CFT$_{d>2}$

1 Exercise: Conformal Algebra

In a CFT the stress tensor $T^{\mu\nu}(x)$ is conserved and traceless

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad T^\mu_{\ \mu}(x) = 0. \tag{1.1}$$

Given a vector field $\epsilon = \epsilon^\mu(x)\partial_\mu$ we can then consider a conserved current $\partial_\mu J^\mu_\epsilon(x) = 0$

$$J^\mu_\epsilon(x) = \epsilon_\mu(x) T^{\mu\nu}(x), \tag{1.2}$$

out of which we can generate the charge (topological surface operator)

$$Q_\epsilon(\Sigma) = -\int_\Sigma dS_\mu J^\mu_\epsilon(x). \tag{1.3}$$

a) Apart from the obvious symmetries that follow from $\partial_\mu T^{\mu\nu}(x) = 0$

$$\begin{aligned} p_\mu &= \partial_\mu, \\ m_{\mu\nu} &= x_\nu \partial_\mu - x_\mu \partial_\nu \end{aligned} \tag{1.4}$$

check that due to tracelessness $T^\mu_{\ \mu}(x) = 0$ there are extra conserved charges

$$\begin{aligned} d &= x^\mu \partial_\mu, \\ k_\mu &= 2x_\mu (x \cdot \partial) - x^2 \partial_\mu \ . \end{aligned} \tag{1.5}$$

b) Using that

$$[Q_{\epsilon_1}, Q_{\epsilon_2}] = Q_{-[\epsilon_1, \epsilon_2]}. \tag{1.6}$$

Check that the new charges satisfy the algebra

$$\begin{aligned} [D, P_\mu] = P_\mu, \quad [D, K_\mu] &= -K_\mu, \\ [K_\mu, P_\nu] &= 2\delta_{\mu\nu} D - 2M_{\mu\nu} \ . \end{aligned} \tag{1.7}$$

c) Conformal primary operators are annihilated by $K_\mu|\mathcal{O}\rangle = K_\mu \mathcal{O}(0)|\Omega\rangle = 0$. Given a primary operator we can construct descendants by acting on it with derivatives $P_{\mu_1}...P_{\mu_n}\mathcal{O}(0)|\Omega\rangle$. In the radial quantization

$$P^\dagger_\mu = K_\mu. \tag{1.8}$$

In a unitary CFT norms of the states are non-negative.[1] Using this and conformal algebra derive the following unitarity bounds:

[1] $\langle\mathcal{O}| = \langle\Omega|\mathcal{O}^\dagger(0) = \lim_{y\to\infty} y^{2\Delta}\mathcal{O}(y)$.

1. For a scalar primary operator $\mathcal{O}$, $\Delta \geq \frac{d-2}{2}$; when $\Delta = \frac{d-2}{2}$, $\mathcal{O}$ is a free scalar.
2. For a spin one primary operator V^μ, $\Delta \geq d-1$; when $\Delta = d-1$, $\partial_\mu V^\mu = 0$.
3. Generalize to an arbitrary representation of SO(d) (section 7.3 in Simmons-Duffin).

d) The conformal Killing vectors found above generate infinitesimal diffeomorphisms $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$. Using the conformal Killing equation show that infinitesemal conformal transformations locally look like a rotation and a scale transformation.

In the same way, finite conformal transformations take the form

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x') R^\mu{}_\nu(x'), \quad R^T R = 1,\ R \in \mathrm{SO}(d). \tag{1.9}$$

Indeed, this changes the metric by a scale factor

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu = \frac{\delta_{\mu\nu} dx'^\mu dx'^\nu}{\Omega(x')^2}\ . \tag{1.10}$$

Show that under conformal transformation (use the fact that special conformal transformation is translation at infinity)

$$x_{ij}^2 = \frac{x'^2_{ij}}{\Omega(x'_i)\Omega(x'_j)}. \tag{1.11}$$

2 Exercise: Correlation Functions

The action of the conformal charges on primary operators is given by

$$\begin{aligned} [Q_\epsilon, \mathcal{O}(x)] &= \left(\epsilon \cdot \partial + \frac{\Delta}{d}(\partial \cdot \epsilon) - \frac{1}{2}(\partial^\mu \epsilon^\nu) \mathcal{S}_{\mu\nu}\right) \mathcal{O}(x), \\ [M_{\mu\nu}, \mathcal{O}^a(0)] &= (\mathcal{S}_{\mu\nu})_b{}^a \mathcal{O}^b(0)\ , \end{aligned} \tag{2.1}$$

where a, b are indices for the SO(d) representation of $\mathcal{O}$ and we kept them implicit in the first line. Conformal invariance of correlation functions is the statement that

$$\langle \Omega | [[Q_\epsilon, \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n)] | \Omega \rangle = 0. \tag{2.2}$$

