# Asymptotic Behaviors of Neutrino Mixing in Matter

Jing-yu Zhu
(TDLI&SJTU)

FLASY2019, Hefei, July 26, 2019

Based on Z.Z. Xing and J.Y. Zhu, 1905.08644; 1603.02002 (JHEP)

# Outline

- Introduction: terrestrial matter effects on neutrino oscillations
- Formulas: sum rules for neutrino mixing in a medium with constant matter density
- Application (1): understanding asymptotic behaviors of neutrino mixing in dense matter
- Application (2): understanding extremes of the effective Jarlskog invariant in matter
- Concluding remarks

# Terrestrial matter effects

L. Wolfenstein (1978)
S. P. Mikheev and A. Yu. Smirnov (1985)

Coherent Forward Scattering

| Type of reaction | Matter potential |
|---|---|
| $V_Z^n$ | $\mp G_F N_n/\sqrt{2}$ |
| $V_Z^p$ | $\pm G_F(1-4\sin^2\theta_W)N_p/\sqrt{2}$ |
| $V_Z^e$ | $\mp G_F(1-4\sin^2\theta_W)N_e/\sqrt{2}$ |
| $V_W^e$ | $\pm\sqrt{2}G_F N_e$ |

$$\mathcal{H}_{\rm m} = \frac{1}{2E} U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} V_W^e + V_Z^n & 0 & 0 \\ 0 & V_Z^n & 0 \\ 0 & 0 & V_Z^n \end{pmatrix} \equiv \frac{1}{2E}\widetilde{U} \begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix} \widetilde{U}^\dagger$$

- In the case of antineutrino beams, *U*➔*U**,matter potentials flip their signs.
- When unitarity violations (sterile neutrinos) are included, the contributions of NC coherent forward scattering are not be trivial.

# Preliminary formulas

$$\mathcal{H}'_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix}0 & 0 & 0\\ 0 & \Delta_{21} & 0\\ 0 & 0 & \Delta_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}A & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] \equiv \frac{1}{2E}\left[\widetilde{U}\begin{pmatrix}0 & 0 & 0\\ 0 & \widetilde{\Delta}_{21} & 0\\ 0 & 0 & \widetilde{\Delta}_{31}\end{pmatrix}\widetilde{U}^\dagger + BI\right]$$

$$\Delta_{ij} \equiv m_i^2 - m_j^2$$ $$\widetilde{\Delta}_{ij} \equiv \widetilde{m}_i^2 - \widetilde{m}_j^2$$ $$A = 2EV_W^e$$ $$B = \widetilde{m}_1^2 - m_1^2 - 2EV_Z^n$$

$$\widetilde{\Delta}_{21} = \frac{2}{3}\sqrt{x^2-3y}\sqrt{3\left(1-z^2\right)} \; ,$$

$$\widetilde{\Delta}_{31} = \frac{1}{3}\sqrt{x^2-3y}\left[3z+\sqrt{3\left(1-z^2\right)}\right] \; ,$$

$$B = \frac{1}{3}x - \frac{1}{3}\sqrt{x^2-3y}\left[z+\sqrt{3\left(1-z^2\right)}\right]$$

NMO

V.D. Barger et al. (1980)
H.W. Zaglauer et al. (1988)

$$B = \frac{1}{3}\left(\Delta_{21}+\Delta_{31}+A-\widetilde{\Delta}_{21}-\widetilde{\Delta}_{31}\right)$$

$$x = \Delta_{21}+\Delta_{31}+A \; ,$$

$$y = \Delta_{21}\Delta_{31} + A\left[\Delta_{21}\left(1-|U_{e2}|^2\right)+\Delta_{31}\left(1-|U_{e3}|^2\right)\right]$$

$$z = \cos\left[\frac{1}{3}\arccos\frac{2x^3-9xy+27A\Delta_{21}\Delta_{31}|U_{e1}|^2}{2\sqrt{\left(x^2-3y\right)^3}}\right] \; .$$

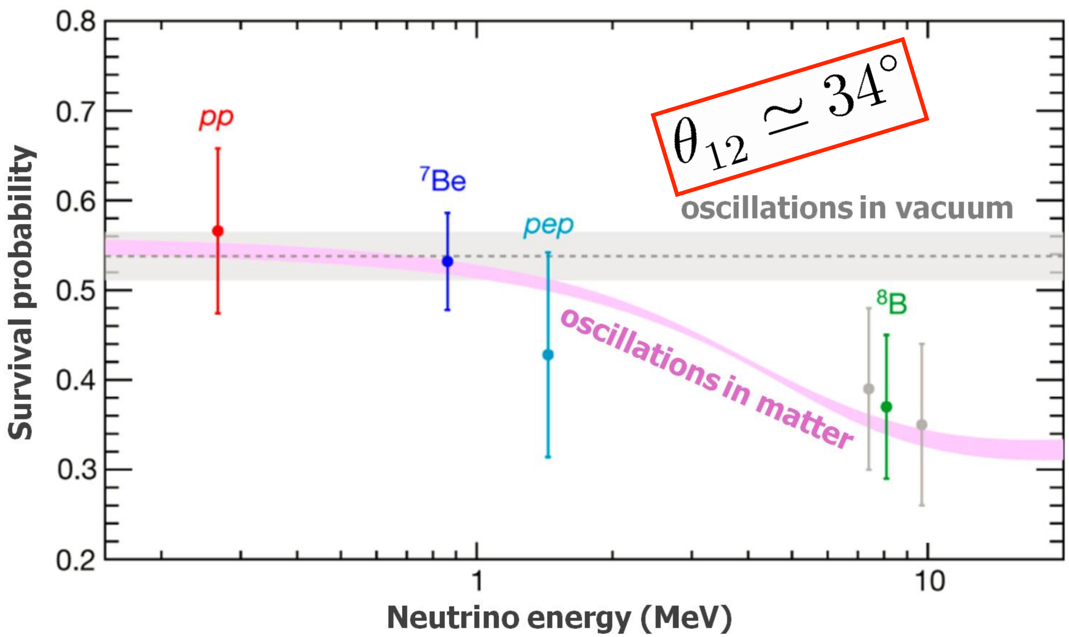
# Matter effects on neutrino oscillations

