# Finite, infinite, and doubly-infinite degrees of freedom and BHs

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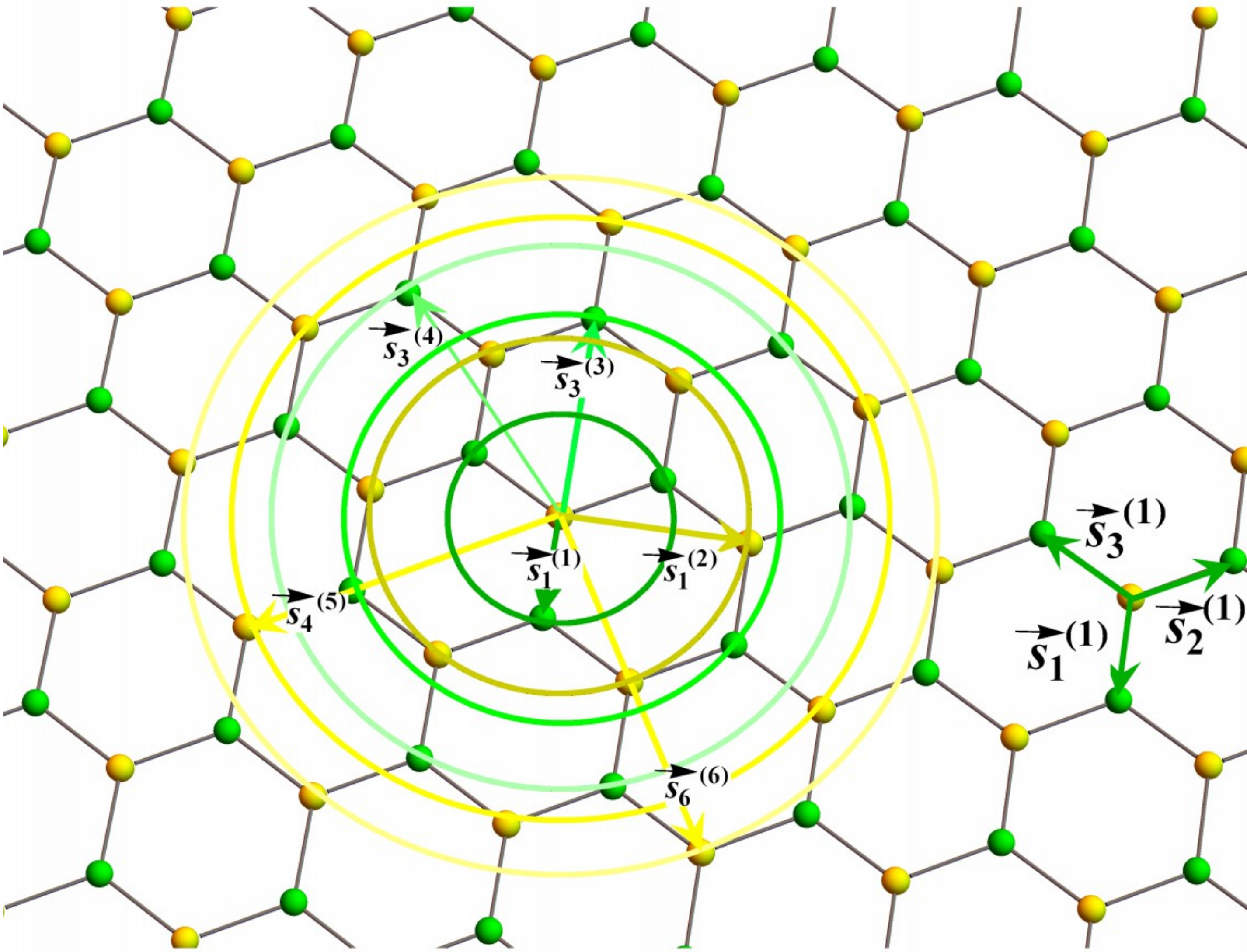
# I will not mention...



