1 CFT and renormalization

1. Consider Maxwell theory in $d = 4$,

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{1}$$

and the operator $O_4 =: F_{\mu\nu}F^{\mu\nu}:$. We would like to compute the 2-point function of this operator. To this aim we introduce a source Φ_0 and consider the coupling $S_{src} = \int d^4x \Phi_0 O_4$. (The subscripts denote the dimension of the operator or source).

   (a) Use dimensional regularisation, $d = 4+\epsilon$, to show that that the regulated 2-point function is given by

$$\langle O_4(p_1)O_4(p_2)\rangle_{reg} = (2\pi)^4\delta(p_1+p_2) \left(\frac{(p_1^2)^2}{(2\pi)^2} \left(\frac{2}{\epsilon} + \ln p_1^2 + \gamma - 2 - \ln 4\pi \right) + O(\epsilon) \right). \tag{2}$$

   (b) Show the infinity may be canceled with a counterterm of the form,

$$S_{ct} = a_{ct} \frac{1}{\epsilon} \int d^{4+\epsilon}x \mu^{-\epsilon} \Phi_0 \Box^2 \Phi_0 \tag{3}$$

   for an appropriate constant a_{ct}. Determine this value.

   (c) Show that the renormalised correlator is

$$\langle O_4(p_1)O_4(p_2)\rangle_{ren} = (2\pi)^4\delta(p_1+p_2) \frac{(p_1^2)^2}{(2\pi)^2} \left(\ln \frac{p_1^2}{\mu^2} + c_0 \right) \tag{4}$$

   where $c_0 = \gamma - 2 - \ln 4\pi$. Show that one can set c_0 to any value by using a local finite counterterm.

   (d) What is the meaning of the μ dependence?

   (e) *(Bonus question)* Fourier transform the answer to position space. Does it agree with what you expect?

The following integral may be useful

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^\alpha((p-k)^2)^\beta} = \frac{\Gamma\left(\alpha+\beta-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\alpha\right)\Gamma\left(\frac{d}{2}-\beta\right)}{(4\pi)^{d/2}\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)} \frac{1}{(p^2)^{\alpha+\beta-d/2}}. \tag{5}$$

2 AdS/CFT

1. Consider AdS in domain-wall coordinates

$$ds^2 = dr^2 + e^{2r} dx^i dx^i \tag{6}$$

Show that the two independent solution of a free massive scalar equation

$$(\Box - m^2)\Phi(r) = 0, \qquad m^2 = \Delta(\Delta - d) \tag{7}$$

that only depend on r are for general Δ

$$\Phi_1 = e^{-(d-\Delta)r}, \qquad \Phi_2 = e^{-\Delta r} \tag{8}$$

Is there any special value of Δ?

Consider $\Delta = d/2+k, k = 0, 1, \ldots$, and check whether these solutions are normalizable, i.e. whether the radial integration in

$$\int_0^\infty dr \sqrt{G} \Phi^2 \tag{9}$$

(G is the determinant of (6)) is convergent or not.

2. Consider a free massless scalar Φ in AdS_5,

$$S = \int d^5x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right) \tag{10}$$

where $g_{\mu\nu}$ is the AdS_5 metric. Work in domain-wall coordinates, as in the previous exercise.

a) What is the dimension of the dual operator?

b) Show that the radial canonical momentum admits the following expansion

$$\pi(\Phi) = \pi_{(2)} + \pi_{(4)} + \tilde{\pi}_{(4)} \log e^{-2\epsilon} + \cdots \tag{11}$$

with $\pi_{(4)}$ undetermined and

$$\tilde{\pi}_{(2)}[\Phi] = -\frac{1}{2}\Box\Phi, \qquad \tilde{\pi}_{(4)} = \frac{1}{8}\Box^2\Phi, \tag{12}$$

where $\Box = \partial^i \partial_i$ and ϵ is the cut-off $r \leq \epsilon$.

c) Compute the counterterm action.

d) Compute the 2-point function of the dual operator.