# Solutions

## Lecture 1

1. Identifying the Schwarzschild metric in the weak field limit as metric (1) with $\Phi = -m/R$, we have that

$$G_{tt} = 2m \cdot 4\pi \, \delta(\underset{\sim}{x})$$

[Recall that $\Delta\left(-\frac{1}{4\pi R}\right) = \delta(\underset{\sim}{x})$ in spherical coords.]

This implies that the only non-vanishing compt of $T_{ab}$ is

$$T_{tt} = m \, \delta(\underset{\sim}{x})$$

Raising indices $\quad T^{tt} = m \, \delta(\underset{\sim}{x}) \qquad$ (to leading order)

$$\therefore \quad T^{ab} = m \, \delta(\underset{\sim}{x}) \, (1, \underset{\sim}{0})^a (1, \underset{\sim}{0})^b$$

$$= \rho U^a U^b \quad \text{as defined in eqn. (3).}$$

2. i)
$$dv = dt + \left(1 - \frac{2m}{r}\right)^{-1} dr$$

$$\Longrightarrow \; -\left(1 - \frac{2m}{r}\right) dt^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2\,dv\,dr - \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

Substituting above eqn into metric (4) gives metric (6).

ii)
$$u = t - \left( r + 2m \log\left|\frac{r}{2m} - 1\right| \right)$$

3. i) Setting $a = 0$, metric (8) becomes

$$ds^2 = -\frac{\Delta}{\Sigma} dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\left(d\theta^2 + \frac{r^4}{\Sigma^2}\sin^2\theta\, d\phi^2\right)$$

with $\Sigma = r^2$, $\Delta = r^2 - 2mr$

$$\therefore\ ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \text{, ie}$$

Schw. soln. in Schw. coords.

ii) The soln. is singular for

$$\Sigma = 0, \quad \Delta = 0$$

$$\Sigma = 0 \implies r = 0, \ \theta = \frac{\pi}{2}$$

$$\Delta = 0 \implies r^2 - 2mr + a^2 = 0 \implies r = \frac{2m \pm \sqrt{4m^2 - 4a^2}}{2}$$

$$r = r_\pm$$

When $m = 0$, $r_+ = r_- = m$. This is know as Extremal Kerr.

iii) $dt = dv - \frac{r^2+a^2}{\Delta} dr$, $d\phi = d\chi - \frac{a}{\Delta} dr$

$$dt^2 = dv^2 - 2\frac{r^2+a^2}{\Delta} dv\,dr + \frac{(r^2+a^2)^2}{\Delta^2} dr^2$$

$$dt\,d\phi = dv\,d\chi - \frac{a}{\Delta} dv\,dr - \frac{r^2+a^2}{\Delta} d\chi\,dr + \frac{a(r^2+a^2)}{\Delta^2} dr^2$$

$$d\phi^2 = d\chi^2 - \frac{2a}{\Delta} d\chi\,dr + \frac{a^2}{\Delta^2} dr^2$$

dvdr compt. : $+2\frac{(r^2+a^2)}{\Delta}\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} + \frac{2a^2\sin^2\theta}{\Delta}\frac{(r^2+a^2-\Delta)}{\Sigma} = \frac{2\Delta\Sigma}{\Delta\Sigma} = 2$

dχ dr compt. : $\frac{2a\sin^2\theta}{\Delta\Sigma}\left[(r^2+a^2-\Delta)(r^2+a^2) - (r^2+a^2)^2 + \Delta a^2\sin^2\theta\right] = -2a\sin^2\theta$

dr² compt. : $\frac{1}{\Sigma\Delta^2}\left[-(r^2+a^2)^2(\Delta - a^2\sin^2\theta) - 2a^2(r^2+a^2)\sin^2\theta(r^2+a^2-\Delta) + a^2\sin^2\theta\left\{(r^2+a^2)^2 - \Delta a^2\sin^2\theta\right\} + \Sigma^2\Delta\right]$

$$= \frac{1}{\Sigma\Delta^2} \Delta \left\{ -(r^2+a^2)^2 + 2a^2(r^2+a^2)\sin^2\theta - a^4\sin^4\theta + \Sigma^2 \right\} = 0 .$$

$$\Rightarrow ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} dv^2 + 2dvdr - 2a\sin^2\theta \frac{(r^2+a^2-\Delta)}{\Sigma} dv d\chi$$

$$- 2a\sin^2\theta \, d\chi dr + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \, d\chi^2 + \Sigma d\theta^2$$

