Black Holes lectures problem questions

Mahdi Godazgar

17–21 June, STFC PhD School, Southampton

<u>*Lecture 1*: Schwarzschild solution</u>

1. Assuming that the only non-zero component of the Einstein tensor of metric

$$ds^2 = -\Big(1 + 2\Phi(R)\Big)dt^2 + \Big(1 - 2\Phi(R)\Big)(dR^2 + R^2 d\Omega^2) \tag{1}$$

in the weak field limit (i.e. to leading order in Φ) is

$$G_{tt} = 2\Delta\Phi, \tag{2}$$

where $\Delta = \nabla^2$ is the three-dimensional Laplacian, show that the Schwarzschild metric in the weak field limit satisfies the Einstein equation $G_{ab} = 8\pi T_{ab}$ with

$$T^{ab} = \rho\, U^a U^b, \quad \rho = m\,\delta(\mathbf{x}),\ U^a = (1, \mathbf{0}). \tag{3}$$

2. i) Starting from the Schwarzschild metric in Schwarzschild coordinates

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2, \tag{4}$$

perform a coordinate transformation

$$v = t + r + 2m\log\left|\frac{r}{2m} - 1\right| \tag{5}$$

to obtain the Schwarzschild solution in ingoing Eddington-Finkelstein coordinates

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{6}$$

ii) Write down the coordinate transformation required in order to put the Schwarzschild metric in outgoing Eddington-Finkelstein form

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)du^2 - 2dudr + r^2 d\Omega^2. \tag{7}$$

3. The Kerr solution represents a rotating black hole with "angular momentum" parameter given by $J = aM$. In Boyer-Lindquist coordinates, the metric is given by

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{8}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2mr + a^2. \tag{9}$$

i) Show that this metric reduces to the Schwarzschild metric when $a = 0$.

ii) Show that the solution is singular for

$$r = 0, \qquad r = r_\pm = m \pm \sqrt{m^2 - a^2}. \tag{10}$$

What is special about the solution when

$$m = a? \tag{11}$$

iii) Show that the surface $r = r_+$ is a coordinate singularity by deriving a new metric in Kerr coordinates (v, r, θ, χ). What happens to the metric in Kerr coordinates for $a = 0$? *Hint: Try the coordinate transformations*

$$dv = dt + \frac{r^2+a^2}{\Delta}dr, \qquad d\chi = d\phi + \frac{a}{\Delta}dr. \tag{12}$$

Lecture 2: ADM formalism and energy

1. The ADM action is given by

$$S[N, N^i, h_{ij}] = \int dtd^3x\, N\sqrt{h}\left({}^{(3)}R + K^{ij}K_{ij} - K^2\right), \tag{13}$$

where

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n g_{ij}, \qquad n = \frac{1}{N}(\partial_t - N^i\partial_i), \tag{14}$$

is the second fundamental form and $K = h^{ij}K_{ij}$ is its trace. Note that all three-dimensional indices are raised/lowered with h^{ij}/h_{ij}, respectively.

i) Show that the canonical momentum conjugate to h_{ij}

$$\pi^{ij} \equiv \frac{\delta S}{\delta \dot{h}_{ij}} = -\sqrt{h}(K^{ij} - Kh^{ij}). \tag{15}$$

ii) From the definition of the Hamiltonian for canonical fields ϕ^I

$$H = \int d^3x \sum_I \pi_{\phi^I} \dot{\phi}^I - \mathcal{L} \tag{16}$$

derive the ADM Hamiltonian

$$H = \int d^3x \left\{ \pi_N \dot{N} + \pi_i \dot{N}^i + \sqrt{h}\left(N\mathcal{H} + N^i \mathcal{H}_i \right) \right\}, \tag{17}$$

where

$$\pi_N \approx 0, \qquad \pi_i \approx 0 \tag{18}$$

and

$$\mathcal{H} = -{}^{(3)}R + h^{-1}\pi^{ij}\pi_{ij} - \frac{1}{2}h^{-1}\pi^2, \tag{19}$$

$$\mathcal{H}_i = -2D^j(h^{-1/2}\pi_{ij}), \tag{20}$$

where D_i is the covariant derivative associated with metric h_{ij}.

