delta m_{31}^2 L}{4E}, \quad \epsilon \equiv \frac{\Delta_{21}}{\Delta_{31}} = \frac{\Delta m_{21}^2}{\Delta m_{31}^2}.$$

HJH & Xu, arXiv:1606.04054, PRD.2017

Oscillation in Matter

- **New LUT oscillation formula in matter:**

$$P_{\mathrm{LUT}}(\nu_\mu \to \nu_e) = \frac{4c^2}{(1-n_E)^2}\sin^2[(1-n_E)\Delta] + \frac{4\epsilon^2 b^2}{n_E^2}\sin^2(n_E\Delta) - \frac{8\epsilon bc \sin[(1-n_E)\Delta]\sin(n_E\Delta)\cos(\Delta+\alpha)}{n_E(1-n_E)}.$$

$$\Delta \equiv \Delta_{31} = \frac{\Delta m_{31}^2 L}{4E}, \quad \epsilon \equiv \frac{\Delta_{21}}{\Delta_{31}} = \frac{\Delta m_{21}^2}{\Delta m_{31}^2}.$$

$$n_E = 2\sqrt{2}\, G_F N_e E / \Delta m_{31}^2 .$$

HJH & Xu, arXiv:1606.04054, PRD.2017

Comparison with PDG Formula

$$P_{\rm PDG}(\nu_\mu\to\nu_e) =$$
$$\frac{1}{(1-n_E)^2}\sin^2\theta_a\sin^2 2\theta_x\sin^2[(1-n_E)\Delta]$$
$$-\frac{\epsilon}{n_E(1-n_E)}\sin 2\theta_s\sin 2\theta_a\sin 2\theta_x\cos\theta_x\sin\delta$$
$$\times\sin\Delta\sin(n_E\Delta)\sin[(1-n_E)\Delta]$$
$$+\frac{\epsilon}{n_E(1-n_E)}\sin 2\theta_s\sin 2\theta_a\sin 2\theta_x\cos\theta_x\cos\delta$$
$$\times\cos\Delta\sin(n_E\Delta)\sin[(1-n_E)\Delta]$$
$$+\frac{\epsilon^2}{n_E^2}\sin^2 2\theta_s\cos^2\theta_a\sin^2(n_E\Delta)\,.$$

$$P_{\rm LUT}(\nu_\mu\to\nu_e) =$$
$$\frac{4c^2}{(1-n_E)^2}\sin^2[(1-n_E)\Delta]+\frac{4\epsilon^2 b^2}{n_E^2}\sin^2(n_E\Delta)$$
$$-\frac{8\epsilon bc\sin[(1-n_E)\Delta]\sin(n_E\Delta)\cos(\Delta+\alpha)}{n_E(1-n_E)}.$$

HJH & Xu, arXiv:1606.04054
PRD.2017

See: PDG.2018, Eqs.(14.74-78)

