國立清華大學
NATIONAL TSING HUA UNIVERSITY

Thermodynamics of f(R) Gravity with Disformal Transformation

Jhih-Rong Lu(NTHU)

FLASY2019: the 8th Workshop on Flavor Symmetries and Consequences in Accelerators and Cosmology,
July 24th , 2019

Reference: C. Q. Geng, W. C. Hsu, JRL and L. W. Luo, Entropy 21, 172 (2019)

Outline

- Introduction
- Thermodynamics in Jordan Frame
  - Non-equilibrium description
  - Equilibrium description
- Thermodynamics in Einstein Frame
- Summary

- Introduction
- Thermodynamics in Jordan Frame
    - Non-equilibrium description
    - Equilibrium description
- Thermodynamics in Einstein Frame
- Summary

What is f(R) theory?

- Einstein equation can be obtained by varying the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,R + S_{\mathrm{M}}[\mathrm{g}_{\mu\nu},\Psi],$$

$$\delta S_{EH} = 0 \;\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}.$$

- General Relativity(GR) becomes f(R) theory by replacing the Lagrangian density of GR, R, with a function of R, $f(R)$:

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,f(R) + S_{\mathrm{M}}[\mathrm{g}_{\mu\nu},\Psi],$$

$$\delta S = 0 \Rightarrow FG_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(f(R) - FR) - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = \kappa T_{\mu\nu}.\ (F := f_{,R})$$

Jordan Frame v.s. Einstein Frame

- f(R) gravity in Jordan frame:

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\;f(R)$$

Conformal transformation: $\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \quad \Omega^2(x) = F(R) := e^{\sqrt{\frac{4\pi G}{3}}\omega}$

- f(R) gravity in Einstein frame:

$$S = \int d^4x\,\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega - V(\omega)\right],$$

where $V(\omega) = \frac{1}{2\kappa}\frac{FR-f}{F^2}$.

Disformal Transformation [arXiv: gr-qc/9211017]

- In 1992, Bekenstein proposed a new gravity theory which is a special kind of bimetric theory.
- One of the metric, $g_{\mu\nu}$, describes the gravitational field of the spacetime, the other, $\gamma_{\mu\nu}$, describes the particle trajectory in gravitational field.
- Bekenstein argued that two metrics should be related through disformal transformation in order to satisfy equivalence principle and causality:

$$\gamma_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X) \partial_\mu \phi \partial_\nu \phi. \qquad \left(X = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi\right)$$

matter (physical) metric

gravitational metric

gravitational scalar field

Thermodynamics and Gravity

國立清華大學
NATIONAL TSING HUA UNIVERSITY

- In BH physics, the temperature and entropy are associated with the surface gravity and area of the horizon. [Comm. Math. Phys., Volume 31, Number 2 (1973), 161-170; Phys. Rev. D. **7**.2333(1973)]
- In 1995, T. Jacobson further showed that Einstein equation can be derived from the thermodynamic behavior of spacetime.[arXiv:gr-qc/9504004v2]
- In 2005, R. G. Cai and S. P. Kim demonstrated that the Friedmann equations can be derived from the first law of thermodynamics on the apparent horizon of the universe.[arXiv:gr-qc/0611071v2]
- Connection between thermodynamics and f(R) gravity and other modified gravity has been widely investigated. [arXiv:0909.2109, 1005.5234]

- Introduction
- Thermodynamics in Jordan Frame
  - Non-equilibrium description
  - Equilibrium description
- Thermodynamics in Einstein Frame
- Summary

Action: $S = \frac{1}{2\kappa}\int \sqrt{-g}\, f(R) d^4x \; + \sum_i S_{\mathrm{M}}^{(i)}[\gamma_{\alpha\beta}]$

國立清華大學
NATIONAL TSING HUA UNIVERSITY

- $\gamma_{\alpha\beta}$ is related with $g_{\alpha\beta}$ by the disformal transformation:

$$\gamma_{\alpha\beta} = A(\phi, X) g_{\alpha\beta} + B(\phi, X)\partial_\alpha\phi\partial_\beta\phi$$

where $X = -\frac{1}{2}\, g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ is the kinetic term of disformal field ϕ.