These are called conformal Ward identities.
a) Show that (2.2) implies that for scalar primaries

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \rangle = \frac{C \delta_{\Delta_1, \Delta_2}}{x_{12}^{2\Delta_1}}. \tag{2.3}$$

For scalar primary operators the statement of conformal invariance takes the following simple form

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \prod_{i=1}^{n} \Omega(x'_i)^{\Delta_i} \langle \mathcal{O}_1(x'_1)...\mathcal{O}_n(x'_n)\rangle. \tag{2.4}$$

b) Using (1.11) check that the famous result by Polyakov transforms in accordance with (2.4)

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{f_{\Delta_1,\Delta_2,\Delta_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_1+\Delta_3-\Delta_2}}. \tag{2.5}$$

c) Write down the most general form of the four-point correlator

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle \sim f(u,v), \tag{2.6}$$

where the cross ratios are

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}) \ . \tag{2.7}$$

d) What is the number of independent cross ratios in the n-point correlator? First, get the answer assuming that all x_{ij}^2 are independent. Second, argue that for large enough n the maximal number of cross ratios is $nd - \frac{(d+2)(d+1)}{2}$.

3 Exercise: Operator Product Expansion

Consider a four-point function of identical scalar operators. It takes the form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi}}. \tag{3.1}$$

Permutation or crossing symmetry implies (check that)

$$g(u,v) = g(\frac{u}{v}, \frac{1}{v}) = \left(\frac{u}{v}\right)^{\Delta_\phi} g(v,u). \tag{3.2}$$

It becomes a powerful tool when combined with the OPE

$$\phi(x_1)\phi(x_2) = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} C_a(x_{12}, \partial_2)\mathcal{O}^a(x_2) \ . \tag{3.3}$$

a) The exact form of $C_a(x_{12}, \partial_2)$ is fixed by conformal symmetry. Show that by evaluating (3.3) inside a three-point function and expanding in x_{12}^μ. For scalar $\mathcal{O}(x_2)$ we get using (2.5)

$$\langle \phi(x_1)\phi(x_2)\mathcal{O}_\Delta(x_3)\rangle = \frac{f_{\Delta_\phi\Delta_\phi\Delta}}{x_{12}^{2\Delta_\phi-\Delta} x_{23}^{\Delta} x_{13}^{\Delta}} = C(x_{12}, \partial_2)\frac{1}{x_{23}^{2\Delta}} = C(x_{12}, \partial_2)\langle \mathcal{O}(x_2)\mathcal{O}(x_3)\rangle. \tag{3.4}$$

Derive first few terms in the small x_{12} expansion of $C_a(x_{12}, \partial_2)$.

b) Argue that the only operators that can appear in the OPE (3.3) are symmetric traceless operators of even spin. To do this consider $\langle \phi | \phi(x) | \mathcal{O}^a \rangle$. To argue that the only even spin is allowed recall that conformal invariance fixes the three-point function to be

$$\langle \phi_1(x_1)\phi_2(x_2)\mathcal{O}^{\mu_1 \cdots \mu_J}(x_3)\rangle = \frac{f_{\phi_1\phi_2\mathcal{O}_J}(Z^{\mu_1}...Z^{\mu_J} - \text{traces})}{x_{12}^{\Delta_1+\Delta_2-\Delta_3+J} x_{23}^{\Delta_2+\Delta_3-\Delta_1-J} x_{13}^{\Delta_1+\Delta_3-\Delta_2-J}},$$
$$Z^\mu \equiv \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{12}^\mu}{x_{12}^2}. \tag{3.5}$$

c) Normalize $\langle \phi(x)\phi(0)\rangle = \frac{1}{x^{2\Delta}}$. Show that

$$f_{\phi\phi T} = -\frac{d\Delta}{d-1}\frac{1}{\text{Vol}_{S^{d-1}}}. \tag{3.6}$$

An instructive way to do that is to consider the Wightman function $\langle \phi(x_1) T_{--}(x_2)\phi(x_3)\rangle$ in the light-cone coordinates and recall that the generator of translations is $P_- = \int dx^- d^{d-2}\vec{x}_\perp T_{--}$.
d) In this way we can write $g(u,v)$ from (3.1) as a sum of conformal blocks

$$g(u,v) = \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} g_{\Delta_\mathcal{O},J_\mathcal{O}}(u,v),$$
$$g_{\Delta_\mathcal{O},J_\mathcal{O}}(u,v) \equiv x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi} C_a(x_{12},\partial_2) C_b(x_{34},\partial_4)\langle \mathcal{O}^a(x_2)\mathcal{O}^b(x_4)\rangle, \tag{3.7}$$

where $g_{\Delta_\mathcal{O},J_\mathcal{O}}(u,v)$ is a conformal block that represents the contribution of a single conformal multiplet to a four-point function.