Assuming that we started off with $r > r_+$, we find that the above metric is regular for $r = r_+$ $\Rightarrow$ we can extend the spacetime through the $r = r_+$ surface

$$a = 0 \Rightarrow ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2$$

i.e. analogue of ingoing Eddington-Finkelstein coords

## Lecture 2

1. i) $$\pi^{ij} = \frac{\delta S}{\delta \dot h_{ij}} = N\sqrt{h}\left(2K^{kl}\frac{\delta K_{kl}}{\delta \dot h_{ij}} - 2K\frac{\delta K}{\delta \dot h_{ij}}\right)$$

$$\frac{\delta K_{kl}}{\delta \dot h_{ij}} = -\frac{1}{2N}\delta^{(i}_{k}\delta^{j)}_{l}$$

$$\Longrightarrow \pi^{ij} = \sqrt{h}\left(-K^{ij} + K h^{ij}\right)$$

ii) $$\frac{\delta S}{\delta \dot N} = \frac{\delta S}{\delta \dot N^{i}} = 0 \Longrightarrow \pi_N \approx 0,\ \pi_i \approx 0$$

$$H = \int d^3x \left\{ \pi^{ij}\dot h_{ij} + \pi_i \dot N^i + \pi_N \dot N - N\sqrt{h}\left({}^{(3)}R + K^{ij}K_{ij} - K^2\right)\right\}$$

$$\dot h_{ij} = -2NK_{ij} + 2D_{(i}N_{j)}\ ,\quad K_{ij} = -h^{-\frac{1}{2}}\left(\pi_{ij} - \tfrac{1}{2}\pi h_{ij}\right)$$

$$\Longrightarrow H = \int d^3x \Big\{ \pi_i \dot N^i + \pi_N \dot N + 2N\pi^{ij}h^{-\frac{1}{2}}\left(\pi_{ij} - \tfrac{1}{2}\pi h_{ij}\right) + 2\pi^{ij}D_iN_j$$
$$- N\sqrt{h}\left[{}^{(3)}R + h^{-1}\left(\pi^{ij} - \tfrac{1}{2}\pi h^{ij}\right)\left(\pi_{ij} - \tfrac{1}{2}\pi h_{ij}\right) - \tfrac{1}{4}h^{-1}\pi^2\right]\Big\}$$

$$H = \int d^3x \Big\{ \pi_i \dot N^i + \pi_N \dot N + \sqrt{h}N\Big[\underbrace{-{}^{(3)}R + h^{-1}\pi^{ij}\pi_{ij} - \tfrac{1}{2}h^{-1}\pi^2}_{\mathcal{H}}\Big]$$
$$+ \sqrt{h}N^i\Big[\underbrace{-2D^j\left(h^{-\frac{1}{2}}\pi_{ij}\right)}_{\mathcal{H}_i}\Big] + \underbrace{2D_i\left(\pi^{ij}N_j\right)}_{\text{total derivative}}\Big\}$$

2. Let $\tilde{g}_{ab} = e^{2\varphi} g_{ab}$, $\tilde{e}^{\mu} = e^{\varphi} e^{\mu}$, $dx^i = \tilde{E}_{\mu}{}^{i} \tilde{e}^{\mu}$
$= E_{\mu}{}^{i} e^{\mu}$

$$d\tilde{e}^{\mu} + \tilde{\omega}^{\mu}{}_{\nu} \wedge \tilde{e}^{\nu} = 0$$

$$\Longrightarrow \underbrace{de^{\varphi} \wedge e^{\mu}}_{e^{\varphi} E_{\rho}{}^{i} \partial_i \varphi\, e^{\rho} \wedge e^{\mu}} + e^{\varphi} \underbrace{de^{\mu}}_{-(\omega^{\mu}{}_{\nu})_{\rho} e^{\rho} \wedge e^{\nu}} + (\tilde{\omega}^{\mu}{}_{\nu})_{\rho}\, \tilde{e}^{\rho} \wedge \tilde{e}^{\nu} = 0$$