vacuum : $P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\left(U_{\alpha i}U^*_{\alpha j}U^*_{\beta i}U_{\beta j}\right)\sin^2\frac{\Delta_{ji}L}{4E} + 8\mathcal{J}\sum_\gamma \epsilon_{\alpha\beta\gamma}\sin\frac{\Delta_{21}L}{4E}\sin\frac{\Delta_{31}L}{4E}\sin\frac{\Delta_{32}L}{4E}$

matter : $\widetilde{P}(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\left(\widetilde{U}_{\alpha i}\widetilde{U}^*_{\alpha j}\widetilde{U}^*_{\beta i}\widetilde{U}_{\beta j}\right)\sin^2\frac{\widetilde{\Delta}_{ji}L}{4E} + 8\widetilde{\mathcal{J}}\sum_\gamma \epsilon_{\alpha\beta\gamma}\sin\frac{\widetilde{\Delta}_{21}L}{4E}\sin\frac{\widetilde{\Delta}_{31}L}{4E}\sin\frac{\widetilde{\Delta}_{32}L}{4E}$

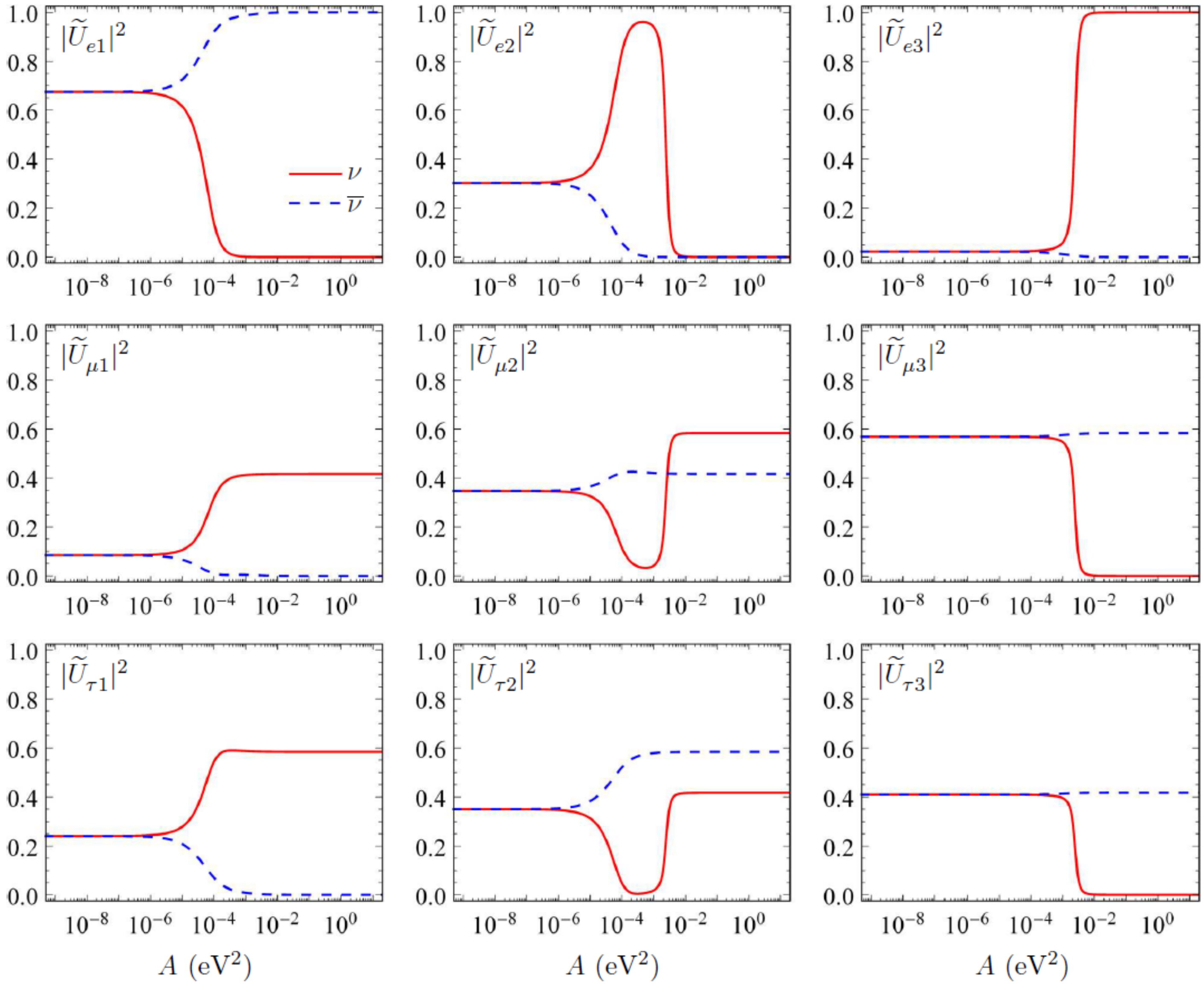
$$\mathrm{Im}\left(U_{\alpha i}U_{\beta j}U^*_{\alpha j}U^*_{\beta i}\right) = \mathcal{J}\sum_\gamma \epsilon_{\alpha\beta\gamma}\sum_k \epsilon_{ijk}$$

$$\mathrm{Im}(\widetilde{U}_{\alpha i}\widetilde{U}_{\beta j}\widetilde{U}^*_{\alpha j}\widetilde{U}^*_{\beta i}) = \widetilde{\mathcal{J}}\sum_\gamma \varepsilon_{\alpha\beta\gamma}\sum_k \varepsilon_{ijk}$$




- Low-energy solar neutrinos: dominated by vacuum oscillations;
- High-energy solar neutrinos: dominated by matter effects.