# I will have in mind, next to each other, a very exotic object

# for which degrees of freedom are supposed to be FINITE…

# …and ordinary objects

# for which degrees of freedom are supposed to be INFINITE or DOUBLE that!

# References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**Ann Phys 387 (2017) 317**

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**Ann Phys 309 (2004) 151**
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# Summary

* Quantum fields and horizons

* Vacuum and entanglement

* The entropy operator

* Two examples

* Finite, infinite, and doubly-infinite degrees of freedom

# * Quantum fields and horizons

*For a quantum system with a finite number of degrees of freedom all the representations of the ccrs are unitarily equivalent to the Shrödinger representation.*

$$[a, a^\dagger] = 1 \quad \text{etc.} \quad a|0\rangle = 0$$

and

$$[b, b^\dagger] = 1 \quad \text{etc.} \quad b|0\rangle\rangle = 0$$

are related by a Bogolubov transformation generated by $G(\epsilon)$

$$b = GaG^{-1} \quad \text{and} \quad |0\rangle\rangle = G|0\rangle$$

and $G$ is well defined and unitary: $G^{-1} = G^\dagger$.

For quantum *fields* ($a \to a_k$, $b \to b_k$) this does not hold any more: $G$ does not exist. Thus $|0\rangle$ and $|0\rangle\rangle$ would not be related any more by a unitary transformation. The infinite set of representations $|0(\epsilon)\rangle$ are what we call UIRs of the CCRs.

Consider a complex massive scalar quantum field $\phi(x)$ in $D$-dimensional Minkowski space–time, with Lagrangian density

$$\mathcal{L}(\phi^*, \phi) = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \,.$$

As usual, $\phi(x)$ can be decomposed in Minkowski modes $\{U_k(x)\}$

$$\phi(x) = \sum_k \left[a_k U_k(x) \, + \, \bar{a}_k^\dagger U_k^*(x)\right] ,$$

with $k = (k_1, \vec{k})$. The quantum Hamiltonian is

$$H_M = \sum_k \omega_k \left(a_k^\dagger a_k \, + \, \bar{a}_k^\dagger \bar{a}_k\right) ,$$

where $\omega_k = \sqrt{k_1^2 + |\vec{k}|^2 + m^2}$, and $a_k, a_k^\dagger$ $(\bar{a}_k, \bar{a}_k^\dagger)$ are the annihilation and creation operators, respectively, for particles (antiparticles)

$$a_k|0_M\rangle \, = \, \bar{a}_k|0_M\rangle \, = 0, \quad \forall \, k \,.$$

and satisfy the usual CCRs

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \,, \quad [\bar{a}_k, \bar{a}_{k'}^\dagger] = \delta_{kk'} \,.$$

To quantize the same $\phi(x)$ in presence of an horizon, one expands the field in terms of the complete set of functions $\{u_p^{(\sigma)}(x)\}$, where $p = (\Omega, \vec{k})$, and $\sigma = \pm$ takes into account the fact that the space-time has two causally disconnected regions. Thus, one has

$$\phi(x) = \sum_\sigma \sum_p \left[b_p^{(\sigma)} u_p^{(\sigma)}(x) + \bar{b}_p^{(\sigma)\,\dagger} u_p^{(\sigma)\,*}(x)\right] ,$$

where $b_p^{(\sigma)}$ and $\bar{b}_p^{(\sigma)}$ are *assumed* to satisfy the usual CCRs.
By introducing

$$d_p^{(\sigma)} = \sum_{k_1} \mathcal{P}_\Omega^{(\sigma)}(k_1) \, a_{k_1 \vec{k}} \,,$$

and similarly for $\bar{d}_p^{(\sigma)}$ in terms of $\bar{a}_k$, where $\{\mathcal{P}_\Omega^{(\sigma)}(k_1)\}$ is a complete set of orthogonal functions, the operators $b_p^{(\sigma)}$, and $\bar{b}_p^{(\sigma)}$ can be expressed in terms of the Bogolubov transformations

$$b_p^{(\sigma)} = d_p^{(\sigma)} \cosh \epsilon(p) + \bar{d}_{\tilde{p}}^{(-\sigma)\dagger} \sinh \epsilon(p) = G(\epsilon) d_p^{(\sigma)} G^{-1}(\epsilon) \,,$$

$$\bar{b}_{\tilde{p}}^{(-\sigma)\dagger} = d_p^{(\sigma)} \sinh \epsilon(p) + \bar{d}_{\tilde{p}}^{(-\sigma)\dagger} \cosh \epsilon(p) = G(\epsilon) \bar{d}_{\tilde{p}}^{(-\sigma)} G^{-1}(\epsilon) \,,$$

where $\tilde{p} = (\Omega, -\vec{k})$, and the generator of the transformations is

$$G(\epsilon) = \exp\left\{ \sum_\sigma \sum_p \epsilon(p) \left[ d_p^{(\sigma)} \bar{d}_{\tilde{p}}^{(-\sigma)} - d_p^{(\sigma)\,\dagger} \bar{d}_{\tilde{p}}^{(-\sigma)\,\dagger} \right] \right\} .$$