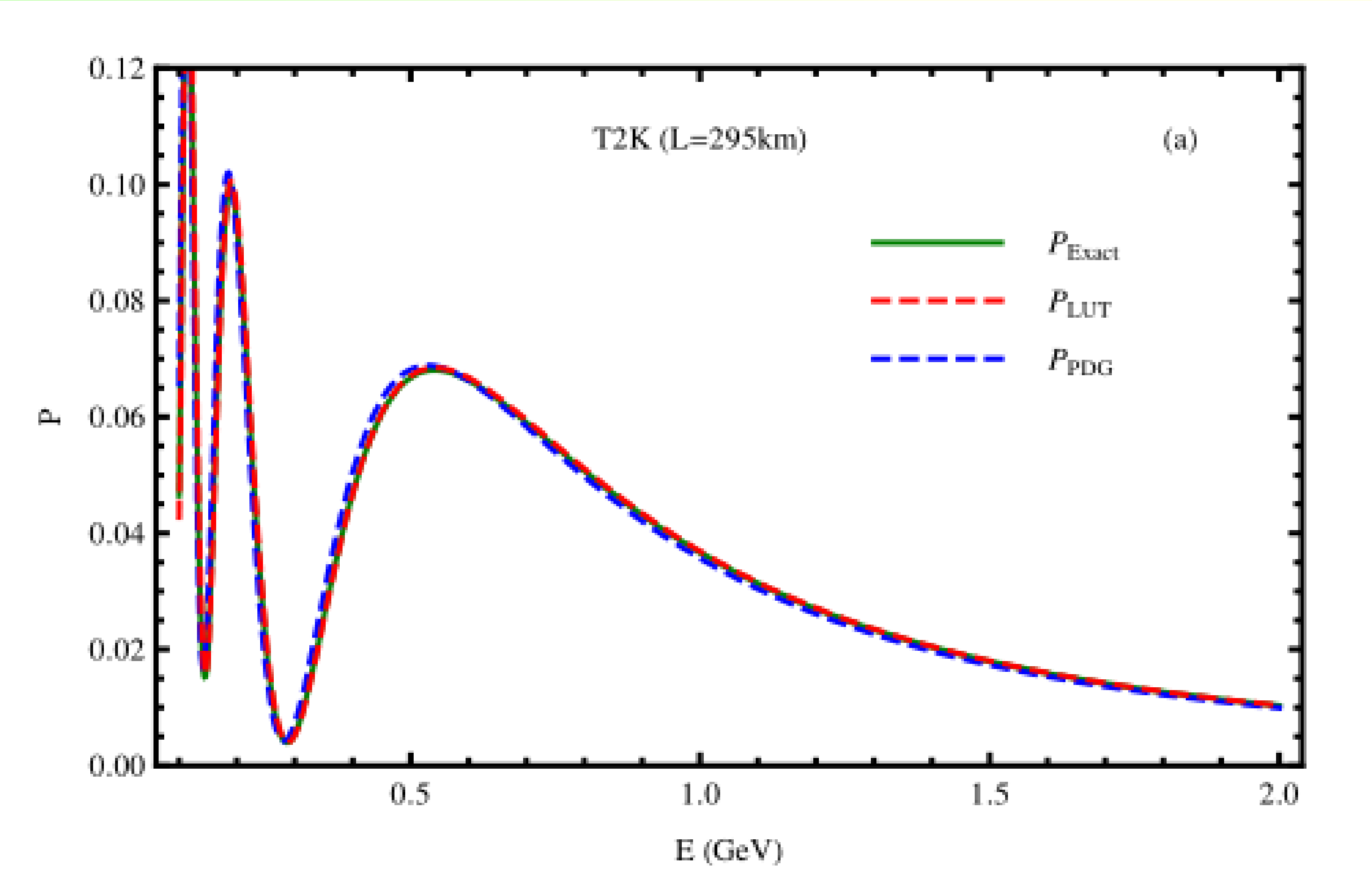
Comparison with PDG Formula

- **More Compact and Simpler.**
- **Has same level of accuracy or better.**
- **Explicit comparisons for T2K, MINOS, NOvA, DUNE: Our analytical LUT formula is precise enough for practical applications !**

HJH & Xu, arXiv:1606.04054, PRD.2017

Comparing ν Probability for T2K:

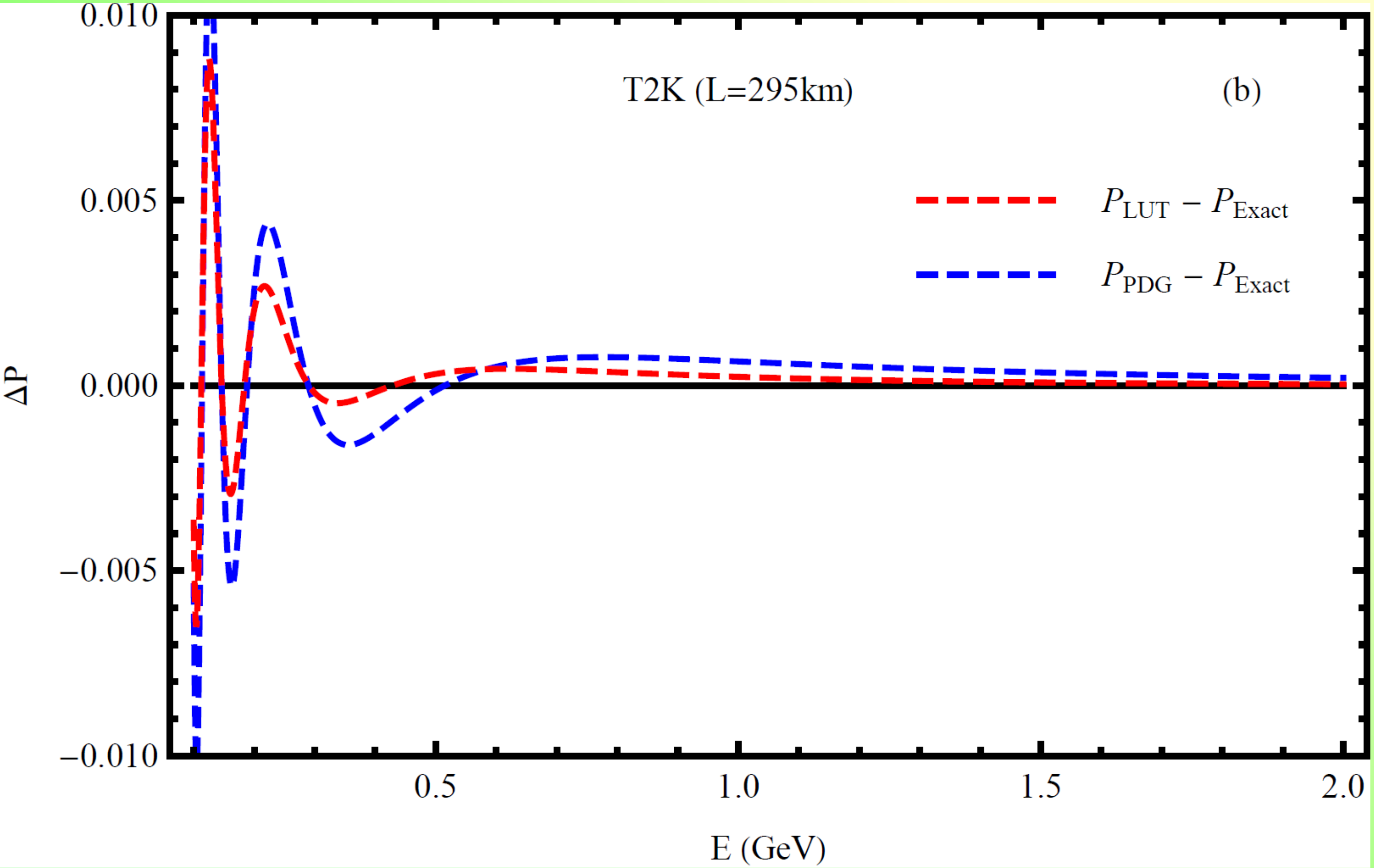
- $\nu_\mu \rightarrow \nu_e$ Oscillation with L = 295 km.