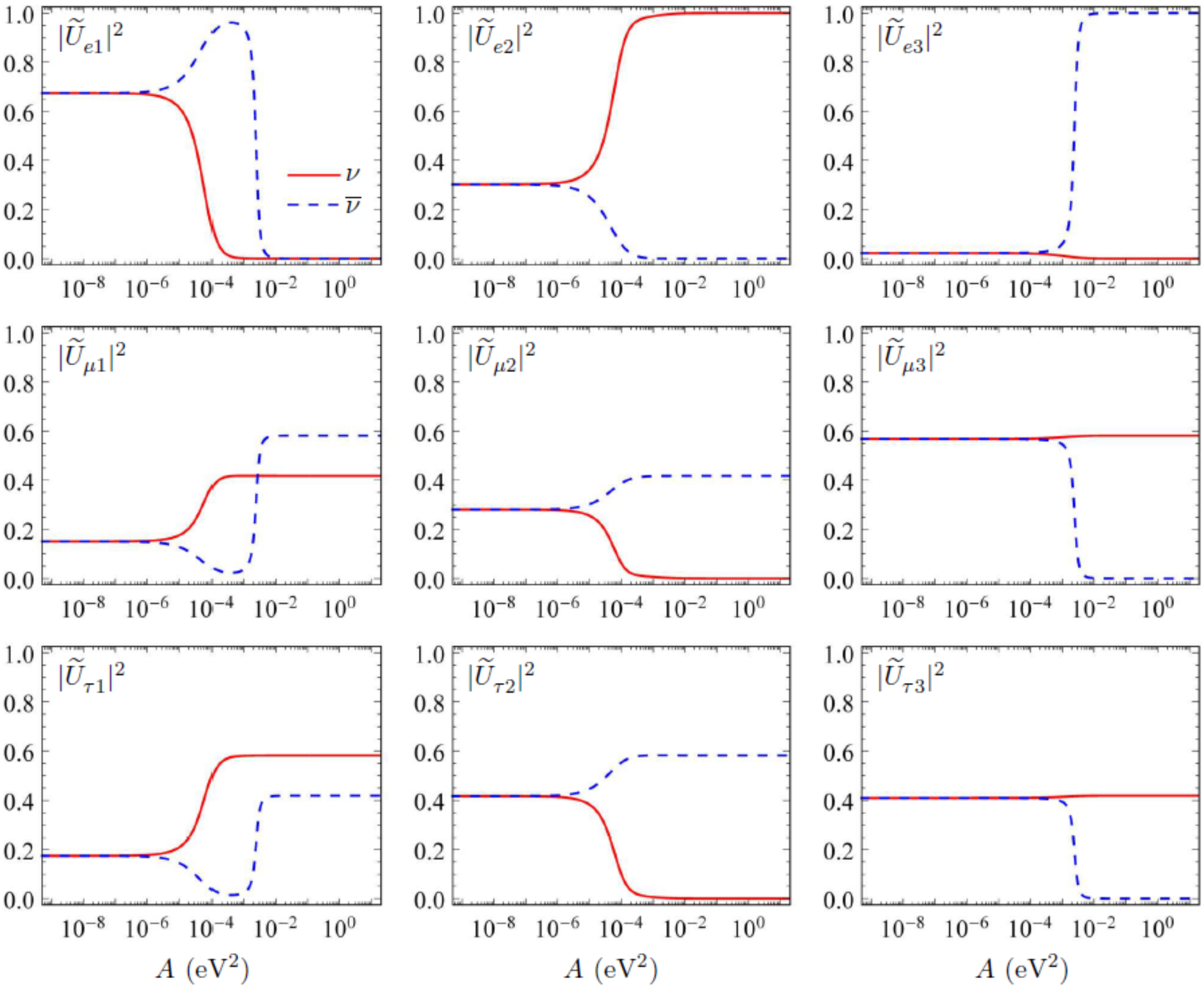
# Asymptotic behaviors (NMO)