At finite volume $G(\epsilon)$ is a unitary operator: $G^{-1}(\epsilon) = G(-\epsilon) = G^\dagger(\epsilon)$. The canonical operators $d_p^{(\sigma)}$ and $\bar{d}_p^{(\sigma)}$ annihilate the Minkowski vacuum $|0_M^{(+)}\rangle \otimes |0_M^{(-)}\rangle$. On the other hand, the operators $b_p^{(\sigma)}$, and $\bar{b}_p^{(\sigma)}$ annihilate the vacuum

$$|0^{(+)}(\epsilon)\rangle \otimes |0^{(-)}(\epsilon)\rangle = G(\epsilon) \left[ |0_M^{(+)}\rangle \otimes |0_M^{(-)}\rangle \right] .$$

In what follows we shall keep the short-hand notation: $|0_M\rangle \equiv |0_M^{(+)}\rangle \otimes |0_M^{(-)}\rangle$, and $|0(\epsilon)\rangle \equiv |0^{(+)}(\epsilon)\rangle \otimes |0^{(-)}(\epsilon)\rangle$.
The parameter $\epsilon$ is given by

$$\sinh \epsilon(p) = \frac{1}{(e^{\Omega/T} - 1)^{1/2}} ,$$

where $T$ is related to the surface gravity of black holes, in the case of Schwarzschild geometry, or to the acceleration, in the case of Rindler geometry. One can also show that the total (not normal-ordered) Hamiltonian is given by

$$\begin{aligned} H_\epsilon &= \sum_\sigma \sum_p \sigma \Omega \, [b_p^{(\sigma)\dagger} b_p^{(\sigma)} + \bar{b}_{\tilde{p}}^{(\sigma)} \bar{b}_{\tilde{p}}^{(\sigma)\dagger}] \\ &= H^{(+)}(\epsilon) - H^{(-)}(\epsilon) \,. \end{aligned}$$

# * Vacuum and Entanglement

We have $G(\epsilon)$: $\mathcal{H} \rightarrow \mathcal{H}_\epsilon$, or $G^{-1}(\epsilon)$: $\mathcal{H}_\epsilon \rightarrow \mathcal{H}$. The physical meaning of this freedom is that we can arbitrarily choose to express Minkowskian quantities in terms of generic $\epsilon$-quantities, or the other way around. We choose, for instance, to express the Minkowskian vacuum in terms of the generic $\epsilon$-vacuum

$$|0_M\rangle \;=\; G^{-1}(\epsilon)|0(\epsilon)\rangle\,.$$

Gaussian decomposing, the Minkowski vacuum can be expressed as a $SU(1,1) \times SU(1,1)$ generalized coherent state of Cooper-like pairs

$$|0_M\rangle = \frac{1}{Z}\exp\left[\sum_\sigma\sum_p \;\tanh\epsilon(p) b_p^{(\sigma)\dagger}\bar{b}_{\bar{p}}^{(-\sigma)\dagger}\right]\;|0(\epsilon)\rangle\,,$$

where $Z = \Pi_p\;\cosh^2\epsilon(p)$.

In the continuum limit in the space of momenta, i.e. in the infinite-volume limit, the number of degrees of freedom becomes uncountable infinite

$$\langle 0(\epsilon)|0_M\rangle \rightarrow 0 \quad \text{as} \quad V\rightarrow\infty, \quad \forall\epsilon$$
$$\langle 0(\epsilon)|0(\epsilon')\rangle \rightarrow 0 \quad \text{as} \quad V\rightarrow\infty, \quad \forall\epsilon,\epsilon',\epsilon\neq\epsilon'\,,$$

where $V$ is the volume of the whole $(D-1)$-dimensional space. This means that in this limit the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_\epsilon$ become unitarily inequivalent, and $\epsilon$ labels the set $\{H_\epsilon, \forall\epsilon\}$ of the infinitely many UIRs of the CCRs.