$$\Longrightarrow \left[ (\tilde{\omega}^{\mu}{}_{[\nu})_{\rho]} - e^{-\varphi} (\omega^{\mu}{}_{[\nu})_{\rho]} + e^{-\varphi} \delta^{\mu}_{[\nu} \nabla_{\rho]} \varphi \right] \tilde{e}^{\rho} \wedge \tilde{e}^{\nu} = 0$$

$$\therefore \quad (\tilde{\omega}^{\mu}{}_{[\nu})_{\rho]} = e^{-\varphi} (\omega^{\mu}{}_{[\nu})_{\rho]} - e^{-\varphi} \delta^{\mu}_{[\nu} \nabla_{\rho]} \varphi \qquad (\dagger)$$

Let $(\tilde{\omega}^{\mu}{}_{\nu})_{\rho} = e^{-\varphi} (\omega^{\mu}{}_{\nu})_{\rho} + e^{-\varphi} \left[ a\, \delta^{\mu}_{\nu} \nabla_{\rho} \varphi + b\, \delta^{\mu}_{\rho} \nabla_{\nu} \varphi + c\, \eta_{\nu\rho} \nabla^{\mu} \varphi \right]$

$$\left.\begin{aligned} (\tilde{\omega}_{(\mu\nu)}) = 0 &\implies a = 0, \quad b + c = 0 \\ (\dagger) &\implies a - b = -1 \end{aligned}\right\} \implies a = 0,\ b = 1,\ c = -1$$

$$\therefore \quad \underline{(\tilde{\omega}^{\mu}{}_{\nu}) = (\omega^{\mu}{}_{\nu})_{\rho} e^{\rho} + \nabla_{\nu} \varphi\, e^{\mu} - \nabla^{\mu} \varphi\, e_{\nu}}$$

$$\begin{aligned}
\tilde{R}^{\mu}{}_{\nu} &= d\tilde{\omega}^{\mu}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\rho} \wedge \tilde{\omega}^{\rho}{}_{\nu} \\
&= R^{\mu}{}_{\nu} + d\left( \nabla_{\nu} \varphi\, e^{\mu} - \nabla^{\mu} \varphi\, e_{\nu} \right) \\
&\quad + (\omega^{\mu}{}_{\rho}) \wedge \left( \nabla_{\nu} \varphi\, e^{\rho} - \nabla^{\rho} \varphi\, e_{\nu} \right) + \left( \nabla_{\rho} \varphi\, e^{\mu} - \nabla^{\mu} \varphi\, e_{\rho} \right) \wedge \omega^{\rho}{}_{\nu} \\
&\quad + \left( \nabla_{\rho} \varphi\, e^{\mu} + \nabla^{\mu} \varphi\, e_{\rho} \right) \wedge \left( \nabla_{\nu} \varphi\, e^{\rho} - \nabla^{\rho} \varphi\, e_{\nu} \right) \\
&= R^{\mu}{}_{\nu} + \nabla_{\rho} \nabla_{\nu} \varphi\, e^{\rho} \wedge e^{\mu} - \nabla_{\rho} \nabla^{\mu} \varphi\, e^{\rho} \wedge e_{\nu} \\
&\quad + \nabla_{\rho} \varphi \nabla_{\nu} \varphi\, e^{\mu} \wedge e^{\rho} - |\nabla \varphi|^2 e^{\mu} \wedge e_{\nu} - \nabla_{\rho} \varphi \nabla^{\mu} \varphi\, e_{\nu} \wedge e^{\rho}
\end{aligned}$$

$$\tilde{R}^{\mu}{}_{\nu} = \Big\{ \tfrac{1}{2} R^{\mu}{}_{\nu\rho\sigma} + \delta^{\mu}_{[\sigma} \nabla_{\rho]} \nabla_{\nu} \varphi - \eta_{\nu[\sigma} \nabla_{\rho]} \nabla^{\mu} \varphi$$