Check that $g_{\Delta_\mathcal{O},J_\mathcal{O}}(u,v)$ does not depend on normalization of $\mathcal{O}$. Using the form $C_a(x_{12},\partial_2)$ derived in **a)** show that

$$\lim_{u\to 0} g_{\Delta_\mathcal{O},0}(u,v) = u^{\Delta/2} + ... \,. \tag{3.8}$$

Using (3.7) and (3.4), argue that $g_{\Delta_\mathcal{O},0}(u,v)$ does not depend on Δ_ϕ.
e) Conformal blocks are eigenfunctions of the conformal Casimir operator[2] $C = -\frac{1}{2}L^{ab}L_{ab}$ that acts with the same eigenvalue on all states in an irreducible representation. This leads to the differential equation satisfied by conformal blocks

$$\mathcal{D}_C g_{\Delta,J}(u,v) = (\Delta(\Delta-d) + J(J+d-2))\, g_{\Delta,J}(u,v)\ ,$$
$$\mathcal{D}_C = 2\left(z^2(1-z)\partial_z^2 - z^2\partial_z\right) + 2\left(\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}\right)$$
$$+\, 2(d-2)\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right). \tag{3.9}$$

Check that the following expressions for the conformal blocks satisfy (3.9) in $d=2$ and $d=4$ with the correct boundary condition (3.8) [3]

[2]Recall that the conformal algebra is isomorphic to $SO(d+1,1)$ with generators L_{ab}.
[3]For operators with the spin the correct boundary condition is $g_{\Delta_\mathcal{O},J}(u,v) = u^{\Delta/2}(1-v)^J + ...$

$$
\begin{aligned}
k_\beta(x) &\equiv x^{\beta/2}\ {}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta,x\right), \\
g^{(2d)}_{\Delta,J}(u,v) &= k_{\Delta+J}(z)k_{\Delta-J}(\bar z)+k_{\Delta-J}(z)k_{\Delta+J}(\bar z)\ , \\
g^{(4d)}_{\Delta,J}(u,v) &= \frac{z\bar z}{z-\bar z}\left(k_{\Delta+J}(z)k_{\Delta-J-2}(\bar z)+k_{\Delta-J-2}(z)k_{\Delta+J}(\bar z)\right)\ .
\end{aligned} \tag{3.10}
$$

For a unit operator we have $g^{(d)}_{0,0}=1$.

4 Exercise: Conformal Bootstrap

Consider a four-point function of identical scalar operators. We get the crossing equation

$$
v^{\Delta_\phi}g(u,v)=v^{\Delta_\phi}\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}}g_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(u,v)=u^{\Delta_\phi}\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}}g_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(v,u)=u^{\Delta_\phi}g(v,u). \tag{4.1}
$$

a) Argue any solution to the crossing equation (4.1) necessarily contains an infinite number of primaries.

b) One of the simplest solutions to (4.1) is called *Generalized Free Field* (GFF). It corresponds to a scalar free field theory in AdS. The correlator takes the form

$$
g(u,v)=1+u^{\Delta_\phi}+\left(\frac{u}{v}\right)^{\Delta_\phi}\ . \tag{4.2}
$$

Using the explicit expression for 2d or 4d conformal blocks read off the low-energy spectrum of conformal primaries in the model. What do they correspond to in AdS?

c) Euclidean bootstrap. Let's set $z=\bar z=e^{-\beta}$ and consider $\beta\to 0$ limit. It corresponds to the short distance limit $x_2\to x_3$ which is dominated by the unit operator. Using conformal block expansion check that

$$
g(z=e^{-\beta},\bar z=e^{-\beta})=\int_0^\infty d\Delta e^{-\beta\Delta}\rho(\Delta)=\frac{1}{\beta^{2\Delta}}(1+O(\beta))\ , \tag{4.3}
$$

where $\rho(\Delta)=\sum_i c_i\delta(\Delta-\Delta_i)$ is a positive spectral density of both primaries and descendants. The Hardy-Littlewood tauberian theorem states that (4.3) implies that[4]

$$
F(E)\equiv\int_0^E dE'\rho(E')\sim\frac{E^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)}. \tag{4.4}
$$

An elegant proof of this result by Karamata can be found in appendix E of Rychkov and Qiao paper. What is the large Δ asymptotic of $\rho(\Delta)$ for a single primary operator? Was it important for this result that we considered GFF? Was it important that we considered CFT$_{d>2}$?