- Under the assumption of Principle of Cosmology :

➢ Metric $g_{\alpha\beta}$ is given by Robertson-Walker metric: (k=0)

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

➢ ϕ depends only on time now, i.e., $\phi = \phi(t)$.

Action: $S = \frac{1}{2\kappa}\int\sqrt{-g}\,f(R)d^4x \ +\sum_i S_{\mathrm{M}}^{(i)}[\gamma_{\alpha\beta}]$

国立清華大學
NATIONAL TSING HUA UNIVERSITY

- $\gamma_{\alpha\beta}$ is related with $g_{\alpha\beta}$ by the disformal transformation:

$$\eta_{\alpha\beta} = A(\phi, X)g_{\alpha\beta} + B(\phi, X)\partial_\alpha\phi\partial_\beta\phi$$

where $X = -\frac{1}{2}\, g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$ is the kinetic term of disformal field ϕ.

- Under the assumption of Principle of Cosmology :

➢ Metric $g_{\alpha\beta}$ is given by Robertson-Walker metric: (k=0)

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

➢ ϕ depends only on time now, i.e., $\phi = \phi(t)$.

國立清華大學
NATIONAL TSING HUA UNIVERSITY

Equation of Motion

- $F G_{\mu\nu} = \kappa \sum_i \left(T^{(i)}_{\mu\nu} + t^{(i)}_{\mu\nu} \right) + \kappa \hat{T}^{(\mathrm{d})}_{\mu\nu}$, $F := \partial f(R)/\partial R$
- $\frac{\partial \mathcal{L}_i}{\partial \phi} + \frac{(\partial_\alpha \phi)(4\partial_\phi A - 2X\partial_\phi B)}{(2X)(-4A_{,X} + 2B + 2XB_{,X})} \frac{\partial \mathcal{L}_i}{\partial(\partial_\alpha \phi)} = 0$

where

$$T^{(i)}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{L}_i}{\delta g^{\mu\nu}},$$

$$t^{(i)}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \left(\frac{-A g_{\mu\nu} - 2A_{,X} \partial_\mu \phi \partial_\nu \phi + X B_{,X} \partial_\mu \phi \partial_\nu \phi}{(2X)(-4A_{,X} + 2B + 2XB_{,X})} \right) (\partial_\alpha \phi) \frac{\partial \mathcal{L}_i}{\partial(\partial_\alpha \phi)},$$

$$\hat{T}^{(\mathrm{d})}_{\mu\nu} = \frac{1}{\kappa} \left[\frac{1}{2} g_{\mu\nu} (f(R) - FR) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F \right]$$

- With the perfect fluid assumption, one is able to express induced matter in terms of ordinary matter:

$$\rho_i^{(\mathrm{in})} = \lambda\rho_i, \ \ P_i^{(\mathrm{in})} = \lambda w^{(\mathrm{in})}\rho_i$$

where

$$\lambda = \frac{(1-a^2)(3\dot{a}\dot{\phi} + a^3\ddot{\phi})}{3\dot{a}\dot{\phi}}$$

and

$$w^{(\mathrm{in})} := \frac{P_i^{(\mathrm{in})}}{\rho_i^{(\mathrm{in})}} = -\frac{a\ddot{\phi}}{3\dot{a}\dot{\phi}+a^3\ddot{\phi}} .$$