HJH & Xu, arXiv:1606.04054, PRD.2017

Comparing Oscillation Accuracy for T2K:

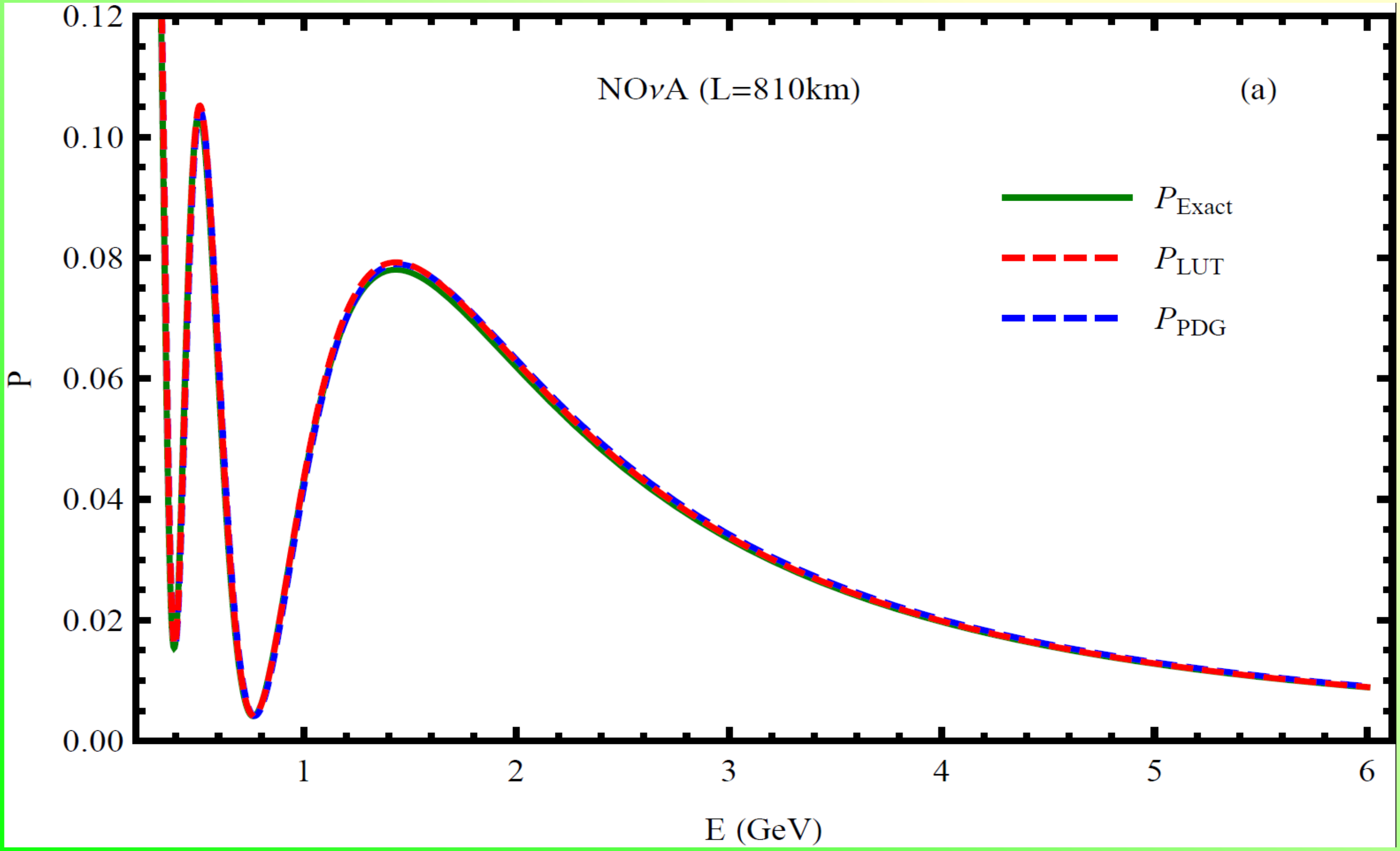
- $\nu_\mu \rightarrow \nu_e$ **Oscillation with L = 295 km.**