$$\delta^{\mu}_{[\rho} \nabla_{\sigma]} \varphi \, \nabla_{\nu} \varphi - \eta_{\nu[\rho} \nabla_{\sigma]} \varphi \, \nabla^{\mu} \varphi - \delta^{\mu}_{[\rho} \eta_{\sigma]\nu} |\nabla\varphi|^{2} \Big\} e^{\rho} \wedge e$$

$$\Longrightarrow \tilde{R}^{\mu}{}_{\nu\rho\sigma} = e^{-2\varphi} \Big\{ R^{\mu}{}_{\nu\rho\sigma} - 2 \Big[ \delta^{\mu}_{[\rho} \nabla_{\sigma]} \nabla_{\nu} \varphi - \eta_{\nu[\rho} \nabla_{\sigma]} \nabla^{\mu} \varphi$$

$$- \delta^{\mu}_{[\rho} \nabla_{\sigma]} \varphi \, \nabla_{\nu} \varphi + \eta_{\nu[\rho} \nabla_{\sigma]} \varphi \, \nabla^{\mu} \varphi + \delta^{\mu}_{[\rho} \eta_{\sigma]\nu} |\nabla\varphi|^{2} \Big] \Big\}$$

$$\tilde{R}_{\nu\sigma} = e^{-2\varphi} \Big\{ R_{\nu\sigma} - \Big[ (n-2) \nabla_{\nu} \nabla_{\sigma} \varphi + \eta_{\nu\sigma} \nabla^{2} \varphi$$

$$- (n-2) \nabla_{\nu} \varphi \nabla_{\sigma} \varphi + (n-2) \eta_{\nu\sigma} |\nabla\varphi|^{2} \Big] \Big\}$$

$$= e^{-2\varphi} \Big\{ R_{\nu\sigma} - (n-2) \big( \nabla_{\nu} \nabla_{\sigma} \varphi - \nabla_{\nu} \varphi \nabla_{\sigma} \varphi \big)$$

$$- \Big[ \nabla^{2} \varphi + (n-2) |\nabla\varphi|^{2} \Big] \eta_{\nu\sigma} \Big\}$$

$$\tilde{R} = \eta^{\nu\sigma} \tilde{R}_{\nu\sigma} = e^{-2\varphi} \Big\{ R - 2(n-1) \nabla^{2} \varphi - (n-1)(n-2) |\nabla\varphi|^{2} \Big\}$$

$$\tilde{R} = e^{-2\varphi} \left[ R - \frac{4(n-1)}{n-2} \, e^{-\frac{(n-2)\varphi}{2}} \nabla^{2} e^{\frac{(n-2)}{2}\varphi} \right]$$

3.

$$ds^2 = -\frac{\left(1-\frac{m}{2\rho}\right)^2}{\left(1+\frac{m}{2\rho}\right)^2}dt^2 + \left(1+\frac{m}{2\rho}\right)^4\left(d\rho^2+\rho^2 d\Omega^2\right)$$

Let $r = \rho\left(1+\frac{m}{2\rho}\right)^2$

$$dr = d\rho\left[\left(1+\frac{m}{2\rho}\right)^2 + 2\rho\left(1+\frac{m}{2\rho}\right)\left(-\frac{m}{2\rho^2}\right)\right]$$

$$= d\rho\left(1+\frac{m}{2\rho}\right)\left(1+\frac{m}{2\rho}-\frac{m}{\rho}\right) = \left(1+\frac{m}{2\rho}\right)\left(1-\frac{m}{2\rho}\right)^2 d\rho$$

$$1-\frac{2m}{r} = 1-\frac{2m}{\rho}\left(1+\frac{m}{2\rho}\right)^{-2} = \left(1+\frac{m}{2\rho}\right)^{-2}\left[\left(1+\frac{m}{2\rho}\right)^2-\frac{2m}{\rho}\right]$$

$$= \left(1-\frac{m}{2\rho}\right)^2\left(1+\frac{m}{2\rho}\right)^{-2}$$

$$\Rightarrow ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \underbrace{\frac{\left(1+\frac{m}{2\rho}\right)^2}{\left(1-\frac{m}{2\rho}\right)^2}}_{\left(1-\frac{2m}{r}\right)^{-1}}dr^2 + r^2 d\Omega^2$$