- Assume that ordinary matter only contain non-relativistic matter(m) and radiation(r).
  - $\rho_i = \rho_\mathrm{m} + \rho_\mathrm{r} := \bar{\rho}_\mathrm{M}$
  - $\rho_i^{(\mathrm{in})} = \rho_\mathrm{m}^{(\mathrm{in})} + \rho_\mathrm{r}^{(\mathrm{in})} = \lambda\,(\rho_\mathrm{m} + \rho_\mathrm{r}) := \lambda\bar{\rho}_\mathrm{M}$
  - $P_i = P_\mathrm{r}$
  - $P_i^{(\mathrm{in})} = P_\mathrm{m}^{(\mathrm{in})} + P_\mathrm{r}^{(\mathrm{in})} = \lambda w^{(\mathrm{in})}(\rho_\mathrm{m}+\rho_\mathrm{r}) := \lambda w^{(\mathrm{in})}\bar{\rho}_\mathrm{M}$

Modified Friedmann Equations (Non-equilibrium case)

- $H^2 = \frac{8\pi G}{3F}(\bar{\rho}_{\mathrm{M}} + \hat{\rho}_{\mathrm{d}} + \lambda\bar{\rho}_{\mathrm{M}}) \coloneqq \frac{8\pi G}{3F}\hat{\rho}_t,$
- $\dot{H} = -\frac{4\pi G}{F}[(\bar{\rho}_{\mathrm{M}} + \hat{\rho}_{\mathrm{d}} + P_{\mathrm{r}} + \hat{P}_{\mathrm{d}}) + \lambda\left(1 + w^{(\mathrm{in})}\right)\bar{\rho}_{\mathrm{M}}$
  $\coloneqq \frac{-4\pi G}{F}(\hat{\rho}_t + \hat{P}_t),$

where

$$\hat{\rho}_{\mathrm{d}} = \frac{1}{8\pi G}\left(\frac{1}{2}(FR - f) - 3H\dot{F}\right),$$

$$\hat{P}_d = \frac{1}{8\pi G}\left(\ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f)\right)$$

are the dark energy density and pressure, respectively.

1st & 2nd Law of Thermodynamics (Non-equilibrium case)

國立清華大學
NATIONAL TSING HUA UNIVERSITY

- $\dot{\hat{\rho}}_{\mathrm{t}} + 3H(\hat{\rho}_{\mathrm{t}} + \hat{P}_{\mathrm{t}}) = \frac{3H^2\dot{F}}{8\pi G}$ ⇒ Non-equilibrium Thermodynamics!

Temp. on the horizon

$\hat{S}$: Horizon entropy in f(R)

$\widehat{W}$: Work density in non-equil. picture

- First law: $T d\hat{S} + T d_i\hat{S} = -d\hat{E} + \widehat{W} dV$,

entropy production term in non-equil. thermodynamics

$\hat{E} = V\hat{\rho}_t$

- Second law: $\frac{d\hat{S}}{dt} + \frac{d_i\hat{S}}{dt} + \frac{d\hat{S}_t}{dt} = \frac{12\pi F\dot{H}^2}{GRH^3} \geq 0$, provided F>0 (viable f(R) condition).

Entropy change rate of ordinary matter and induced matter within the horizon

- Introduction
- Thermodynamics in Jordan Frame
    - Non-equilibrium description
    - Equilibrium description
- Thermodynamics in Einstein Frame
- Summary

Modified Friedmann Equation (Equilibrium case)

- $H^2 = \frac{8\pi G}{3}(\bar{\rho}_{\rm M} + \rho_{\rm d} + \lambda\bar{\rho}_{\rm M}) := \frac{8\pi G}{3}\rho_t$
- $\dot{H} = -4\pi G[(\bar{\rho}_{\rm M}+\rho_{\rm d} + P_r + P_{\rm d}) + \lambda(1 + w^{(\rm in)})\bar{\rho}_{\rm M}]$
  $:= -4\pi G(\rho_t + P_t)$

where

$$\rho_{\rm d} = \frac{1}{8\pi G}\left(\frac{1}{2}(FR - f) - 3H\dot{F} + 3H^2(1 - F)\right),$$

$$P_d = \frac{1}{8\pi G}\left(\ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f) - (1 - F)(2\dot{H} + 3H^2)\right)$$

are the dark energy density and pressure in equilibrium picture, respectively.