HJH & Xu, arXiv:1606.04054, PRD.2017

Comparing ν Probability for NOνA:

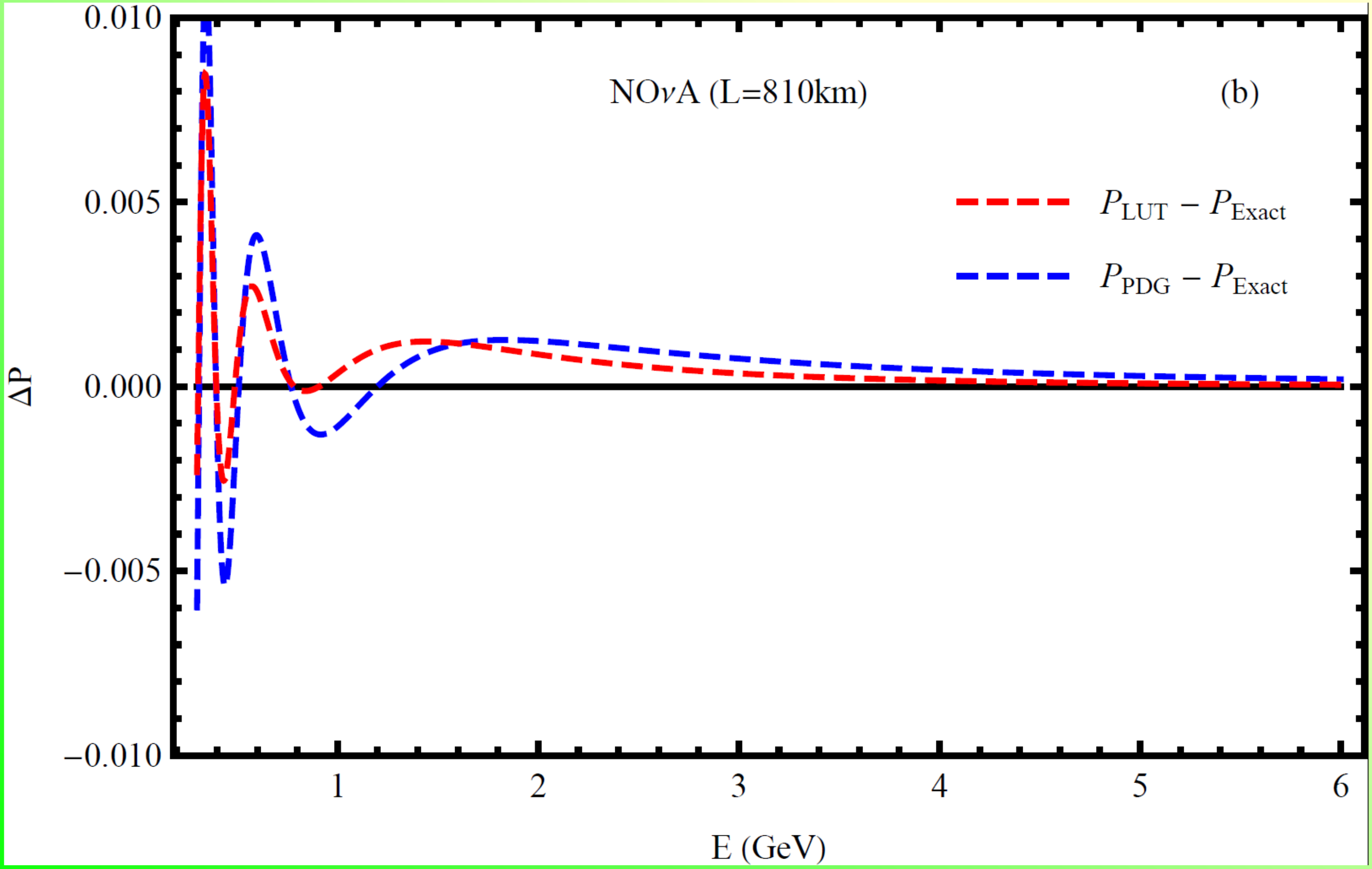
➢ $\nu_\mu \rightarrow \nu_e$ Oscillation with L = 810 km.