[4]Here $a\sim b$ means $a/b\to 1$ in the assumed limit.

d) Lorentzian/analytic bootstrap. Consider the limit $1 - \bar{z} \ll z \ll 1$, check that in this limit point 2 becomes light-like separated from points 1 and 3. The limit is again dominated by the unit operator. The leading asymptotic of the correlator takes the form

$$g(z, \bar{z}) = \frac{z^{\Delta}}{(1-\bar{z})^{\Delta}} + \dots \ , \quad 1 - \bar{z} \ll z \ll 1 \ . \tag{4.5}$$

We would like to show that this result is reproduced by the large spin operators in the dual channel of dimension $\Delta = 2\Delta_\phi + J$. To do this first show (using the Casimir equation (3.9)) that in the limit $z \to 0$ conformal blocks take the form

$$\lim_{z \to 0} g_{\Delta,J}(z, \bar{z}) = z^{\frac{\Delta - J}{2}} k_{\Delta+J}(\bar{z}) + \dots \ , \tag{4.6}$$

in any d. In this way the z-dependence of (4.5) is easily reproduced. To reproduce the $\bar{z}$ dependence show that for large $J \gg 1$

$$k_{2J}(1 - \frac{y^2}{J^2}) = \frac{\Gamma(2J)}{\Gamma(J)^2} \left(2K_0(2y) + O(J^{-1}) \right) \ . \tag{4.7}$$

Using this to make a prediction for the large spin behavior of the three-point couplings $f_{\phi\phi\mathcal{O}_{2\Delta_\phi+J,J}}$ by reproducing the result (4.5).

Was it important for this result that we considered GFF? Was it important that we considered $\mathrm{CFT}_{d>2}$? What is the interpretation of this result for Quantum Gravity in AdS?
e) Bound on the gap (Numerical bootstrap). Let us introduce the so-called ρ-coordinate that is obtained by mapping the cut plane $\mathbb{C}\backslash(1, \infty)$ inside the unit disc

$$z = \frac{4\rho}{(1+\rho)^2} \ . \tag{4.8}$$

Expanding conformal blocks in terms ρ-variable converges faster. Let us approximate the conformal blocks by the first term in their ρ-expansion for $0 < z = \bar{z} < 1$

$$g_{\Delta,J}(z) \simeq \rho(z)^{\Delta}. \tag{4.9}$$

Estimate the error of this approximation around $z = \bar{z} = \frac{1}{2}$. We rewrite the crossing equation (4.1) as follows

$$(1-z)^{2\Delta_\phi} - z^{2\Delta_\phi} + \sum_{\Delta} f_\Delta^2 \left((1-z)^{2\Delta_\phi} \rho(z)^{\Delta} - z^{2\Delta_\phi} \rho(1-z)^{\Delta} \right) = 0, \tag{4.10}$$

where we isolated the contribution of the unit operator from the rest. We will this equation to show that there is an upper bound on the lowest primary dimension $\Delta_{\min}$ that appears in the $\phi \times \phi$ OPE. To do this expand the crossing equation (4.10) around $z = \frac{1}{2} + x$ and collect x and x^3 terms. Use these equations and unitarity (or the fact that $f_\Delta^2 > 0$) to show that

$$\Delta_{\min} \le \sqrt{(\Delta_\phi - 1)(2\Delta_\phi - 1)}. \tag{4.11}$$

5 Literature

The basic lecture notes are

- S. Rychkov, "EPFL Lectures on Conformal Field Theory in $d \geq 3$ Dimensions," [1]
- D. Simmons-Duffin, "The Conformal Bootstrap," [2]
- D. Poland, S. Rychkov and A. Vichi, "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications," [3]

We mostly used [2] in our lectures and problems. Some more AdS/CFT-oriented readers might benefit from

- J. Kaplan, "Lectures on AdS/CFT from the Bottom Up" [4]
- J. Penedones, "TASI lectures on AdS/CFT," [5]

Many of the standard results about conformal blocks can be found in the classic papers by Dolan and Osborn, see e.g [6]. The ρ-coordinate, tauberian theorems and convergence of OPE is well-explained in [7]. For bootstrap at large N the standard reference is [8]. For the analytic bootstrap, see [9, 10].

References

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