1st & 2nd Law of Thermodynamics (Equilibrium case)

- $\rho_t + 3H(\rho_t + P_t) = 0$ ⇒ Equilibrium Thermodynamics

Horizon entropy in equil. description

W: Work density in equil. picture

- First law: $TdS = -dE + WdV$

Temp. on the horizon

$E = \rho_t V$ (total energy in equil. picture)

Entropy change rate of ordinary matter and induced matter within the horizon

- Second law : $\frac{d}{dt}(S + S_t) = \frac{12\pi \dot{H}^2}{GRH^3} \geq 0.$

Thermodynamics in Jordan Frame

国立清華大學
NATIONAL TSING HUA UNIVERSITY

- Non-equilibrium case
  - First law: $T d\hat{S} + T d_i \hat{S} = -d\hat{E} + \widehat{W} dV$
  - Second law: $\frac{d\hat{S}}{dt} + \frac{d_i \hat{S}}{dt} + \frac{d\hat{S}_t}{dt} = \frac{12\pi F \dot{H}^2}{GRH^3} \geq 0$ provided F>0.

- Equilibrium case
  - First law: $TdS = -dE + WdV$
  - Second law: $\frac{d}{dt}(S + S_t) = \frac{12\pi \dot{H}^2}{GRH^3} \geq 0.$

- Introduction
- Thermodynamics in Jordan Frame
  - Non-equilibrium description
  - Equilibrium description
- Thermodynamics in Einstein Frame
- Summary

Thermodynamics in Einstein Frame

- $\tilde{g}_{\alpha\beta}(x) = \Omega^2(x) g_{\alpha\beta}(x)$
- The action becomes

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \omega \partial_\nu \omega - V(\omega) \right] + \sum_i S_{\mathrm{M}}^{(i)}[\omega, \tilde{g}_{\mu\nu}, \Psi_{\mathrm{M}}]$$

- The FRW metric becomes

$$d\tilde{t} = \Omega \, dt$$
$$\tilde{a}(\tilde{t}) = \Omega \, a(t)$$

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})(dx^2 + dy^2 + dz^2)$$

➢ The EoMs

$$\tilde{G}_{\mu\nu} = \sum \kappa \left(\tilde{T}_{\mu\nu}^{(i)} + \tilde{t}_{\mu\nu}^{(i)} \right) + \kappa \tilde{\mathcal{T}}_{\mu\nu} \,,$$

$$\frac{\partial \mathcal{L}_i}{\partial \phi} + \frac{(\partial_\alpha \phi)\left(4\partial_\phi A - 2X\partial_\phi B\right)}{(2X)(-4A_{,X} + 2B + 2XB_{,X})} \frac{\partial \mathcal{L}_i}{\partial(\partial_\alpha \phi)} = 0,$$

$$\frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left(\sqrt{-\tilde{g}} \, \tilde{g}^{\mu\nu} \partial_\nu \omega \right) - \frac{\partial V}{\partial \omega} + \sum_i \left(\frac{1}{\sqrt{-\tilde{g}}} \frac{\delta \mathcal{L}_i}{\delta \omega} - \frac{1}{\alpha} \tilde{g}^{\mu\nu} \tilde{t}_{\mu\nu}^{(i)} \right) = 0.$$

where

$$T_{\mu\nu}^{(i)} = \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{L}_i}{\delta \tilde{g}^{\mu\nu}},$$

$$t_{\mu\nu}^{(i)} = \frac{-2}{\sqrt{-g}} \frac{\partial \mathcal{L}_i}{\partial(\partial_\alpha \phi)} \left(\frac{-A\Omega^{-2}\tilde{g}_{\mu\nu} - 2A_{,X}\partial_\mu\phi\partial_\nu\phi + XB_{,X}\partial_\mu\phi\partial_\nu\phi}{(2X)(-4A_{,X} + 2B + 2XB_{,X})} \right) e^{2\omega/\alpha} (\partial_\alpha \phi),$$

$$\tilde{\mathcal{T}}_{\mu\nu} = \tilde{g}_{\mu\nu} \left(-\frac{1}{2} \tilde{g}^{\alpha\beta} \partial_\alpha \omega \, \partial_\beta \omega - V(\omega) \right) + \partial_\mu \omega \, \partial_\nu \omega$$