HJH & Xu, arXiv:1606.04054, PRD.2017

Comparing Oscillation Accuracy for NOνA:

➢ $\nu_\mu \rightarrow \nu_e$ **Oscillation with L = 810 km.**



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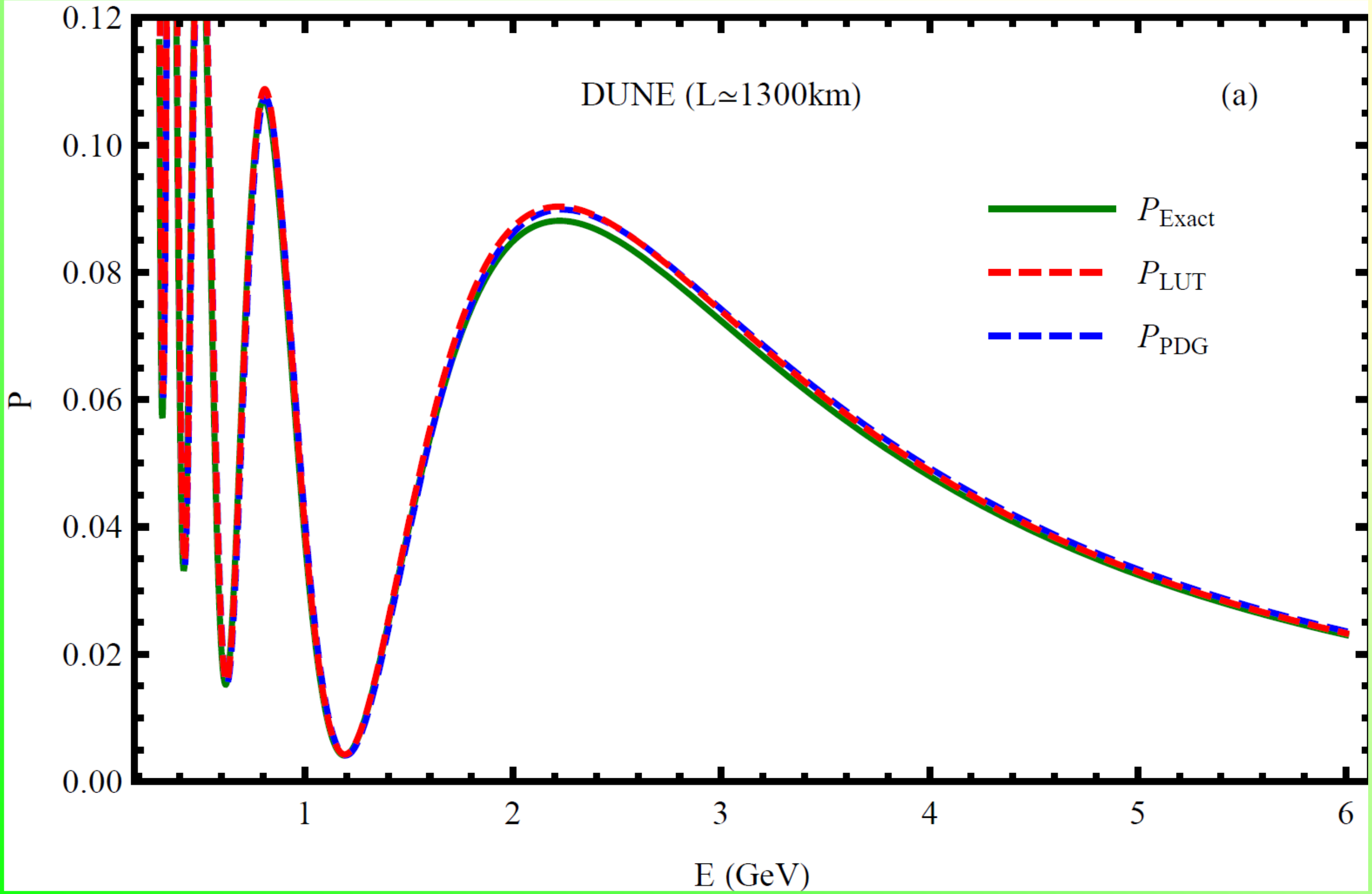
Comparing ν Probability for DUNE:

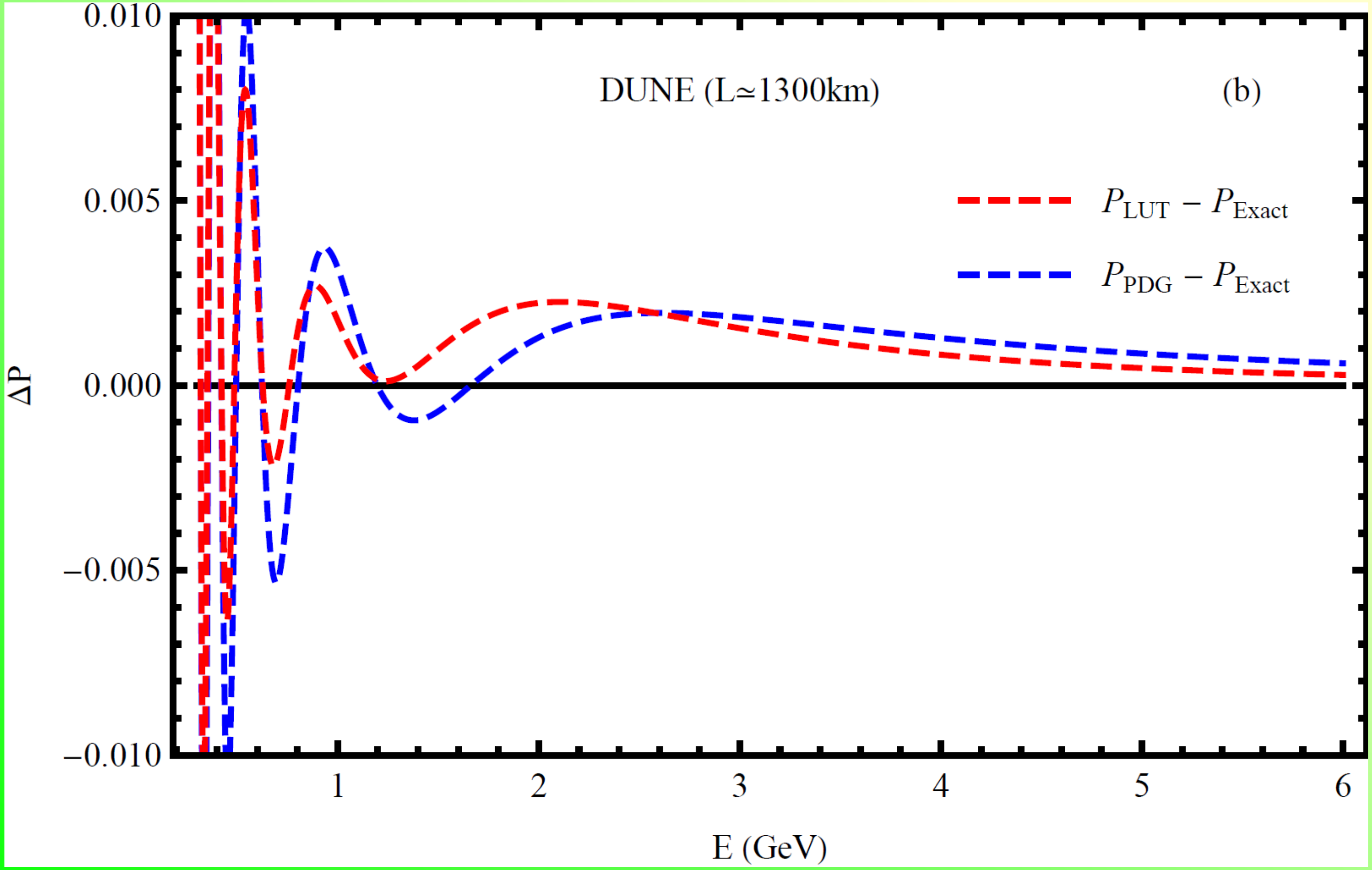
➢ $\nu_\mu \to \nu_e$ Oscillation with L = 1300 km.



