SOLUTIONS

1.

a) $p_4 \to \epsilon^2 p_4$: $|4\rangle \to \epsilon|4\rangle$

$|4] \to \epsilon|4]$

$$A(1^-2^-3^+4^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \underbrace{\frac{\langle 31\rangle}{\langle 34\rangle\langle 41\rangle}}_{\text{Soft factor}} \cdot \underbrace{\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}}_{A(1^-2^-3^+)}$$

Soft factor should be $\dfrac{\epsilon_4^+\cdot p_1}{p_4\cdot p_1} - \dfrac{\epsilon_4^+\cdot p_3}{p_4\cdot p_3}$

Ignore $\sqrt{2}$ factor in ϵ_4^+:

$$\frac{1}{\langle r4\rangle}\left(\frac{\langle r1\rangle\cancel{[14]}}{\langle 41\rangle\cancel{[14]}} - \frac{\langle r3\rangle\cancel{[34]}}{\langle 43\rangle\cancel{[34]}}\right) =$$

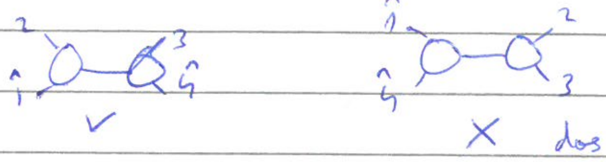
$$= \frac{1}{\langle 34\rangle\langle 41\rangle}\left(\frac{\langle r1\rangle\langle 34\rangle + \langle r3\rangle\langle 41\rangle}{\langle r4\rangle}\right) =$$

$\Big(\langle r1\rangle\langle 34\rangle + \langle r3\rangle\langle 41\rangle = -\langle r4\rangle\langle 13\rangle$ Schouten id.$\Big)$

$$= \frac{\langle 31\rangle}{\langle 34\rangle\langle 41\rangle} \quad \checkmark$$

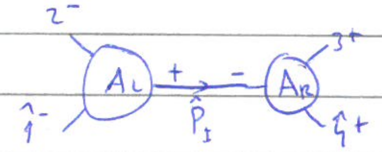
1.

b) Potential channels of partial amplitude (colour ordered)



✗ does not appear in recursion because internal line is undeformed $\hat p_1 + \hat p_4 = p_1 + p_4$

Then

$A(1^- 2^- 3^+ 4^+) =$  $=$ $\quad P_I = p_3 + p_4$

$$= A(\hat 1^-, 2^-, \hat P_I^+)\,\frac{1}{P_I^2}\,A(-\hat P_I^-, 3^+, \hat 4^+) =$$

$|\hat 1\rangle = |1\rangle$, $|\hat 4] = |4]$

$$= \left(\frac{\langle 12\rangle^3}{\langle 2\hat P_I\rangle\langle \hat P_I 1\rangle}\right)\frac{1}{s_{34}}\left(\frac{[43]^3}{[-\hat P_I 4][3 -\hat P_I]}\right) =$$

$\langle ij\rangle^* = [ji]$

$\left(-\hat P_I = |-\hat P_I\rangle[-\hat P_I| = |\hat P_I\rangle(-[\hat P_I|)\right)$ choice

$$= \frac{\langle 12\rangle^3[43]^3}{s_{34}\,\langle 1|\hat P_I|4]\,\langle 2|\hat P_I|3]}$$ $\hat P_I = p_3 + \hat p_4$

$$= \frac{\langle 12\rangle^3[43]^3}{s_{34}\,\langle 1|p_3|4]\,\langle 2|\hat p_4|3]} =$$

$$= \frac{\langle 12\rangle^3[43]^3}{s_{34}\,\langle 13\rangle[34]\left(\langle 24\rangle[43] - z_I\langle 21\rangle[43]\right)}$$

$Q_I = -|1\rangle[4|$

- $z_I = -\frac{P_I^2}{2P_I\cdot Q_I} = \frac{s_{34}}{\langle 1|P_I|4]} = \frac{\langle 34\rangle[43]}{\langle 13\rangle[34]} = +\frac{\langle 34\rangle}{\langle 31\rangle}$
- $\langle 24\rangle[43] - z_I\langle 21\rangle[43] = [43]\left(\langle 24\rangle - \frac{\langle 34\rangle\langle 21\rangle}{\langle 31\rangle}\right) =$

$$= \frac{[43]}{\langle 31\rangle}\left(\langle 31\rangle\langle 24\rangle + \langle 34\rangle\langle 12\rangle\right) = \frac{[43]}{\langle 31\rangle}\left(-\langle 32\rangle\langle 41\rangle\right)$$ Schouten

$$= \frac{\langle 12\rangle^3[43]^3}{s_{34}\,\langle 13\rangle[34]\,\frac{[43]}{\langle 31\rangle}\,\langle 23\rangle\langle 41\rangle} = \frac{\langle 12\rangle^3\,\cancel{[43]^3}}{\langle 34\rangle\cancel{[43]^3}\langle 23\rangle\langle 41\rangle} = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

