Exercises on Scattering Amplitudes in QFT and String Theory

Note: If you get the expected result up to a simple numerical factor, consider yourself correct.

1. Consider the four-point MHV (partial) amplitude in Yang-Mills theory, $A(1^-2^-3^+4^+)$.

a) Show that it satisfies the appropriate soft theorem, say for particle 4^+.

b) Derive this amplitude via the BCFW recursion relation, using the 2-particle deformation

$$p_1 = |1\rangle[1| \to \hat{p}_1 = |1\rangle[\hat{1}| = |1\rangle([1| + z[4|)\,, \quad p_4 = |4\rangle[4| \to \hat{p}_4 = |4\rangle[4| = (|4\rangle - z|1\rangle)[4|\,, \quad z \in \mathbb{C}\,.$$

[This is the hardest exercise in this sheet.]

c) Write down the full (i.e., colour-dressed) amplitude, using colour traces. Show that, if one of the particles is a photon (its Lie algebra generator is the identity), then the full amplitude vanishes – this is the so-called U(1) decoupling identity.

2. Up to overall factors, the tree-level scattering amplitude for n tachyons in closed string theory ($m^2 = -4/\alpha'$) is

$$\int \frac{\prod_{i=1}^4 d^2 z_i}{\text{vol SL}(2,\mathbb{C})} \prod_{j<k} |z_{jk}|^{\alpha' p_j \cdot p_k}\,, \tag{1}$$

where $z_{ij} = z_i - z_j$. For $n = 4$, use the SL$(2,\mathbb{C})$ freedom to fix 3 punctures at 0, 1 and $R \to \infty$, and then use the result

$$\int d^2 z\, |z|^{2a-2} |1-z|^{2b-2} = \frac{2\pi \Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}\,, \qquad \text{with} \quad a+b+c = 1\,, \tag{2}$$

to reproduce the Virasoro-Shapiro amplitude.

3. Sphere integrals such as

$$\frac{\alpha'}{4\pi} \int \frac{\prod_{i=1}^4 d^2 z_i}{\text{vol SL}(2,\mathbb{C})} \prod_{j<k} |z_{jk}|^{\alpha' p_j \cdot p_k} \frac{1}{z_{12} z_{23} z_{34} z_{41}} \frac{1}{\bar{z}_{12} \bar{z}_{24} \bar{z}_{43} \bar{z}_{31}} \tag{3}$$

appear in closed string amplitudes with four massless external states. For reference, the α' expansion of this expression is

$$\frac{1}{s_{12}} \exp\left(-2 \sum_{k=1}^{\infty} \frac{\zeta_{2k+1}}{2k+1} \left(\frac{-\alpha'}{4} \right)^{2k+1} \left(s_{12}^{2k+1} + s_{13}^{2k+1} + s_{14}^{2k+1} \right) \right)\,, \tag{4}$$

where $s_{ij} = -(p_i + p_j)^2$ and $\zeta_k = \sum_{n=0}^{\infty} n^{-k}$.

In the CHY formalism for tree-level scattering amplitudes in theories of massless particles, there are integrals such as

$$\int d\mu_4^{(\mathrm{CHY})} \, \frac{1}{z_{12}z_{23}z_{34}z_{41}} \, \frac{1}{z_{12}z_{24}z_{43}z_{31}} \,, \tag{5}$$

where the measure (after the $\mathrm{SL}(2,\mathbb{C})$-fixing of punctures 1,2,3) is

$$d\mu_4^{(\mathrm{CHY})} = dz_4 \, (z_{12}z_{23}z_{31})^2 \, \delta(E_4) \,. \tag{6}$$

The n-particle CHY measure localises a moduli space integral on the solutions of the so-called *scattering equations*,

$$E_i = \sum_{j \neq i} \frac{2p_i \cdot p_j}{z_{ij}} = 0 \,. \tag{7}$$

Only $n-3$ of the equations are linearly independent, and there exist $(n-3)!$ solutions up to $\mathrm{SL}(2,\mathbb{C})$ transformations.

The exercise is to calculate the CHY integral above. Compare the result to the limit $\alpha' \to 0$ of the closed string integral. [This is a particular case of a general result. Using integrals of these types (with different orderings in the 'Parke-Taylor' denominators) it is possible to reveal a deep connection between field theory and string theory amplitudes. For instance, we can use such integrals to write type II supergravity amplitudes (using CHY integrals) and type II superstring amplitudes (using α'-dependent sphere integrals), so that the kinematic coefficients of the integrals are the same in both theories.]

4. Contributions like

$$\frac{\mathrm{tr}(T^{a_1}T^{a_2}\cdots T^{a_n})}{z_{12}z_{23}\cdots z_{n1}} \tag{8}$$

arise directly from correlators of 2d CFTs, namely from 'current algebras', $\langle j^{a_1}(z_1) j^{a_2}(z_2)\cdots\rangle$.

a) Show that the 'Parke-Taylor' denominator above gives rise to the denominator of an MHV amplitude (the Parke-Taylor formula), up to a factor independent of the particle ordering, if we take

$$z_i = \frac{\langle ia\rangle}{\langle ib\rangle} \,, \tag{9}$$

where $|a\rangle$ and $|b\rangle$ are arbitrary spinors (not proportional to each other or to any $|i\rangle$).

b) Show that this choice of the z_i solves the scattering equations (7) for any n in four spacetime dimensions. [For $n = 4$, (9) relates to the solution of the scattering equations that you hopefully found in the previous exercise by an $\mathrm{SL}(2,\mathbb{C})$ transformation, since there are $(n-3)!$ solutions at n points. The observations above are related to how (ambi)twistor strings – chiral 2d CFTs – reproduce QFT amplitudes.]