Worldsheet construction of gauge instantons

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Outline

$D = 4$ $SU(N)$ Yang-Mills

Gauge potential: $A_\mu = A_\mu^a T^a$, where $T^a \in \mathfrak{su}(N)$

Covariant derivative: $\mathcal{D} = d + A$

Field strength: $F \equiv \mathcal{D}A$

Action:

$$S[A] = -\frac{1}{2g^2} \int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge *F\,,$$

Gauge transformations:

$$A \longrightarrow U^{-1}(d + A)U\,, \qquad U(x) \in SU(N)$$

Instanton: a classical solution (vacuum) such that $S[A_{\mathrm{cl}}] < \infty$.

$\Longrightarrow$ need pure gauge at ∞

$$A_{\mathrm{cl}} \overset{r\to\infty}{\sim} U^{-1}dU\,, \qquad \text{for some } U(x) \in SU(N)$$

Instanton number

Claim 1: Instantons classified by their **charge** (number) k, where

$$k \equiv -\frac{1}{16\pi^2} \int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F \;\in\; \pi_3(SU(N)) \;\cong\; \mathbb{Z}$$

Proof:

Have

$$\mathrm{Tr}\, F \wedge F = dK \,, \quad \text{where} \quad \frac{1}{2} K = \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

and $dA \overset{r\to\infty}{\sim} -A \wedge A$, so $K \overset{r\to\infty}{\sim} -\frac{2}{3} \mathrm{Tr}\, A \wedge A \wedge A$, so by Stokes

$$k = \frac{1}{24\pi^2} \int_{S^3_\infty} \mathrm{Tr}\, U^{-1} dU \wedge U^{-1} dU \wedge U^{-1} dU \;\in\; \mathbb{Z}$$

$\Longrightarrow$ Pontryagin index (winding number)

(Anti-)selfduality condition

Claim 2: All instantons satisfy

$$*F = (\operatorname{sgn} k) F$$

Proof:

Fix a topological sector $k \in \mathbb{Z}$ and use the Bogomolnyi trick:

$$S = \underbrace{-\frac{1}{4g^2}\int \mathrm{Tr}\,(F \mp *F) \wedge *(F \mp *F)}_{\text{non-negative}} \mp \frac{1}{2g^2}\underbrace{\int \mathrm{Tr}\, F \wedge F}_{-16k\pi^2} \geqslant \pm \frac{8k\pi^2}{g^2}$$

$\implies$ S minimized iff $*F = (\operatorname{sgn} k) F$ with

$$S[A_{\mathsf{cl}}] = \frac{8\pi^2}{g^2}|k|$$

Moduli spaces

Moduli (collective coords / zero modes):

fluctuations $A_{\text{cl}} \to A_{\text{cl}} + \delta A$ such that $\delta S = 0$ modulo gauge trafos

One obtains ($\mathcal{D}_{\text{cl}} \equiv d + A_{\text{cl}}$)

$$*\mathcal{D}_{\text{cl}}\, \delta A = (\operatorname{sgn} k)\, \mathcal{D}_{\text{cl}}\, \delta A\,, \qquad *\mathcal{D}_{\text{cl}} * \delta A = 0 \tag{1}$$

Denote by $\mathfrak{M}_k$ the space of solutions of (1)

Claim 3:

$$\dim \mathfrak{M}_k = 4kN$$

Proof: Use Atiyah-Singer index theorem.

Define a metric

$$\gamma(\delta_1 A, \delta_2 A) \equiv -2 \int \operatorname{Tr} \delta_1 A \wedge * \delta_2 A$$

Claim 4 (ADHM): With respect to γ, the spaces $\mathfrak{M}_k$ are hyper-Kähler manifolds with conical singularities.

Example: $k = 1$

$\underline{N = 2}$: we have (in regular gauge)

$$A_\mu(x; x_0, \rho, g) = g\, a^a_\mu(x; x_0, \rho) T^a\, g^{-1}$$

where

$$a^a_\mu(x; x_0, \rho) = 2\eta^a{}_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^2 + \rho^2}\,,$$

with $\eta^a{}_{\mu\nu}$ the selfdual t'Hooft symbol

8 moduli for $k = 1$, $N = 2$
$$\begin{cases} x_0 \in \mathbb{R}^4 & \text{instanton centre (4)} \\ \rho \in \mathbb{R}^+ & \text{instanton size (1)} \\ g \in SU(2) & \text{global rotation (3)} \end{cases}$$