$$\underline{ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2}$$

# Lecture 3

1. Want $\xi$ s.t.

$$\delta_\xi g_{uu} = O\left(\tfrac{1}{r}\right), \quad \delta_\xi g_{ur} = O\left(\tfrac{1}{r^2}\right), \quad \delta_\xi g_{uI} = O(1)$$

$$\delta_\xi g_{rr} = \delta_\xi g_{rI} = 0, \quad \delta g_{IJ} = O(r), \quad g^{IJ}\delta_\xi g_{IJ} = 0.$$

- $$\delta_\xi g_{rr} = \mathcal{L}_\xi g_{rr} = \xi^a \partial_a g_{rr} + 2 g_{ar}\partial_r \xi^a$$
$$= 2 g_{ur}\partial_r \xi^u = 0$$
$$\Longrightarrow \underline{\partial_r \xi^u = 0} \Longrightarrow \underline{\xi^u = f(u, x^I)} \qquad (+)$$

- $$\delta_\xi g_{rI} = \xi^a \partial_a g_{rI} + 2 g_{a(r}\partial_{I)}\xi^a$$
$$= g_{ur}\partial_I \xi^u + g_{aI}\partial_r \xi^a = g_{ur}\partial_I \xi^u + g_{uI}\underbrace{\partial_r \xi^u}_{\substack{=0 \\ \Leftarrow (+)}} + g_{JI}\partial_r \xi^J$$
$$= 0$$
$$\Longrightarrow \partial_r \xi^I = + \frac{h^{IJ}}{r^2} e^{2\beta}\partial_J f$$
$$\Longrightarrow \underline{\xi^I = Y^I(u, x^I) - \int_r^\infty dr' \frac{h^{IJ} e^{2\beta}}{r'^2}\partial_J f}$$

- $$g^{IJ}\delta_\xi g_{IJ} = g^{IJ}\left(\xi^a \partial_a g_{IJ} + 2 g_{a(I}\partial_{J)}\xi^a\right)$$
$$= \frac{\omega^{-1}}{r^4}\xi^a \partial_a (r^4 \omega) + 2 g^{IJ} g_{uI}\partial_J \xi^u + 2 g^{IJ} g_{KI}\partial_J \xi^K$$
$$= \frac{1}{r^4}\xi^r \partial_r r^4 + \frac{1}{\omega}\xi^I \partial_I \omega - 2C^I \partial_I f + 2\partial_K \xi^K$$
$$= \frac{4}{r}\xi^r - 2\left(C^I D_I f - D_K \xi^K\right)$$
$$\Longrightarrow \underline{\xi^r = -\frac{r}{2}\left(D_I \xi^I - C^I D_I f\right)}$$

$$\therefore\ \xi = f\partial_u + \Big[\underbrace{Y^I - \int_r^\infty dr'\, \frac{h^{IJ}e^{2\beta}}{r'^2} D_J f}_{\xi^I}\Big]\partial_I - \frac{r}{2}\left(D_I\xi^I - C^I D_I f\right)\partial_r$$

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$$\xi^I = Y^I - \frac{D^I f}{r} + \cdots$$

$$\xi^r = -\frac{r}{2}D_I Y^I + \frac{1}{2}D^2 f + \cdots$$

- $$\delta_\xi g_{ur} = \mathcal{L}_\xi g_{ur} = \xi^a\partial_a g_{ur} + 2 g_{a(u}\partial_{r)}\xi^a$$

$$= \underbrace{\xi^a\partial_a}_{O(1)}\big(\underbrace{-e^{2\beta}}_{-1+O(\frac{1}{r^2})}\big) + g_{au}\partial_r\xi^a + g_{ur}\partial_u\xi^u$$

$$= -\partial_r\xi^r - \partial_u f + O(\tfrac{1}{r^2})$$

$$= -\left(\partial_u f - \tfrac{1}{2}D_I Y^I\right) + O(\tfrac{1}{r^2})$$

$$= O(\tfrac{1}{r^2}) \implies \underline{\partial_u f = \tfrac{1}{2}D_I Y^I} \qquad (\#)$$