$A \to 0$
**vacuum**

$A \to \infty$
**dense matter**

# Asymptotic behaviors (IMO)




$A \to \infty$

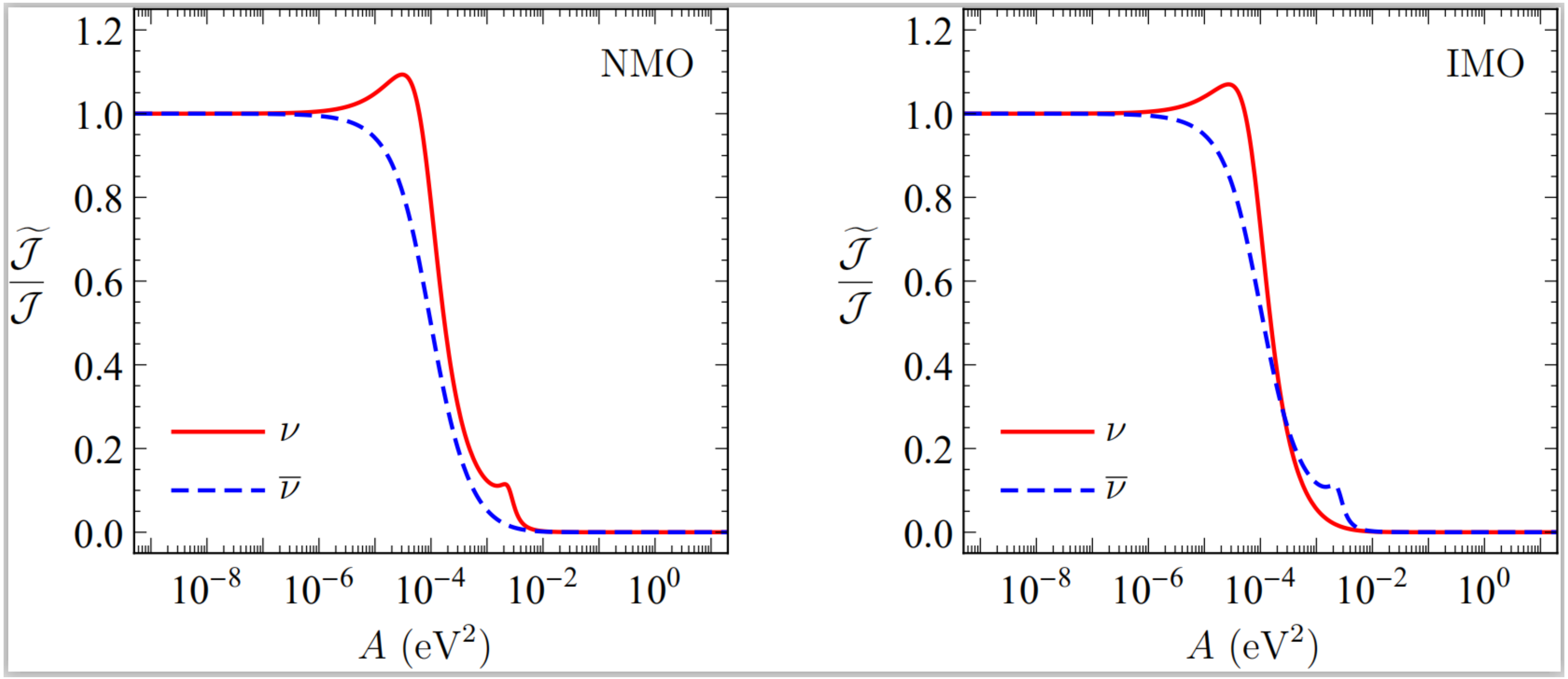
finite values

**Unitarity** **assures asymptotic behaviors**

# Effective Jarlskog invariant in matter

$$\frac{\tilde{\mathcal{J}}}{\mathcal{J}} = \frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\tilde{\Delta}_{21}\tilde{\Delta}_{31}\tilde{\Delta}_{32}}$$

V.A. Naumov (1992)




Motivation: to analytically understand the asymptotic behaviors in the $A \to \infty$ limit and Jarlskog' s peaks.

# Sum rules

A full set of linear equations of 3 unknown variables: they' re solvable!

$$\sum_{i=1}^{3} \widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i}\widetilde{\Delta}_{i1} = \sum_{i=1}^{3} U_{\alpha i}U^*_{\beta i}\Delta_{i1} + A\delta_{e\alpha}\delta_{e\beta} - B\delta_{\alpha\beta}$$

$$\sum_{i=1}^{3} \widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i}\widetilde{\Delta}_{i1}(\widetilde{\Delta}_{i1} + 2B) = \sum_{i=1}^{3} U_{\alpha i}U^*_{\beta i}\Delta_{i1}\left[\Delta_{i1} + A(\delta_{e\alpha} + \delta_{e\beta})\right] + A^2\delta_{e\alpha}\delta_{e\beta} - B^2\delta_{\alpha\beta}$$

$$\sum_{i=1}^{3} \widetilde{U}_{\alpha i}\widetilde{U}^*_{\beta i} = \sum_{i=1}^{3} U_{\alpha i}U^*_{\beta i} = \delta_{\alpha\beta}$$

Setting $\alpha = \beta$, we obtain 9 moduli of the PMNS matrix elements in matter.

Analytical results

$$|\widetilde{U}_{\alpha 1}|^2 = \frac{\zeta - 2\xi B - \xi\widetilde{\Delta}_{21} - \xi\widetilde{\Delta}_{31} + \widetilde{\Delta}_{21}\widetilde{\Delta}_{31}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}}$$

$$|\widetilde{U}_{\alpha 2}|^2 = \frac{\xi\widetilde{\Delta}_{31} + 2\xi B - \zeta}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{32}} ,$$

$$|\widetilde{U}_{\alpha 3}|^2 = \frac{\zeta - 2\xi B - \xi\widetilde{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}} ,$$

$$\xi = \Delta_{21}|U_{\alpha 2}|^2 + \Delta_{31}|U_{\alpha 3}|^2 + A\delta_{e\alpha} - B ,$$

$$\zeta = \Delta_{21}(\Delta_{21} + 2A\delta_{e\alpha})|U_{\alpha 2}|^2 + \Delta_{31}(\Delta_{31} + 2A\delta_{e\alpha})|U_{\alpha 3}|^2 + A^2\delta_{e\alpha} - B^2$$