$\underline{\text{General } N}$:

$$A^{SU(N)}_\mu = g \begin{pmatrix} 0 & 0 \\ 0 & a^{SU(2)}_\mu \end{pmatrix} g^{-1}\,, \qquad g \in \frac{SU(N)}{SU(N-2) \times U(1)}$$

$$\underbrace{N^2 - 1}_{SU(N)} \quad - \quad [\underbrace{(N-2)^2 - 1}_{SU(N-2)} \quad + \quad \underbrace{1}_{U(1)}] + \underbrace{5}_{a^{SU(2)}_\mu \text{ moduli}} \quad = \quad \underline{4N \text{ moduli}}$$

Supersymmetry and non-commutativity

$\mathcal{N} = 4$ super Yang-Mills:

field content: $A^a_\mu,\ \phi^i,\ \lambda^a,\ \bar{\lambda}^a$

$$\begin{aligned} \mu &= 0, \ldots, 3 \\ i &= 1, \ldots, 6 \\ a &= 1, \ldots, N^2 - 1 \end{aligned}$$

lagrangian:

$$\mathcal{L}_{\mathsf{SYM}} = \mathcal{L}_{\mathsf{kin}} + \frac{1}{g^2} \mathrm{Tr} \left\{ -\frac{1}{2}[\phi, \phi]^2 - i\bar{\Sigma}\lambda[\phi, \lambda] - i\Sigma\bar{\lambda}[\phi, \bar{\lambda}] \right\}$$

$\rightarrow$ instantons are 1/2-BPS states of the extended $\mathcal{N} = 4$ superalgebra

$\rightarrow$ $2kN$ fermionic zero modes $\Longrightarrow$ supermoduli space with dim $= 6kN$

Non-commutative $\mathbb{R}^4$: $[x^\mu, x^\nu] = i\theta^{\mu\nu}$

$\Longrightarrow$ replace all products by the **Moyal product:**

$$f(x)g(x) \longrightarrow f(x) \star g(x) \equiv e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\zeta^\nu}} f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0}$$

$\rightarrow$ conical singularities in $\mathfrak{M}_k$ resolved

$\rightarrow$ non-trivial instantons for $k = 1$

Outline

Type II superstrings: action

Type II RNS superstring propagating in D dims ($\mu, \nu, \ldots = 0, \ldots, D-1$) with (const.) background metric $g_{\mu\nu}$ (with fixed reparametrization gauge symm.):

$$S_{\mathsf{gf}} = S_{\mathsf{m}} + S_{\mathsf{gh}}$$

where

$$\begin{aligned} S_{\mathsf{m}} &= \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z\, g_{\mu\nu} \left(\partial X^\mu \bar{\partial} X^\nu + \psi^\mu \overline{\partial} \psi^\nu + \overline{\psi}^\mu \partial \overline{\psi}^\nu\right) \\ S_{\mathsf{gh}} &= \frac{1}{2\pi} \int_\Sigma d^2 z\, (b \overline{\partial} c + \overline{b} \partial \overline{c} + \beta \overline{\partial} \gamma + \overline{\beta} \partial \overline{\gamma}) \end{aligned}$$

for a Riemann surface Σ (worldsheet) and

X^μ	...	Grassmann-even target coordinates
ψ^μ	...	Grassmann-odd target coordinates
b, c	...	reparametrization ghosts
β, γ	...	reparametrization superghosts

Type II superstrings: symmetries

$\mathcal{N} = (1,1)$ (local) superconformal symmetry:

$$\begin{aligned}
\delta_{\eta,\overline{\eta}} X^\mu(z,\overline{z}) &= -\eta(z)\psi^\mu(z) - \overline{\eta}(\overline{z})\overline{\psi}^\mu(\overline{z}) \\
\delta_{\eta,\overline{\eta}} \psi^\mu(z) &= \eta(z)\partial X^\mu(z) \\
\delta_{\eta,\overline{\eta}} \overline{\psi}^\mu(\overline{z}) &= \overline{\eta}(\overline{z})\overline{\partial} X^\mu(\overline{z})
\end{aligned}$$

Generators:

$$G(z) = -\frac{1}{2} g_{\mu\nu}\psi^\mu \partial X^\nu \qquad \overline{G}(\overline{z}) = -\frac{1}{2} g_{\mu\nu}\overline{\psi}^\mu \overline{\partial} X^\nu$$

$$T(z) = -\frac{1}{2} g_{\mu\nu}(\partial X^\mu \partial X^\nu + \psi^\mu \partial \psi^\nu) \qquad \overline{T}(\overline{z}) = -\frac{1}{2} g_{\mu\nu}(\overline{\partial} X^\mu \overline{\partial} X^\nu + \overline{\psi}^\mu \overline{\partial \psi}^\nu)$$

with OPE $G(z)G(0) \sim Dz^{-3} + 2T(0)z^{-1}$

Superconformal anomaly:

$$c = \underbrace{D}_{=c_{\mathsf{m},X}} + \underbrace{\frac{1}{2}D}_{=c_{\mathsf{m},\psi}} + \underbrace{-26}_{=c_{\mathsf{gh},bc}} + \underbrace{11}_{=c_{\mathsf{gh},\beta\gamma}} = \frac{3}{2}D - 15$$

$$c = 0 \implies D = 10$$

Closed superstring: boundary conditions

Σ with punctures (in and out states) and holes (loops)