HJH & Xu, arXiv:1606.04054, PRD.2017

Comparing Oscillation Accuracy for DUNE:

- $\nu_\mu \rightarrow \nu_e$ **Oscillation with L = 1300 km.**



HJH & Xu, arXiv:1606.04054, PRD.2017

Summary

- **We proved: Each ν Oscillation channel (with CPV) can be fully described by the corresponding geometrical LUT.**
- **LUT is Rephasing Invariant.**
- **We proposed a New Geometrical Formulation**

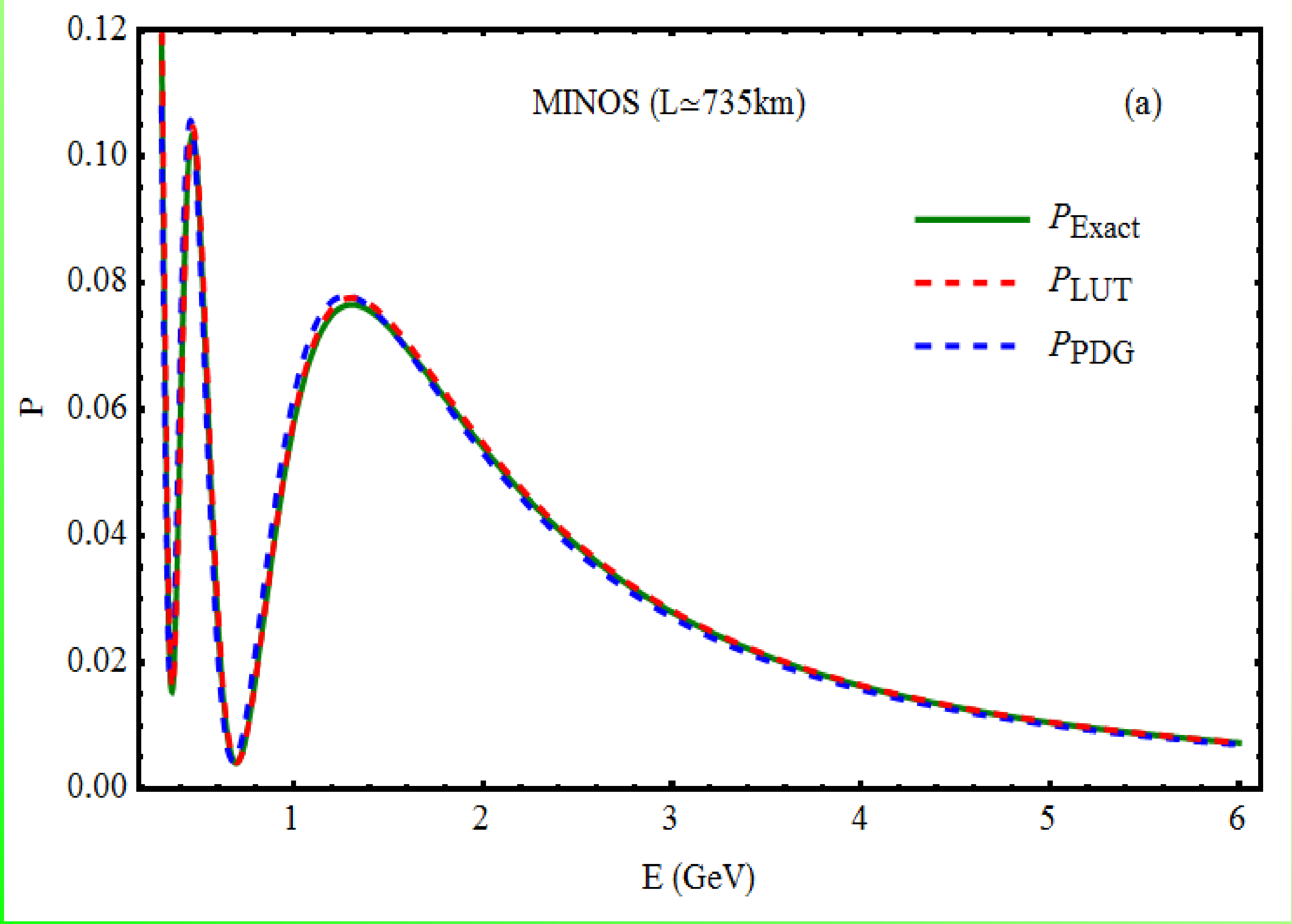
  for 3ν Oscillations: Probability P equals the squared distance between 2 circling points around 2 vertices of the vacuum LUT, expressed in terms of only 3 LUT parameters, (b, c, α).
- **We included Matter Effects by introducing Effective LUT and derived analytical solutions, as accurate as (or better than) PDG-formula.**
- **We applied our LUT formula to study $\nu_\mu \rightarrow \nu_e$ Oscillations in all key experiments: T2K, MINOS, NOvA, DUNE.**


Thank You !