What about the entanglement of the vacuum $|0_M\rangle$? First we rewrite it as

$$|0_M\rangle = \frac{1}{Z}[|0(\epsilon)\rangle + \sum_p \tanh\epsilon(p)(|1_p^{(+)},\bar{0}\rangle \otimes |0,\bar{1}_{\bar{p}}^{(-)}\rangle + |0,\bar{1}_{\bar{p}}^{(+)}\rangle \otimes |1_p^{(-)},\bar{0}\rangle) + \ldots],$$

where $|n_p^{(\sigma)},\bar{0}\rangle \equiv |1_{p_1}^{(\sigma)},\ldots,1_{p_n}^{(\sigma)},\bar{0}\rangle$, and similarly for antiparticles. If $\uparrow$ stands for a particle, and $\downarrow$ for an antiparticle, the two-particle state in $|0_M\rangle$ can be written as

$$|\uparrow^{(+)}\rangle \otimes |\downarrow^{(-)}\rangle + |\downarrow^{(+)}\rangle \otimes |\uparrow^{(-)}\rangle,$$

which is an entangled state of particle and antiparticle living in the two causally disconnected regions $(\pm)$.

This mechanism takes place at all orders in the expansion, thus the whole vacuum $|0_M\rangle$ is an infinite superposition of entangled states

$$|0_M\rangle = \sum_{n=0}^{+\infty} \sqrt{W_n}|\mathrm{Entangled}\rangle_n$$

where

$$W_n = \prod_p \frac{\sinh^{2n_p}\epsilon(p)}{\cosh^{2(n_p+2)}\epsilon(p)},$$

with

$$0 < W_n < 1 \quad \text{and} \quad \sum_{n=0}^{+\infty} W_n = 1\,.$$

# * The Entropy Operator

The number of modes of the type $b_p^{(\sigma)}$ in $|0_M\rangle$ is

$$\mathcal{N}_b^{(\sigma)} \equiv \langle 0_M | b_p^{(\sigma)\dagger} b_p^{(\sigma)} | 0_M \rangle = \sinh^2 \epsilon(p) \; , \quad \sigma = \pm \; ,$$

and similarly for the modes $\bar{b}_{\bar{p}}^{(\sigma)}$.
Let us define $S^{(+)}(\epsilon)$ and $S^{(-)}(\epsilon)$ as

$$\begin{aligned} S^{(+)}(\epsilon) &= \mathcal{S}^{(+)}(\epsilon) + \bar{\mathcal{S}}^{(+)}(\epsilon) \\ &= -\sum_p [b_p^{(+)\dagger} b_p^{(+)} \ln \sinh^2 \epsilon(p) - b_p^{(+)} b_p^{(+)\dagger} \ln \cosh^2 \epsilon(p) \\ &\quad +(b \rightarrow \bar{b})] \; , \end{aligned}$$

$$\begin{aligned} S^{(-)}(\epsilon) &= \mathcal{S}^{(-)}(\epsilon) + \bar{\mathcal{S}}^{(-)}(\epsilon) \\ &= -\sum_p [b_p^{(-)\dagger} b_p^{(-)} \ln \sinh^2 \epsilon(p) - b_p^{(-)} b_p^{(-)\dagger} \ln \cosh^2 \epsilon(p) \\ &\quad +(b \rightarrow \bar{b})] \; . \end{aligned}$$

By counting the number of occupation states in the vacuum $|0_M\rangle$ with the number operator for particles $N_b^{(\sigma)} \equiv b_p^{(\sigma)\,\dagger} b_p^{(\sigma)}$, we must subtract those occupation states counted by the operator $\bar{b}_{\bar{p}}^{(\sigma)} \bar{b}_{\bar{p}}^{(\sigma)\,\dagger} = 1 + N_{\bar{b}}^{(\sigma)}$, where $N_{\bar{b}}^{(\sigma)}$ is the number operator for the antiparticles. This accounts for our definitions of the entropy operators

$$S^{(\sigma)}(\epsilon) = -\sum_p \left[ N_b^{(\sigma)} \ln N_b^{(\sigma)} - \left(1 + N_{\bar{b}}^{(\sigma)}\right) \ln \left(1 + N_{\bar{b}}^{(\sigma)}\right) + (b \to \bar{b}) \right]$$

In terms of the coefficients $W_n$

$$\langle 0_M | S^{(\sigma)}(\epsilon) | 0_M \rangle = -\sum_{n \geq 0} W_n \ln W_n$$