# Moduli of the matrix elements

$$|\widetilde{U}_{e1}|^2 = \frac{1}{9}\left[\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{31}}\cdot\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{21}-\Delta_{32}-A}{\widetilde{\Delta}_{21}}|U_{e1}|^2 + \frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{31}}\cdot\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}-\Delta_{21}-\Delta_{31}-A}{\widetilde{\Delta}_{21}}|U_{e2}|^2 + \frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}+\Delta_{21}-\Delta_{32}-A}{\widetilde{\Delta}_{31}}\cdot\frac{\widetilde{\Delta}_{21}+\widetilde{\Delta}_{31}-\Delta_{21}-\Delta_{31}-A}{\widetilde{\Delta}_{21}}|U_{e3}|^2\right]$$

$$|\widetilde{U}_{e2}|^2 = \frac{1}{9}\left[\frac{\widetilde{\Delta}_{32}-\widetilde{\Delta}_{21}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{32}}\cdot\frac{\widetilde{\Delta}_{21}-\widetilde{\Delta}_{32}+\Delta_{32}-\Delta_{21}+A}{\widetilde{\Delta}_{21}}|U_{e1}|^2 + \frac{\widetilde{\Delta}_{32}-\widetilde{\Delta}_{21}+\Delta_{31}+\Delta_{32}-A}{\widetilde{\Delta}_{32}}\cdot\frac{\widetilde{\Delta}_{21}-\widetilde{\Delta}_{32}+\Delta_{21}+\Delta_{31}+A}{\widetilde{\Delta}_{21}}|U_{e2}|^2 + \frac{\widetilde{\Delta}_{32}-\widetilde{\Delta}_{21}-\Delta_{32}+\Delta_{21}-A}{\widetilde{\Delta}_{32}}\cdot\frac{\widetilde{\Delta}_{21}-\widetilde{\Delta}_{32}+\Delta_{21}+\Delta_{31}+A}{\widetilde{\Delta}_{21}}|U_{e3}|^2\right]$$

# Asymptotic results (1)

**The effective neutrino mass-squared differences + PMNS matrix elements in the $A \to \infty$ limit:**

| | NMO (neutrinos) | IMO (neutrino) |
|---|---|---|
| $\widetilde{\Delta}_{21}$ | $\Delta_{31}\left(1-\lvert U_{e3}\rvert^2\right)-\Delta_{21}\lvert U_{e1}\rvert^2$ | $A$ |
| $\widetilde{\Delta}_{31}$ | $A$ | $\Delta_{31}\left(1-\lvert U_{e3}\rvert^2\right)-\Delta_{21}\lvert U_{e1}\rvert^2$ |
| $\widetilde{\Delta}_{32}$ | $A$ | $-A$ |
| $\widetilde{U}$ | $\begin{pmatrix} 0 & 0 & 1 \\ \sqrt{1-\lvert U_{\mu 3}\rvert^2} & \lvert U_{\mu 3}\rvert & 0 \\ -\lvert U_{\mu 3}\rvert & \sqrt{1-\lvert U_{\mu 3}\rvert^2} & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 0 \\ \sqrt{1-\lvert U_{\mu 3}\rvert^2} & 0 & \lvert U_{\mu 3}\rvert \\ -\lvert U_{\mu 3}\rvert & 0 & \sqrt{1-\lvert U_{\mu 3}\rvert^2} \end{pmatrix}$ |

**Illustration:**

$$\left(\lvert\widetilde{U}_{\alpha i}\rvert^2\right)\Big|_{A\to\infty}^{(\mathrm{NMO},\ \nu)} = \begin{pmatrix} 0 & 0 & 1 \\ 0.417 & 0.583 & 0 \\ 0.583 & 0.417 & 0 \end{pmatrix} \quad\Bigg|\quad \left(\lvert\widetilde{U}_{\alpha i}\rvert^2\right)\Big|_{A\to\infty}^{(\mathrm{IMO},\ \nu)} = \begin{pmatrix} 0 & 1 & 0 \\ 0.418 & 0 & 0.582 \\ 0.582 & 0 & 0.418 \end{pmatrix}$$

## Asymptotic results (2)

**The effective antineutrino mass-squared differences + PMNS matrix elements in the $A \to \infty$ limit:**

| | NMO (antineutrino) | IMO (antineutrino) |
|---|---|---|
| $\widetilde{\Delta}_{21}$ | $A$ | $-\Delta_{31}\left(1-\lvert U_{e3}\rvert^2\right)+\Delta_{21}\lvert U_{e1}\rvert^2$ |
| $\widetilde{\Delta}_{31}$ | $A$ | $-A$ |
| $\widetilde{\Delta}_{32}$ | $\Delta_{31}\left(1-\lvert U_{e3}\rvert^2\right)-\Delta_{21}\lvert U_{e1}\rvert^2$ | $-A$ |
| $\widetilde{U}$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1-\lvert U_{\mu 3}\rvert^2} & \lvert U_{\mu 3}\rvert \\ 0 & -\lvert U_{\mu 3}\rvert & \sqrt{1-\lvert U_{\mu 3}\rvert^2} \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\ \lvert U_{\mu 3}\rvert & \sqrt{1-\lvert U_{\mu 3}\rvert^2} & 0 \\ -\sqrt{1-\lvert U_{\mu 3}\rvert^2} & \lvert U_{\mu 3}\rvert & 0 \end{pmatrix}$ |