✓

1.

c) non-cyclic permutations + reflection property

$$A_4 = A(1234)\left[\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \mathrm{tr}(T^{a_4}T^{a_3}T^{a_2}T^{a_1})\right]$$
$$+ A(1342)\left[\mathrm{tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) + \mathrm{tr}(T^{a_2}T^{a_4}T^{a_3}T^{a_1})\right]$$
$$+ A(1423)\left[\mathrm{tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) + \mathrm{tr}(T^{a_3}T^{a_2}T^{a_4}T^{a_1})\right]$$

$$A(1^-2^-3^+4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$A(1^-3^+4^+2^-) = \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle}$$

$$A(1^-4^+2^-3^+) = \frac{\langle 12\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}$$

Take $T^{a_4} = \mathbb{1}$,

$$A_4 = \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\left[A(1234) + A(1342) + A(1423)\right]$$
$$+ \mathrm{tr}(T^{a_1}T^{a_3}T^{a_2})\left[\qquad \text{''} \qquad\right]$$

$$\frac{A(1234) + A(1342) + A(1423)}{\langle 12\rangle^4} =$$

$$= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(1 + \frac{-\langle 23\rangle\langle 41\rangle}{\langle 13\rangle\langle 42\rangle} + \frac{-\langle 12\rangle\langle 34\rangle}{\langle 42\rangle\langle 31\rangle}\right) =$$

$$= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 13\rangle\langle 42\rangle}\underbrace{\left(\langle 31\rangle\langle 24\rangle + \langle 23\rangle\langle 14\rangle + \langle 12\rangle\langle 34\rangle\right)}_{=0 \ \text{Schouten}} =$$

$$= 0$$

2.

Fix z_1, z_2, z_3,

$$\int d^2 z_4 \; |z_{12}|^2 |z_{23}|^2 |z_{31}|^2 \times |z_{12}|^{\alpha' p_1\cdot p_2} \, |z_{13}|^{\alpha' p_1\cdot p_3} \, |z_{14}|^{\alpha' p_1\cdot p_4} \times$$

$$\times \, |z_{23}|^{\alpha' p_2\cdot p_3} \, |z_{24}|^{\alpha' p_2\cdot p_4} \, |z_{34}|^{\alpha' p_3\cdot p_4}$$

Take $z_1 = 0$, $z_2 = 1$, $z_3 = R \to \infty$, and set $z_4 = z$:

$$\int d^2 z \; \cancel{|R|^4} \; \cancel{|R|^{\alpha'(p_1+p_2+p_4)\cdot p_3}} \; |z|^{\alpha' p_1\cdot p_4} \, |1-z|^{\alpha' p_2\cdot p_4}$$

since $\alpha'(p_1+p_2+p_4)\cdot p_3 \underset{\uparrow \; \sum_i p_i = 0}{=} -\alpha' p_3^2 = \alpha' m^2 \overset{m^2 = -\frac{4}{\alpha'}}{=} -4$

Then we can use

- $2a - 2 = \alpha' p_1\cdot p_4 \Leftrightarrow$

  $\Leftrightarrow a = 1 + \frac{\alpha'}{2} p_1\cdot p_4 \underset{\uparrow \text{ use } s_{ij} = -(p_i+p_j)^2}{=} 1 + \frac{\alpha'}{4}\left(-s_{14} - p_1^2 - p_4^2\right) \overset{p_i^2 = -m^2 = \frac{4}{\alpha'}}{=} -1 - \frac{\alpha'}{4} s_{14}$

- $2b - 2 = \alpha' p_2\cdot p_4 \Leftrightarrow b = -1 - \frac{\alpha'}{4} s_{24} = -1 - \frac{\alpha'}{4} s_{13}$

- $c = 1 - a - b = -1 - \frac{\alpha'}{2}\left(p_1\cdot p_4 + p_2\cdot p_4\right) \overset{\sum_i p_i = 0}{=}$

  $= -1 + \frac{\alpha'}{2}\left(p_4^2 + p_4\cdot p_3\right) = 1 + \frac{\alpha'}{2} p_4\cdot p_3$