It is easy now to prove that $W_n = \Pi_p \sinh^{2n_p} \epsilon(p) / \cosh^{2(n_p+2)} \epsilon(p)$ The vacuum can be written in the form

$$\begin{aligned} |0_M\rangle &= e^{-S^{(+)}/2} \sum_{n_p=0}^{\infty} \left[ |n_p^{(+)}, \bar{0}\rangle \otimes |0, \bar{n}_{\bar{p}}^{(-)}\rangle + |0, \bar{n}_{\bar{p}}^{(+)}\rangle \otimes |n_p^{(-)}, \bar{0}\rangle \right] \\ &\equiv e^{-S^{(+)}/2} \sum_{n_p=0}^{\infty} \sum_{\sigma=\pm} |n_p^{(\sigma)}; \bar{n}_{\bar{p}}^{(-\sigma)}\rangle \,, \end{aligned}$$

where, according to the notation introduced above

$$\sum_{\sigma=\pm} |n_p^{(\sigma)}; \bar{n}_{\bar{p}}^{(-\sigma)}\rangle \equiv |\mathrm{Entangled}\rangle_{n_p} \,.$$

Hence,

$$\begin{aligned} |0_M\rangle &= \sum_{n_p=0}^{\infty} e^{\sum_{p'} [n_{p'} \ln \sinh \epsilon(p') - (1+n_{p'}) \ln \cosh \epsilon(p')]} \sum_{\sigma=\pm} |n_p^{(\sigma)}; \bar{n}_{\bar{p}}^{(-\sigma)}\rangle \\ &= \sum_{n_p=0}^{\infty} \prod_p \tanh^{n_p} \epsilon(p) \cosh^{-1} \epsilon(p) \sum_{\sigma=\pm} |n_p^{(\sigma)}; \bar{n}_{\bar{p}}^{(-\sigma)}\rangle \\ &= \sum_{n_p=0}^{\infty} \sqrt{W_{n_p}} \sum_{\sigma=\pm} |n_p^{(\sigma)}; \bar{n}_{\bar{p}}^{(-\sigma)}\rangle \,, \end{aligned}$$

The operator $S^{(+)}(\epsilon) = \mathcal{S}^{(+)}(\epsilon) + \bar{\mathcal{S}}^{(+)}(\epsilon)$ is the sum of the entropy operators for the boson gas of particles and antiparticles in the sector $(+)$, similarly for $S^{(-)}(\epsilon)$ in the sector $(-)$. The total entropy operator is given by

$$S_\epsilon = S^{(+)}(\epsilon) - S^{(-)}(\epsilon)\,,$$

and, as for the Hamiltonian, it is the *difference* of the two operators.

Also, $[S_\epsilon, G(\epsilon)] = 0$, hence $S_\epsilon|0_M\rangle = 0$. This means that one can arbitrarily choose one of the two sectors, $\sigma = \pm$, to "measure" the correspondent entropy $S^{(\pm)}(\epsilon)$ relative to the ground state $|0_M\rangle$. Let us work in the sector $\sigma = +$.

One computes

$$\langle 0(\epsilon)|S^{(+)}(\epsilon)|0(\epsilon)\rangle = -2\sum_{\Omega,\vec{k}} \ln\cosh^2\epsilon(\Omega)\,,$$

which diverges due to $\Sigma_{\vec{k}}$. If the entropy operator is normal-ordered, $\ln\cosh^2\epsilon(\Omega)$ is removed and the expectation value of the entropy *vanishes*.

With $S^{(+)}(\epsilon)$ we have to look at the entropy of $|0_M\rangle$, and not of $|0(\epsilon)\rangle$! The result of such a "cross measurement" $\langle 0_M|S^{(+)}(\epsilon)|0_M\rangle$ is

$$-2\sum_{\Omega,\vec{k}}\left[\sinh^2\epsilon(\Omega)\ln\sinh^2\epsilon(\Omega) - \cosh^2\epsilon(\Omega)\ln\cosh^2\epsilon(\Omega)\right]\,.$$

This diverges too, but this time even if the expression is normal-ordered the result is *never zero*.

# * Two examples

*Schwarzschild.*

We shall derive the thermodynamical properties of black holes described by the Schwarzschild geometry

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where the space-time is taken to be 4-dimensional. The event horizon is given by $r_S = 2GM$, and the Bekenstein-Hawking temperature is $T \simeq r_S^{-1} = (2GM)^{-1}$.

We want to compute the finite part of the entropy by moving to the continuum limit, taken to be formally identical to the Minkowskian one

$$\sum_{\Omega,\vec{k}} \to \frac{V}{(2\pi)^3} \int_0^\infty d\Omega \int d^2k\,,$$

where $V$ is the 3-dimensional volume of the whole space-time.