2. In $n > 2$ dimensions, given a conformal transformation of the metric of the form

$$\tilde{g}_{ab} = e^{2\varphi} g_{ab}, \tag{21}$$

show that the Ricci scalars of the two metrics are related by the following equation

$$\tilde{R} = e^{-2\varphi}\left[R - \frac{4(n-1)}{(n-2)} e^{-(n-2)\varphi/2} \nabla^2 \left(e^{(n-2)\varphi/2} \right) \right]. \tag{22}$$

3. Starting from the Schwarzschild metric in isotropic coordinates (t, ρ, θ, ϕ)

$$ds^2 = -\frac{(1 - \frac{m}{2\rho})^2}{(1 + \frac{m}{2\rho})^2} dt^2 + \left(1 + \frac{m}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right), \tag{23}$$

show that, we recover the Schwarzschild metric in Schwarzschild coordinates via the coordinate transformation

$$r = \rho\left(1 + \frac{m}{2\rho}\right)^2. \tag{24}$$

<u>*Lecture 3*: Bondi energy and BMS charges</u>

1. Asymptotically flat spacetimes are defined as those for which Bondi coordinates $(t, r, x^I = \{\theta, \phi\})$ can be introduced such that the metric takes the Bondi form

$$ds^2 = -Fe^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{IJ}(dx^I - C^I du)(dx^J - C^J du), \tag{25}$$

with the metric functions satisfying the following fall-off conditions at large r

$$
\begin{aligned}
F(u,r,x^I) &= 1 + \frac{F_0(u,x^I)}{r} + \ldots, \\
\beta(u,r,x^I) &= \frac{\beta_0(u,x^I)}{r^2} + \ldots, \\
C^I(u,r,x^I) &= \frac{C_0^I(u,x^I)}{r^2} + \ldots, \\
h_{IJ}(u,r,x^I) &= \omega_{IJ} + \frac{C_{IJ}}{r} + \ldots,
\end{aligned} \tag{26}
$$

as well as the gauge condition

$$
h = \omega, \tag{27}
$$

where $h \equiv \det(h_{IJ})$ and $\omega \equiv \det(\omega_{IJ}) = \sin^2\theta$.

Consider all diffeomorphisms that leave the form of the Bondi metric invariant and show that these take the form

$$
\xi = f\partial_u + \xi^I\partial_I - \frac{r}{2}\left(D_I\xi^I - C^I D_I f\right)\partial_r, \tag{28}
$$

where

$$
\xi^I = Y^I(x^I) - \int_r^\infty dr' \frac{e^{2\beta}}{r'^2} h^{IJ} D_J f, \qquad f = s(x^I) + \frac{u}{2} D_I Y^I, \qquad D_{(I}Y_{J)} = \frac{1}{2} D_K Y^K \omega_{IJ} \tag{29}
$$

and D_I is the metric associated with the metric ω_{IJ}. The set of vectors ξ defined above generate the BMS algebra.

Hint: You may use the fact that a conformal Killing vector on the two-sphere, Y^I, satisfies the following equation

$$
D^2(D_I Y^I) + 2 D_I Y^I = 0. \tag{30}
$$

2. The general expression for the variation of an asymptotic charge for asymptotically flat spacetimes is

$$
\begin{aligned}
\delta\!\!\!/ \mathcal{Q}_\xi[\delta g, g] = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S^2} (d^2x)_{ab} \sqrt{-g}\, \Big\{ & \xi^b g^{cd} \nabla^a \delta g_{cd} - \xi^b g^{ac} \nabla^d \delta g_{cd} + \xi^c g^{ad} \nabla^b \delta g_{cd} \\
& + \frac{1}{2} g^{cd} \delta g_{cd} \nabla^b \xi^a + \frac{1}{2} g^{bd} \delta g_{cd} (\nabla^a \xi^c - \nabla^c \xi^a) \Big\},
\end{aligned} \tag{31}
$$

where

$$
(d^2x)_{ab} = \frac{1}{4} \eta_{abIJ}\, dx^J \wedge dx^J, \tag{32}
$$

Take the background metric to be a subset of the class of asymptotically flat spacetimes with metric

$$ds^2 = -F du^2 - 2du dr + r^2 \Omega^2, \tag{33}$$

so that the only non-zero component of δg_{ab} is

$$\delta g_{uu} = -\frac{\delta F_0}{r} + \dots \ . \tag{34}$$

Furthermore, choose

$$\xi = \partial_u . \tag{35}$$

Using the fact that

$$\Gamma^u_{IJ} = r\,\omega_{IJ}, \tag{36}$$

show that the right hand side of the above expression becomes integrable and conclude that

$$\mathcal{Q}_\xi = -\frac{1}{8\pi}\int_{S^2} d\Omega \ F_0, \tag{37}$$

i.e. we recover the Bondi energy.