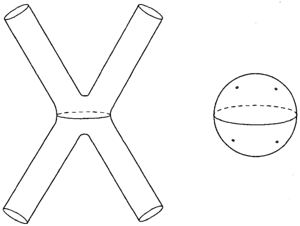


Figure: tree-level closed 4-string amp

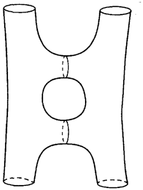


Figure: 1-loop closed 4-string amp

Going around non-contractible cycles:

worldsheet bosons: $X^\mu(e^{2i\pi} z, \overline{z}) = X^\mu(z, e^{2i\pi}\overline{z}) = X^\mu(z, \overline{z})$

worldsheet fermions: $\psi^\mu(e^{2i\pi} z) = \begin{cases} +\psi^\mu(z) & \text{Ramond} \\ -\psi^\mu(z) & \text{Neveu-Schwarz} \end{cases}$

$$\overline{\psi}^\mu(e^{-2i\pi}\overline{z}) = \begin{cases} +\overline{\psi}^\mu(\overline{z}) & \text{R} \\ -\overline{\psi}^\mu(\overline{z}) & \text{NS} \end{cases}$$

Closed superstring: low-energy physics

Consistency on higher-genus $\Sigma \implies$ GSO projection

Massless spectrum:

$$\text{Type IIA:} \quad g,\ B_{(2)},\ \phi, \quad C_{(1)},\ C_{(3)}$$

$$\text{Type IIB:} \quad \underbrace{g,\ B_{(2)},\ \phi,}_{\text{NSNS sector}} \quad \underbrace{C_{(0)},\ C_{(2)},\ C_{(4)}}_{\text{RR sector}}$$

Low-energy effective action:

$\mathcal{N} = (1,1)$ / $\mathcal{N} = (2,0)$ type IIA / IIB supergravity in $D = 10$

Classical solutions: 1/2-BPS (black) p-branes

- $\rightarrow$ p even / odd in IIA / IIB
- $\rightarrow$ sources for g and $C_{(p+1)}$
- $\rightarrow$ 1/2-BPS multiplets of IIA/IIB $D = 10$ super-Poincarè

Open superstring: boundary conditions

Σ with boundaries: G, T conserved across $\partial\Sigma$ $\Longrightarrow$ gluing conditions

$$G(z)|_{\partial\Sigma} = \overline{G}(\bar{z})|_{\partial\Sigma}$$

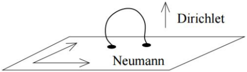
$$T(z)|_{\partial\Sigma} = \overline{T}(\bar{z})|_{\partial\Sigma}$$

Solutions $\equiv$ D-branes (# solutions = # irreps of the $c_{\mathrm{m}} = 15$ $\mathcal{N} = (1,1)$ SCA)

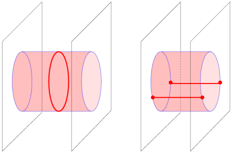
In particular have Dp-branes:

$$\partial X^{\mu}(z)|_{\partial\Sigma} = \pm\overline{\partial} X^{\nu}(\bar{z})|_{\partial\Sigma}$$

$$\psi^{\mu}(z)|_{\partial\Sigma} = \pm\overline{\psi}^{\mu}(\bar{z})|_{\partial\Sigma}$$



Open-closed duality:



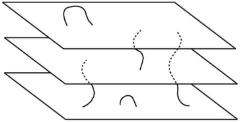
$\Longrightarrow$

D-branes are sources for g and $C_{(p)}$

Dp-branes $\equiv$ black p-branes

Example: D3 branes in type IIB

Stack of N D3 branes



Massless spectrum:

$$\underbrace{A^a_\mu,\ \phi^i,}_{\text{NS sector}}\quad \underbrace{\lambda^a,\ \overline{\lambda}^a}_{\text{R sector}} \qquad \begin{array}{l} \mu = 0,\ldots,3 \\ i = 1,\ldots,6 \\ a = 1,\ldots,N^2-1 \end{array}$$

Low-energy effective action:

$\mathcal{N} = 4\ SU(N)$ super Yang-Mills in $D = 4$

Non-zero NSNS B-field $\Longrightarrow$ non-commutativity with

$$\theta^{\mu\nu} = \left(\frac{1}{g+B}\right)^{\mu\nu}_{\mathsf{A}}$$

Outline

Instantons on D3 branes

N D3 branes with k instanton gauge field configuration F

$$S_{\text{LEEA}} \supset S_{\text{WZ}} = \int C_{(0)} \wedge F \wedge F$$

$\Longrightarrow$ N D3 / k D(-1) bound state

SUSY counting:

$$\left.\begin{array}{c} F \text{ is 1/2-BPS w.r.t. } D=4 \text{ SYM} \\ \text{D3 is 1/2-BPS w.r.t. } D=10 \text{ IIB SUGRA} \end{array}\right\} \Longrightarrow \begin{array}{c} \text{D3/D}(-1) \text{ must be 1/4-BPS} \\ \text{w.r.t. } D=10 \text{ IIB SUGRA} \end{array}$$

Indeed, $\exists$ 1/4-BPS multiplets of $D=10$ $\mathcal{N}=(2,0)$ IIB super-Poincarè

$$\{Q_\alpha, Q_\beta\} \supset (C\Gamma^{11})_{\alpha\beta} Z + \frac{1}{4!}(C\Gamma_{\mu\nu\rho\sigma}\Gamma^{11})_{\alpha\beta} Z^{\mu\nu\rho\sigma}$$

with $Z=k$, $Z^{0\mu\nu\lambda}=\varepsilon^{\mu\nu\lambda}N$ and mass

$$\mathcal{M}^2_{\text{D3/D}(-1)} = k^2\mathcal{M}^2_{\text{D}(-1)} + N^2\mathcal{M}^2_{\text{D3}} + 2|kN| + 2kN\,\text{Pf}\,B \leqslant (k\mathcal{M}_{\text{D}(-1)} + N\mathcal{M}_{\text{D3}})^2$$

Q: what are the D3/D(-1) worldsheet boundary conditions?

Gepner-like construction

1. replace $\mathbb{R}^4 \to T^4 \cong T^2 \times T^2$ with T^2 the $SU(3)$ torus with $R = \sqrt{\alpha'}$
2. turn on $B_{\mu\nu}$ background s.t. $\int_{T^2} B = +\frac{1}{2}$

$$G_{T^4}(z) = \sum_{a=1}^{6} (G_a^+(z) + G_a^-(z)) \,, \quad T_{T^4}(z) = \sum_{a=1}^{6} T_a(z) \,,$$

where $\mathcal{W}_a = \{G_a^\pm, T_a, J_a\}$ generate $\mathcal{N} = 2$ SCA with $c = 1$ and

$$T_a(z)T_b(0) \sim T_a(z)G_b^\pm(0) \sim \ldots \sim 0 \quad \text{for } a \neq b$$

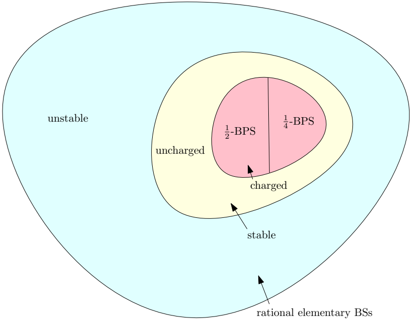
finite # irreps of $\mathcal{W}_a \implies$ finite # irreps of $\mathcal{W}_{T^4} = \bigoplus_{a=1}^{6} \mathcal{W}_a$!

GCs on $\mathcal{W}_{T^4}$ consistent with the total $\mathcal{N} = (1,1)$ SCA ($\pi \in S_6$, $\omega_a = \pm$)

$$G_a^\pm(z)|_{\partial\Sigma} = \overline{G}_{\pi(a)}^{\mp\omega}(\bar{z})|_{\partial\Sigma}$$
$$T_a(z)|_{\partial\Sigma} = \overline{T}_{\pi(a)}(\bar{z})|_{\partial\Sigma}$$
$$J_a(z)|_{\partial\Sigma} = \omega_a \overline{J}_{\pi(a)}(\bar{z})|_{\partial\Sigma}$$

Results

We classified D-branes w.r.t. to all $6! \times 2^6 = 46\,080$ GCs imposed on $\mathcal{W}_{T^4}$



Interesting solutions:

→ 1/4-BPS objects with $Z = k$, $Z^{0123} = N$, $4kN$ massless string modes and mass $\mathcal{M}_{\mathrm{D3/D}(-1)}$ $\Longrightarrow$ instanton boundary conditions !

→ non-BPS objects with no tachyons $\Longrightarrow$ new stable D-branes !

Thank you!