The entropy *density* $\langle s^{(+)}(\epsilon)\rangle_M \equiv V^{-1}\langle 0_M|S^{(+)}(\epsilon)|0_M\rangle$ is

$$\frac{-2}{(2\pi)^3} \int_0^\infty d\Omega \left[\sinh^2\epsilon(\Omega) \ln \sinh^2\epsilon(\Omega) - \cosh^2\epsilon(\Omega) \ln \cosh^2\epsilon(\Omega)\right] \int d^2k\,.$$

By expressing $\epsilon(\Omega)$ as a function of $\Omega$ one can compute the integral to obtain

$$\langle s^{(+)}(\epsilon)\rangle_M = \frac{\pi^2 T}{3(2\pi)^3} \int d^2k\,,$$

which is again divergent, as expected. We use a cutoff $k_C$ of the order of the Planck momentum $k_C \simeq k_P = l_P^{-1} = G^{-1/2}$. Our entropy density is then given by

$$\langle s^{(+)}(\epsilon)\rangle_M = \frac{k_C^2 T}{24\pi}\,.$$

Being the proper volume in the Schwarzschild geometry

$$V_{\rm prop} = \int \sqrt{-g_{rr}g_{\theta\theta}g_{\varphi\varphi}}\, dr d\theta d\varphi$$

only defined for $r > r_S$, we now have to compute the entropy for the spherical shell of radii $r_S$ and $r_S + \delta$. The volume of the shell is given by

$$\mathcal{V} = 4\pi r_S^3 \int_1^{1+h} \frac{x^{5/2}}{\sqrt{x-1}}\, dx \propto r_S^3\,,$$

where $h = \delta/r_S$ is chosen by requiring the numerical factor of proportionality to be $\mathcal{O}(1)$. Since $k_C \lesssim k_P$, and recalling that the Bekenstein-Hawking temperature is $T \sim r_S^{-1}$, we obtain the upper bound for the entropy

$$\langle S^{(+)}(\epsilon)\rangle_M = \mathcal{V}\,\langle s^{(+)}(\epsilon)\rangle_M \lesssim \frac{\mathcal{A}}{l_P^2}.$$

Hence the entropy is proportional to the horizon *area* $\mathcal{A}$ of the black hole, and is bounded from above.

*Rindler.*

The Rindler flat space-time (corresponding to an observer moving with constant acceleration $a$) is described by the line element

$$ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2) - dy^2 - dz^2\,,$$

which reduces to Minkowski space-time letting

$$t = \frac{e^{a\xi}}{a}\sinh a\tau\,, \quad x = \frac{e^{a\xi}}{a}\cosh a\tau\,.$$

As the surface gravity of the black hole is the gravitational *acceleration* at radius $r$ measured at infinity, for the Rindler space-time results are formally equivalent to the Schwarzschild case.

The metric covers a portion of Minkowski space-time with $x > |t|$. The boundary planes are determined by $x \pm t = 0$.
Davies and Unruh have shown that the vacuum state for an inertial observer is a canonical ensemble for the Rindler observer. The temperature $T_R$ characterizing this ensemble is

$$T_R = \frac{a}{2\pi} .$$

This is the thermalization theorem, in a nutshell.
The proper volume is given by

$$V_{\rm prop} = \int \sqrt{g_{\xi\xi} g_{yy} g_{zz}}\, d\xi dy dz = \int_{-\infty}^{0} e^{a\xi}\, d\xi \int dy dz = \frac{\mathcal{A}}{a} ,$$

where $\mathcal{A} = \int dydz$ is the area of a surface of constant $\xi$ and $\tau$. The entropy density is computed for a cutoff on the momenta $k_C \lesssim l_P^{-1}$, and is given by

$$\langle s^{(+)}(\epsilon) \rangle_M = \frac{a k_C^2}{48\pi^2} .$$

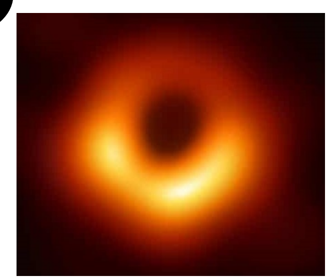
Hence, the entropy computed in the volume $V_{\rm prop}$ (by considering that due to the Unruh effect $T \sim a$) is given by