Backup Slides

Comparing ν Probability for MINOS:

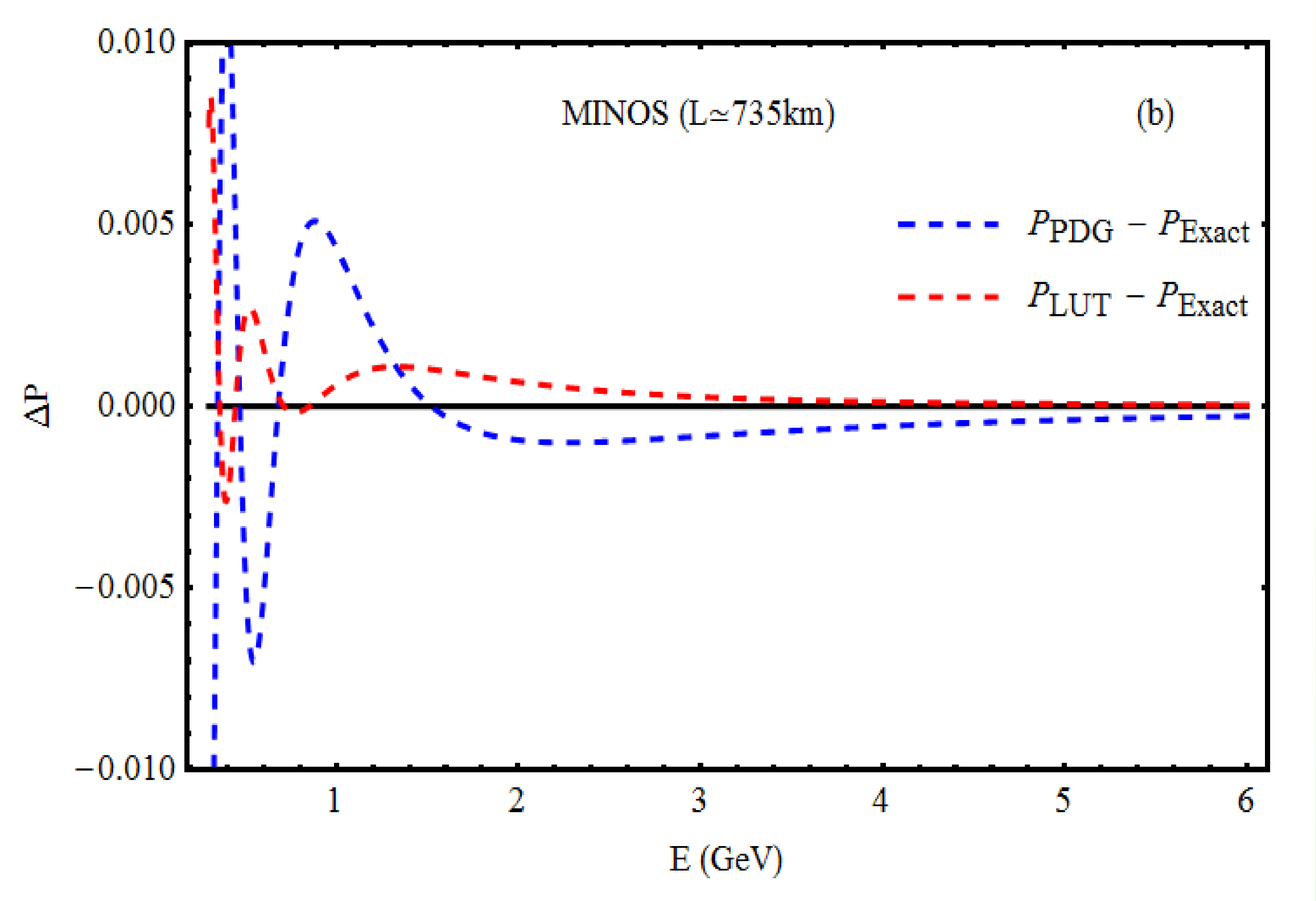
- $\nu_\mu \rightarrow \nu_e$ **Oscillation with L = 735 km.**



HJH & Xu, arXiv:1606.04054

Comparing Oscillation Accuracy for MINOS:

- $\nu_\mu \rightarrow \nu_e$ Oscillation with L = 735 km.



HJH & Xu, arXiv:1606.04054