Modified Friedmann equations

- $\widetilde{H}^2 = \frac{8\pi G}{3}\left((\tilde{\bar{\rho}}_{\mathrm{M}} + \tilde{\rho}_{\omega}) + \tilde{\lambda}\tilde{\bar{\rho}}_{\mathrm{M}}\right) := \frac{8\pi G}{3}\tilde{\rho}_t$
- $\widetilde{H}' = -4\pi G\left((\tilde{\bar{\rho}}_{\mathrm{M}} + \tilde{\rho}_{\omega} + \tilde{P}_r + \tilde{P}_{\omega}) + \tilde{\lambda}(1+\widetilde{\omega}^{(\mathrm{in})})\tilde{\bar{\rho}}_{\mathrm{M}}\right)$
  $:= -4\pi G(\tilde{\rho}_t + \tilde{P}_t)$

with $\tilde{\bar{\rho}}_{\mathrm{M}} = \tilde{\rho}_{\mathrm{m}} + \tilde{\rho}_{\mathrm{r}}$, $\tilde{\rho}_{\omega} = \frac{1}{2}\omega'^2 + V(\omega)$, and $\hat{P}_{\omega} = \frac{1}{2}\omega'^2 - V(\omega)$.

➢ It can be shown that the total energy and pressure obeys the continuity equation:

$$\tilde{\rho}_t' + 3\widetilde{H}(\tilde{\rho}_t + \tilde{P}_t) = 0.$$

➢ Thermodynamics in Einstein frame can be considered as an equilibrium description.

1st & 2nd Law of Thermodynamics (in Einstein frame)

$\tilde{S}$: Horizon entropy in Einstein frame

$\widetilde{W}$: work density in Einstein frame

- First law: $\tilde{T}d\tilde{S} = -d\tilde{E} + \widetilde{W}d\tilde{V}$

$\tilde{T}$: Temperature in Einstein frame

$\tilde{E} = \tilde{\rho}_t\tilde{V}$ (Total energy within the horizon in Einstein frame)

- Second law: $\frac{d}{d\tilde{t}}\left(\tilde{S} + \tilde{S}_t\right) = \frac{12\pi\widetilde{H}'^2}{G\tilde{R}\widetilde{H}^3} \geq 0$

$d\tilde{S}_t$: Entropy change rate of ordinary matter and induced matter within the horizon

國立清華大學
NATIONAL TSING HUA UNIVERSITY

Summary

- Consider the physical metric directly coupled to matter.
- Consider the simple case: $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$.
- Interpret the effects of f(R) deviated from GR as dark energy.
- Assume Principle of Cosmology.
- Assume all the matter including dark energy as perfect fluids.

➡ Verify the first and second laws of thermodynamics in f(R) gravity with disformal transformation in both Jordan and Einstein frames.

THANK YOU FOR YOUR ATTENTION!

Back-up slides

- Consider only simple case : $\gamma_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1)$.