- $$\delta_\xi g_{uI} = \mathcal{L}_\xi g_{uI} = \xi^a\partial_a g_{uI} + g_{au}\partial_I\xi^a + g_{aI}\partial_u\xi^a$$

$$= \underbrace{\xi^a\partial_a}_{O(1)}\big(-r^2\underbrace{h_{IJ}C^J}_{O(1)}\big) + \underbrace{g_{uu}\partial_I\xi^u}_{O(1)} + g_{ur}\partial_I\xi^r + \underbrace{g_{uJ}\partial_I\xi^J}_{O(1)}$$

$$+ \underbrace{g_{uI}\partial_u\xi^u}_{O(1)} + g_{JI}\partial_u\xi^J$$

$$= r^2 h_{IJ}\partial_u\left(Y^J - \frac{D^J f}{r}\right) + \frac{r}{2}D_I\left(D_J Y^J\right) + O(1)$$

$O(r^2)$ : $\partial_u Y^J = 0$

$O(r)$ : $D_I\left[-\partial_u f + \tfrac{1}{2}D_J Y^J\right] = 0$ ✓ $\Leftarrow$ (#)

$$\implies \underline{Y^I = Y^I(x)}$$

Now, (#) $\Longrightarrow$

$$\underline{f = \frac{u}{2} D_I Y^I + s(x)}$$

- $$\delta_\xi g_{IJ} = \mathcal{L}_\xi g_{IJ} = \xi^a \partial_a g_{IJ} + 2 g_{a(I} \partial_{J)} \xi^a$$

$$= \xi^u \partial_u \underbrace{g_{IJ}}_{r^2\omega_{IJ} + O(r)} + \xi^r \partial_r g_{IJ} + \underbrace{2 g_{u(I} \partial_{J)} \xi^u}_{O(1)} + r^2 \mathcal{L}_{\hat\xi} h_{IJ}$$

where $\hat\xi^I = \xi^I$

$$= r^2 \left( \mathcal{L}_Y \omega_{IJ} - D_K Y^K \omega_{IJ} \right) + O(r)$$

$$\Longrightarrow \mathcal{L}_Y \omega_{IJ} = D_K Y^K \omega_{IJ}$$

or $$\underline{D_{(I} Y_{J)} = \frac{1}{2} D_K Y^K \omega_{IJ}}$$

- $$\delta_\xi g_{uu} = \mathcal{L}_\xi g_{uu} = \underbrace{\xi^a \partial_a}_{O(1)} \underbrace{g_{uu}}_{(1+O(\frac{1}{r}))} + 2 g_{ua} \partial_u \xi^a$$

$$= 2 g_{uu} \partial_u \xi^u + 2 g_{ur} \partial_u \xi^r + 2 \underbrace{g_{uI}}_{O(1)} \underbrace{\partial_u \xi^I}_{O(\frac{1}{r})} + O\left(\frac{1}{r}\right)$$

$$= -2 \partial_u f - D^2 \partial_u f + O\left(\frac{1}{r}\right)$$

$$= - \left( \underbrace{D\cdot Y + \frac{1}{2} D^2 (D\cdot Y)}_{=0} \right) + O\left(\frac{1}{r}\right)$$

$[D_I, D_J] Y^J = R_{IJ}{}^J{}_K Y^K = -Y_I$ $\qquad R_{IJKL} = 2\omega_{I[K}\omega_{L]J}$

$\Rightarrow D_I D\cdot Y - D_J D_I Y^J = -Y_I$

$\Rightarrow D^2(D\cdot Y) - D^I D^J D_I Y_J = -D\cdot Y$ ①

$D^I D^J D_I Y_J = D^I D^J \left( D_{(I} Y_{J)} + D_{[I} Y_{J]} \right) = \frac{1}{2} D^2(D\cdot Y) + \frac{1}{2} D\cdot Y - \frac{1}{2} D\cdot Y = \frac{1}{2} D^2 (D\cdot Y)$ ②