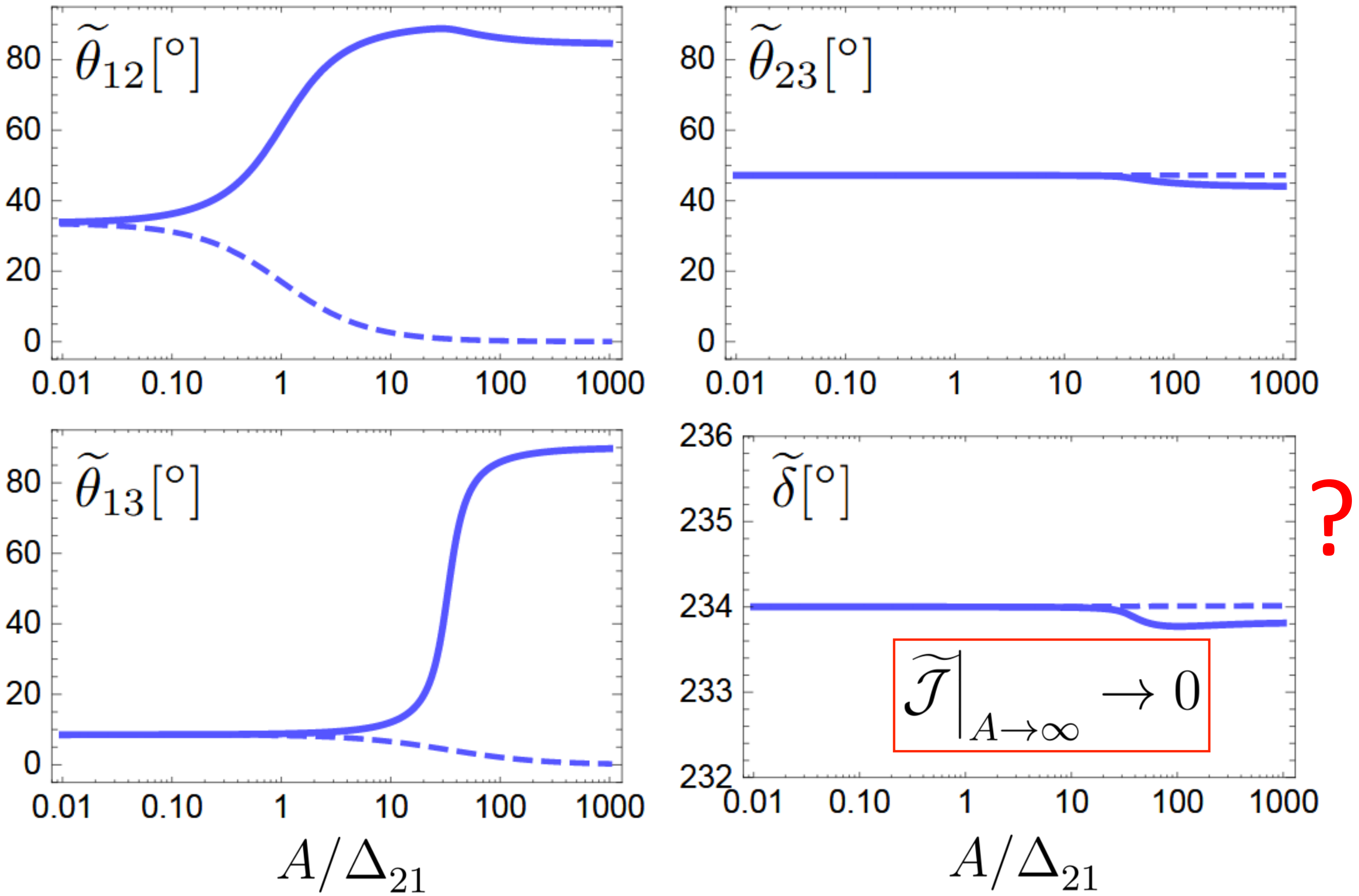
**Illustration:**

$$\left(\lvert\widetilde{U}_{\alpha i}\rvert^2\right)\Big|_{A\to\infty}^{(\mathrm{NMO},\ \overline{\nu})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.417 & 0.583 \\ 0 & 0.583 & 0.417 \end{pmatrix} \quad \Bigg| \quad \left(\lvert\widetilde{U}_{\alpha i}\rvert^2\right)\Big|_{A\to\infty}^{(\mathrm{IMO},\ \overline{\nu})} = \begin{pmatrix} 0 & 0 & 1 \\ 0.582 & 0.418 & 0 \\ 0.418 & 0.582 & 0 \end{pmatrix}$$

# Something misleading?

parameter redundancy

Why the CP-violating phase remains finite in the $A \to \infty$ limit?



# Extremes of the effective Jarlskog invariant in matter

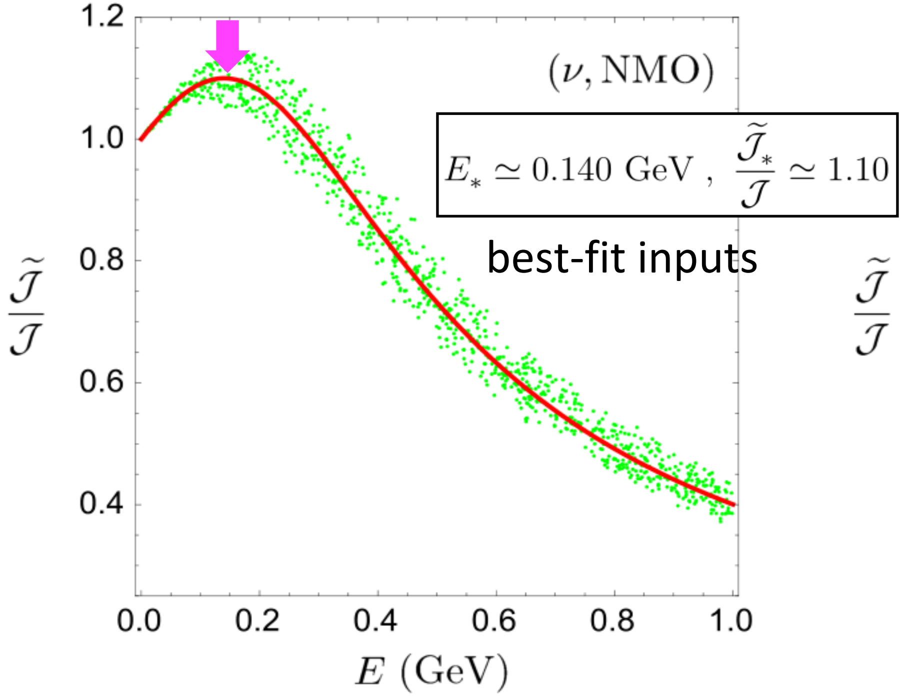
$$E_* \simeq \frac{\Delta_{21}}{2\sqrt{2}\, G_{\rm F} N_e}\left[\cos 2\theta_{12}\left(1+\sin^2\theta_{13}\right)+\frac{3}{2}\sin^2 2\theta_{12}\frac{\Delta_{21}}{\Delta_{31}}\right]$$ resonant energy