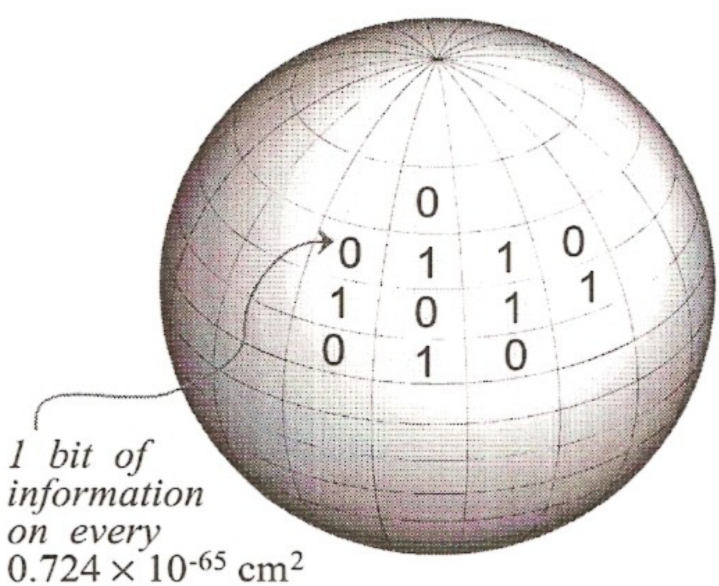
$$\langle S^{(+)}(\epsilon) \rangle_M \lesssim \frac{\mathcal{A}}{l_P^2} .$$

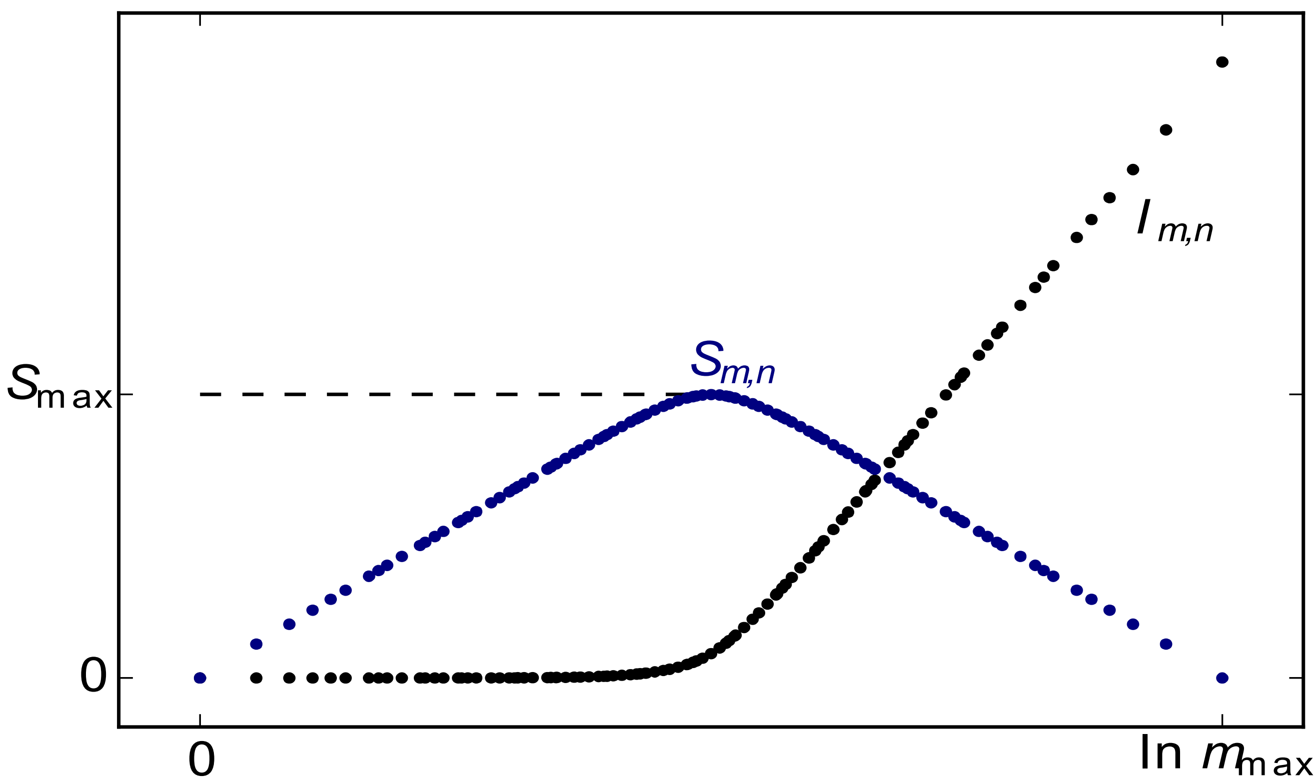
Thus, also in the Rindler case the entropy is proportional to the *area* of the event horizon and bounded from above.