  $\Rightarrow c = -1 - \frac{\alpha'}{4} s_{34} = -1 - \frac{\alpha'}{4} s_{12}$

Up to numerical factor, get

$$\frac{\Gamma(-1 - \alpha' s_{12}/4)\,\Gamma(-1 - \alpha' s_{13}/4)\,\Gamma(-1 - \alpha' s_{14}/4)}{\Gamma(2 + \alpha' s_{12}/4)\,\Gamma(2 + \alpha' s_{13}/4)\,\Gamma(2 + \alpha' s_{14}/4)}$$

✓

3.

$$\int dz_4 \, (z_{12} z_{23} z_{31})^2 \, \delta(E_4) \frac{1}{z_{12} z_{23} z_{34} z_{41}} \frac{1}{z_{12} z_{24} z_{43} z_{31}}$$

Take $z_1 = 0$, $z_2 = 1$, $z_3 = R \to \infty$, and set $z_4 = z$:

$$\int dz \, \cancel{R^4} \, \delta(E_4) \frac{1}{\cancel{R^2} z} \frac{1}{\cancel{R^2} (1 - z)}$$

- $E_4 = 2 \frac{p_4 \cdot p_1}{z_{41}} + 2 \frac{p_4 \cdot p_2}{z_{42}} + 2 \frac{p_4 \cdot p_3}{z_{43}} = 2 \frac{p_4 \cdot p_1}{z} + \frac{2 p_4 \cdot p_2}{z - 1}$
- $\delta(E_4) = \frac{\delta(z - z_*)}{\partial_z E_4}$
- $E_4(z_*) = 0 \Leftrightarrow p_4 \cdot p_1 (z_* - 1) = - p_4 \cdot p_2 \, z_* \Leftrightarrow$

  $\Leftrightarrow z_* \underbrace{p_4 \cdot (p_1 + p_2)}_{= - p_3 \cdot p_4} = p_4 \cdot p_1 \Leftrightarrow z_* = - \frac{p_4 \cdot p_1}{p_4 \cdot p_3}$

So we get

$$\frac{1}{- 2 \frac{p_4 \cdot p_1}{z_*^2} - \frac{2 p_4 \cdot p_2}{(z_* - 1)^2}} \times \frac{1}{z_*} \frac{1}{(1 - z_*)} =$$

$$= - \frac{1}{2} \frac{1}{p_4 \cdot p_1 \left(\frac{1 - z_*}{z_*} \right) + p_4 \cdot p_2 \left(\frac{z_*}{1 - z_*} \right)} = - \frac{1}{2} \frac{1}{p_4 \cdot p_2 + p_4 \cdot p_1} = \frac{1}{2 p_4 \cdot p_3} =$$

$$\left(\frac{1 - z_*}{z_*} = \frac{p_4 \cdot p_3 + p_4 \cdot p_1}{- p_4 \cdot p_1} = \frac{p_4 \cdot p_2}{p_4 \cdot p_1} \right)$$

$$= - \frac{1}{s_{34}} = - \frac{1}{s_{12}} \qquad \left(s_{ij} = - (p_i + p_j)^2 \right)$$

Up to overall sign, agrees with closed string integral for $\alpha' \to 0$.

4.

a) $$z_{ij} = z_i - z_j = \frac{1}{\langle ib\rangle\langle jb\rangle}\left(\underbrace{\langle ia\rangle\langle jb\rangle + \langle ib\rangle\langle aj\rangle}_{\substack{= -\langle ij\rangle\langle ba\rangle \\ \text{Schouten}}}\right) = \frac{\langle ij\rangle\langle ab\rangle}{\langle ib\rangle\langle jb\rangle}$$

$$\Rightarrow \frac{1}{z_{12}z_{23}\cdots z_{m1}} = \frac{\prod_{i=1}^{m}\langle ib\rangle^2}{\langle ab\rangle^m}\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle m1\rangle}$$

b)
$$\sum_{j\neq i}\frac{2p_i\cdot p_j}{z_{ij}} = \frac{\langle ib\rangle}{\langle ab\rangle}\sum_{j\neq i}\frac{\langle jb\rangle\cancel{\langle ij\rangle}[ji]}{\cancel{\langle ij\rangle}} = -\frac{\langle ib\rangle}{\langle ab\rangle}\sum_{j\neq i}\langle b|p_j|i] =$$

$$\overset{\sum_i p_i = 0}{=} \frac{\langle ib\rangle}{\langle ab\rangle}\langle b|\underbrace{p_i|i]}_{=0} = 0$$