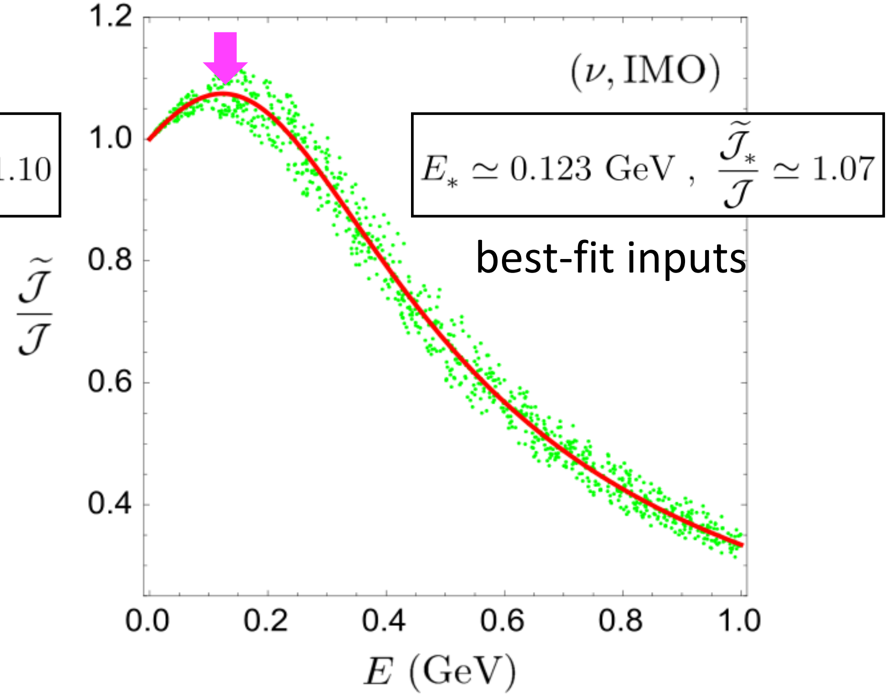
$$\frac{\widetilde{\mathcal{J}}_*}{\mathcal{J}} \simeq \frac{1}{\sin 2\theta_{12}}\left[1+\frac{1}{2}\left(1+3\cos 2\theta_{12}\right)\frac{\Delta_{21}}{\Delta_{31}} + \text{smaller terms}\right]$$ enhancive peak

**We find a unique energy upper limit for $\widetilde{\mathcal{J}}/\mathcal{J} \gtrsim 1$ :**

$$E_0 \simeq \Delta_{21}\cos 2\theta_{12}/(\sqrt{2}\, G_{\rm F} N_e) \simeq 0.25\ \text{GeV}$$