  Thus, the disformal transformation becomes

$$\eta_{\alpha\beta} = A(\phi,X)g_{\alpha\beta} + B(\phi,X)\partial_\alpha\phi\partial_\beta\phi$$

- $\delta\eta_{\alpha\beta} = 0 \;\Rightarrow\; \delta\left(\partial_\beta\phi\right) = \bar{V}_\beta A_{\mu\nu}\,\delta g^{\mu\nu} + \bar{V}_\beta\,\delta\phi$

where $\bar{V}_\beta = (\partial_\beta\phi)\dfrac{4\partial_\phi A - 2X\,\partial_\phi B}{(2X)(-4A_{,X}+2B+2XB_{,X})}$,

$$A_{\mu\nu} = \frac{-Ag_{\mu\nu}-A_{,X}\partial_\mu\phi\partial_\nu\phi+XB_{,X}\partial_\mu\phi\partial_\nu\phi}{4\partial_\phi A-2X\partial_\phi B}.$$

1st & 2nd Law of Thermodynamics (Non-equilibrium case)

- $\dot{\hat{\rho}}_t + 3H(\hat{\rho}_t + \hat{P}_t) = \frac{3H^2\dot{F}}{8\pi G}$ ⟹ Non-equilibrium Thermodynamics!

$$T = \frac{\kappa_s}{2\pi} = \frac{1}{2\pi \bar{r}_A}\left(1 - \frac{\dot{\bar{r}}_A}{2H\bar{r}_A}\right)$$

$$\hat{S} = \frac{FA}{4G}$$

$$W = \frac{1}{2}(\hat{\rho}_t - \hat{P}_t)$$

- First law: $T d\hat{S} + T d_i\hat{S} = -dE + \widehat{W} dV,$

$$d_i\hat{S} = -\frac{1}{T}\frac{\bar{r}_A}{2G}(1 + 2\pi\bar{r}_A T)dF$$

$$\hat{E} = V\hat{\rho}_t$$

- Second law: $\frac{d\hat{S}}{dt} + \frac{d_i\hat{S}}{dt} + \frac{d\hat{S}_t}{dt} = \frac{12\pi F\dot{H}^2}{GRH^3} \geq 0.$

$$d\hat{S}_t = \frac{1}{T}[d(\hat{\rho}_t V) + \hat{P}_t dV]$$

1st & 2nd Law of Thermodynamics (Equilibrium case)

- $\dot{\hat{\rho}}_\mathrm{t} + 3H\left(\hat{\rho}_\mathrm{t} + \hat{P}_\mathrm{t}\right) = 0$ ⟹ Equilibrium Thermodynamics!

$S = \frac{A}{4G}$

$W = \frac{1}{2}(\hat{\rho}_t - \hat{P}_t)$

- First law: $TdS = -dE + WdV$

$\hat{E} = V\hat{\rho}_t$

- $\dot{\hat{\rho}}_t + 3H(\hat{\rho}_t + \hat{P}_t) = 0$ Equilibrium Thermodynamics!
- First law: $TdS = -dE + WdV$
- Second law of thmodynamics: $\frac{d}{dt}(S + S_t) = \frac{12\pi\dot{H}^2}{GRH^3} \geq 0.$

- Second law of thmodynamics: $\frac{d}{dt}(S + S_t) = \frac{12\pi\dot{H}^2}{GRH^3} \geq 0.$

$dS_t = \frac{1}{T}[d(\rho_t V) + P_t dV]$

1st & 2nd Law of Thermodynamics (in Einstein frame)

$$\tilde{S} = \frac{\tilde{A}}{4G}$$

$$\tilde{W} = \frac{1}{2}(\tilde{\rho}_t - \tilde{P}_t)$$

- First law: $\tilde{T}d\tilde{S} = -d\tilde{E} + \tilde{W}d\tilde{V}$

$$\tilde{T} = \frac{1}{2\pi\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2\tilde{H}\tilde{r}_A}\right)$$

$$\tilde{E} = \tilde{\rho}_t\tilde{V}$$

- Second law: $\frac{d}{d\tilde{t}}(\tilde{S} + \tilde{S}_t) = \frac{12\pi\tilde{H}'^2}{G\tilde{R}\tilde{H}^3} \geq 0$

$$d\tilde{S}_t = \frac{1}{\tilde{T}}(d(\tilde{\rho}_t\tilde{V}) + \tilde{P}_t d\tilde{V}),$$