It is an interesting question investigating the relation between the derivation of these formulae and the holographic principle.

# * Finite, infinite and doubly-infinite degrees of freedom

$S$  $\leq$ $S$ 



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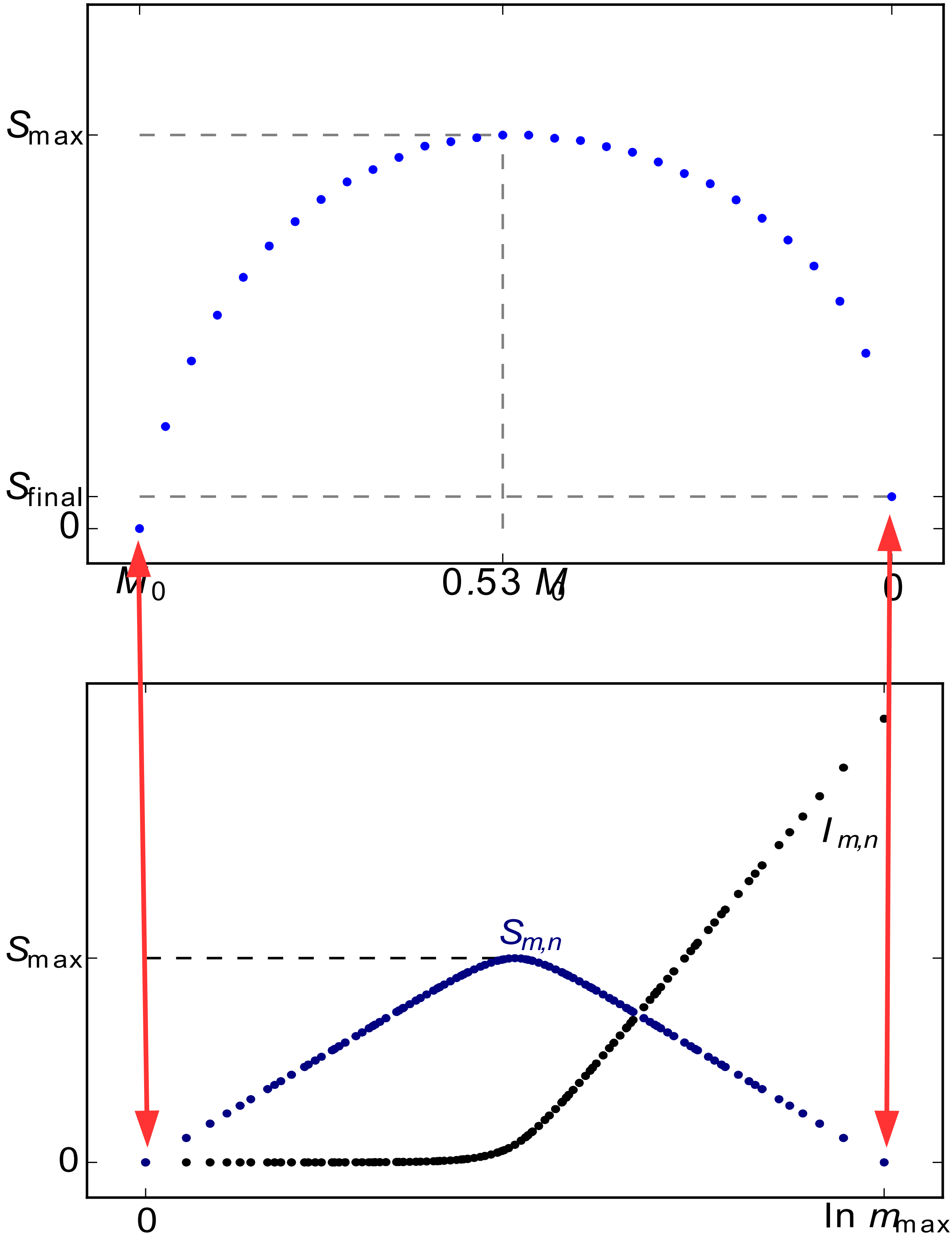


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