best-fit inputs



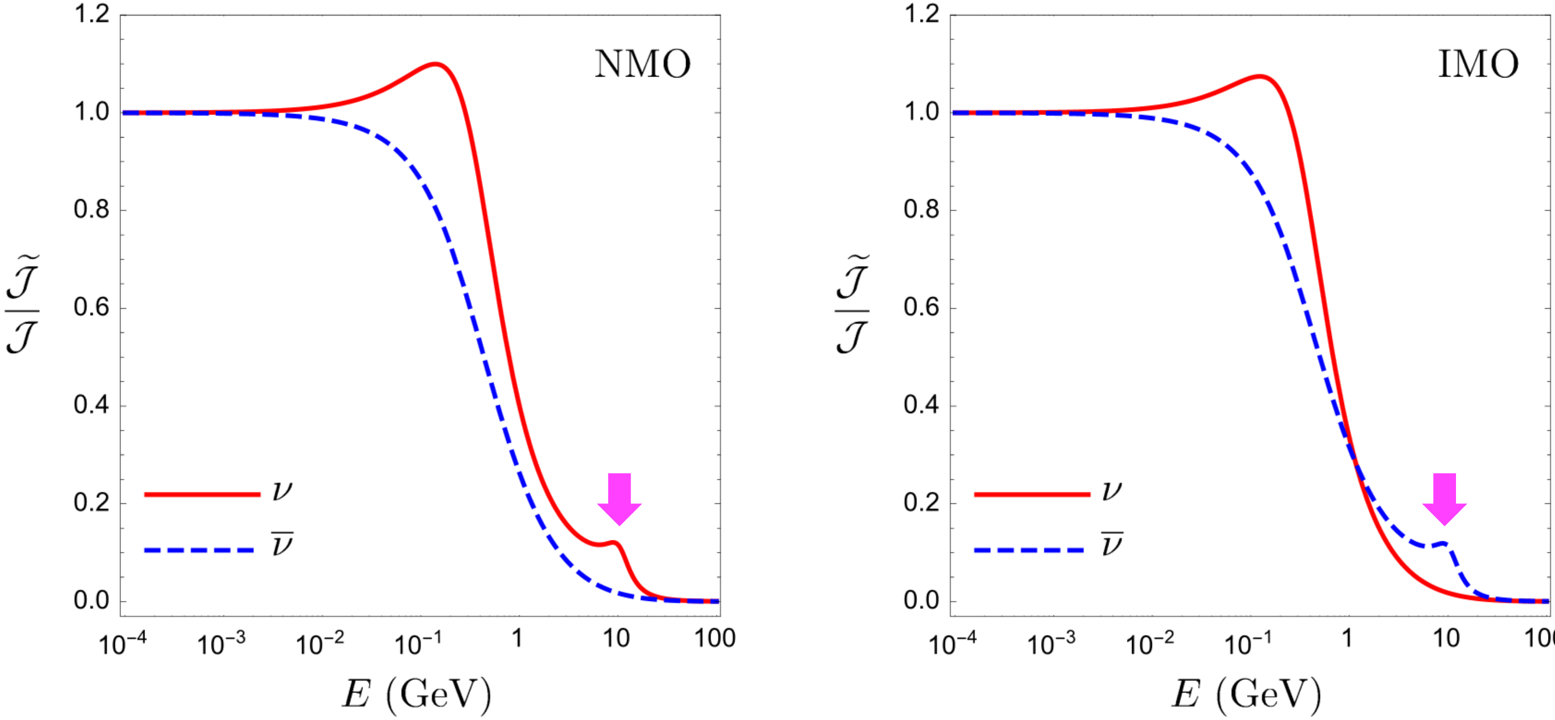



$$\nu \text{ beam } (\Delta_{31} > 0): \quad E_* \simeq 0.140 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}_*}{\mathcal{J}} \simeq 1.10; \quad E'_* \simeq 8.906 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}'_*}{\mathcal{J}} \simeq 0.12$$

$$\nu \text{ beam } (\Delta_{31} < 0): \quad E_* \simeq 0.123 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}_*}{\mathcal{J}} \simeq 1.07;$$

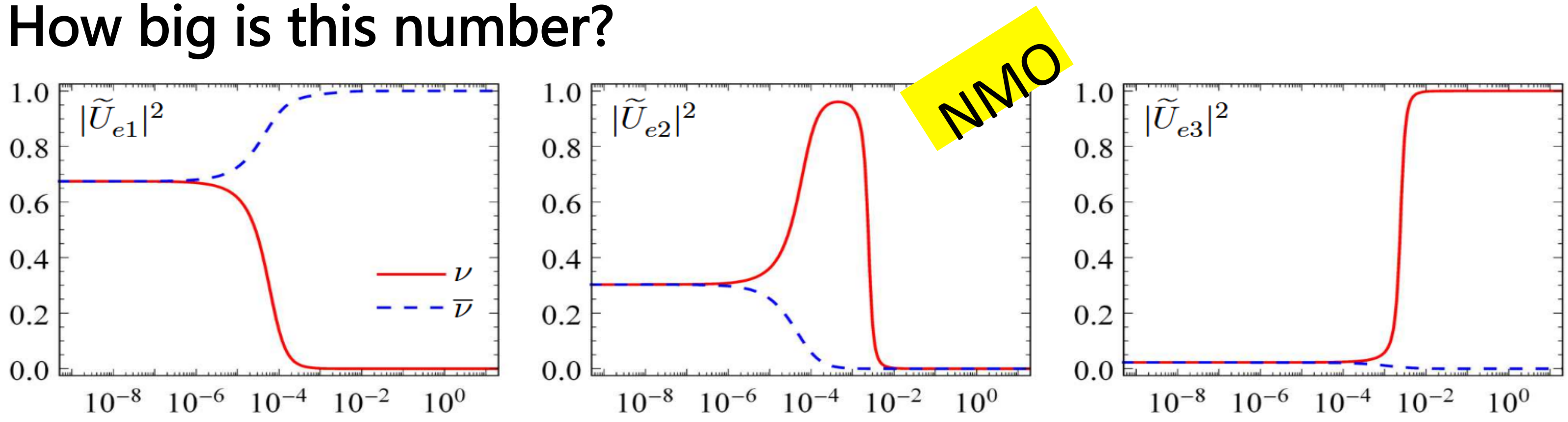
$$\overline{\nu} \text{ beam } (\Delta_{31} < 0): \quad E'_* \simeq 8.828 \text{ GeV}, \quad \frac{\widetilde{\mathcal{J}}'_*}{\mathcal{J}} \simeq 0.12.$$

$$\beta'_* = \frac{\pm A'_*}{\Delta_{31}} \simeq \frac{3\cos 2\theta_{13} + \sqrt{1 - 9\sin^2 2\theta_{13}}}{4}$$



# Concluding remarks

- The $A \to \infty$ limit is at least conceptually interesting, and in practice it is approximately equivalent to $A \gtrsim 10^{-2}\ \mathrm{eV}^2$, as shown in the figures. How big is this number?




In the very center of the Sun, the matter density is not that big at all!

$$N_e(0) \approx 6 \times 10^{25}\ \mathrm{cm}^{-3}$$

$$A \simeq 7.5 \times 10^{-6}\,(E/\mathrm{MeV})\,\mathrm{eV}^2$$

- There is obvious enhancements of the effective Jarlskog invariant in the region with low neutrino beam energy or low matter density. There are also much smaller terrestrial matter effects in this region.

So a possible experiment focusing on the low energy $\nu$-oscillation to explore CP violation, for example :MOMENT (150 ~ 200 MeV)

J. Cao et al, 1401.8125
M. Blennow et al, 1511.02859

Thank you for your attention!