①&② $\Rightarrow$ $\underline{\frac{1}{2} D^2(D\cdot Y) + D\cdot Y = 0}$

2. Taking the metric to be

$$g_{ab} = \begin{pmatrix} -F & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2\omega_{IJ} \end{pmatrix}$$

with $F = 1 + \frac{F_0}{r} + \dots$

the only non-zero compt. of $\delta g_{ab}$ is $\delta g_{uu} = \frac{-\delta F_0}{r} + \dots$

The inverse metric is $g^{ab} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & F & 0 \\ 0 & 0 & \frac{1}{r^2}\omega^{IJ} \end{pmatrix}$

& $\sqrt{-g} = r^2\sqrt{\omega}$

Now, $(d^2x)_{ab}\sqrt{-g} = \eta_{ur\theta\phi}\, d\theta d\phi\, \delta^{ur}_{ab}\, r^2\sqrt{\omega} = \underline{d\Omega\, r^2\, \delta^{ur}_{ab}}$

Taking $\xi = \partial_u$

$$\$Q_\xi = \frac{1}{8\pi}\int_{S^2_\infty} d\Omega\, r^2 \Big\{ g^{cd}\xi^{[r}\nabla^{u]}\delta g_{cd} - \xi^{[r}g^{u]c}\nabla^d\delta g_{cd} + \xi^c g^{d[u}\nabla^{r]}\delta g_{cd} + \tfrac{1}{2}\cancel{g^{uu}}\delta g_{uu}\nabla^{[r}\xi^{u]} + \tfrac{1}{2}g^{u[r}\delta g_{uu}\cancel{\left(\nabla^{u]}\xi^u - \nabla^{|u|}\xi^{u]}\right)} \Big\}$$

$$= \frac{1}{16\pi}\int_{S^2_\infty} d\Omega\, r^2 \Big\{ -g^{cd}\nabla^r\delta g_{cd} + g^{rc}\nabla^d\delta g_{cd} + g^{du}\nabla^r\delta g_{ud} - g^{dr}\nabla^u\delta g_{ud} \Big\}$$

$$= \frac{1}{16\pi}\int_{S^2_\infty} d\Omega\, r^2 \Big\{ -\nabla^r\left(g^{cd}\delta g_{cd} - g^{rc}\delta g_{cr} - g^{du}\delta g_{ud}\right) + \cancel{g^{rc}\nabla^u\delta g_{cu}} + g^{rc}\nabla^I\delta g_{cI} - \cancel{g^{dr}\nabla^u\delta g_{ud}} \Big\}$$

$$= \frac{1}{16\pi}\int_{S^2_\infty} d\Omega\, r^2 \left\{ -\nabla^r \left( g^{IJ}\delta g_{IJ} \right) + g^{rc}\nabla^I \delta g_{cI} \right\}$$

$$= \frac{1}{16\pi}\int_{S^2_\infty} d\Omega\, r^2 \left\{ -g^{IJ} g^{rc} \underbrace{\nabla_c \delta g_{IJ}}_{=0} + g^{rc} g^{IJ} \nabla_J \delta g_{cI} \right\}$$

$$= \frac{1}{16\pi}\int_{S^2_\infty} d\Omega\, r^2\, g^{rc} g^{IJ} \left( \underbrace{\partial_J \delta g_{cI} - \Gamma^d_{Jc}\delta g_{dI}}_{=0} - \Gamma^d_{JI}\delta g_{cd} \right)$$

$$= \frac{1}{16\pi}\int_{S^2_\infty} d\Omega\, r^2\, \underbrace{g^{ur} g^{IJ}}_{-\frac{1}{r^2}\omega^{IJ}} \underbrace{\left( -\Gamma^u_{IJ} \right)}_{-r\omega_{IJ}} \underbrace{\delta g_{uu}}_{-\delta F_0/r + \cdots}$$

$$= \frac{1}{16\pi}\int_{S^2_\infty} d\Omega \left[ -2\delta F_0 + \underbrace{\cancel{\cdots}}_{0} \right]$$

$$= -\frac{1}{8\pi}\int_{S^2_\infty} d\Omega\, \delta F_0$$

$$\Longrightarrow \quad Q_\xi = -\frac{1}{8\pi}\int_{S^2_\infty} d\Omega\, F_0$$