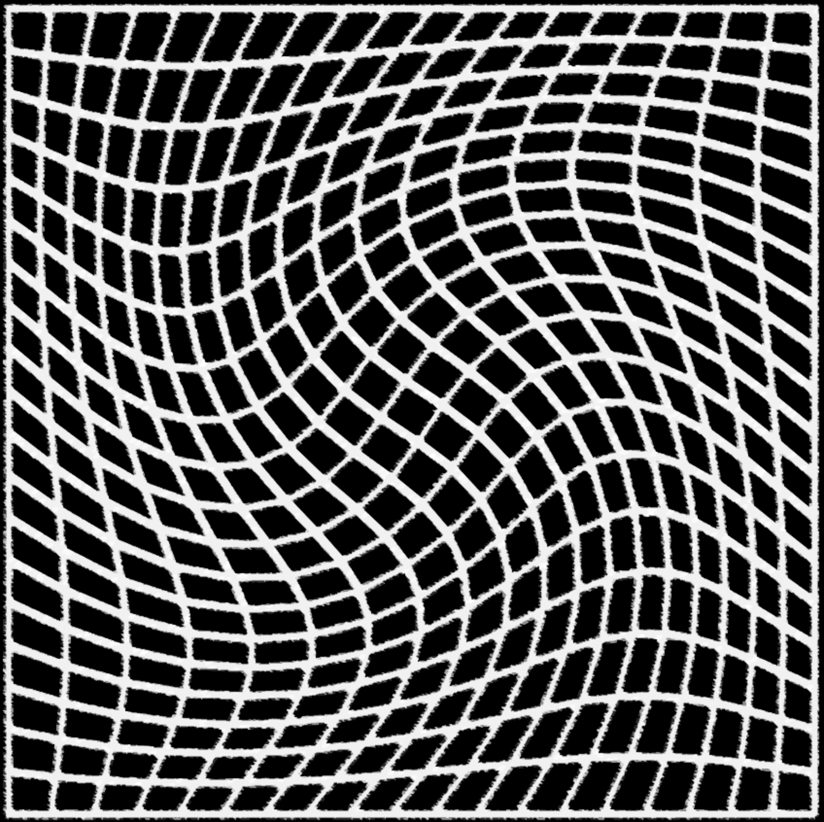
Geometry



of parametric quantum systems

Pavel Cejnar

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Closed bound quantum system

Hamiltonian

$$\widehat{H}$$

Eigenvalues

$$E_i$$

Eigenvectors

$$|i\rangle$$

Closed bound quantum system with variable parameters

$\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$

Hamiltonian

$$\hat{H}(\boldsymbol{\lambda})$$

Eigenvalues

$$E_i(\boldsymbol{\lambda})$$

Eigenvectors

$$|i(\boldsymbol{\lambda})\rangle$$

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$$\hat{H}(\boldsymbol{\lambda})$$

Eigenvalues

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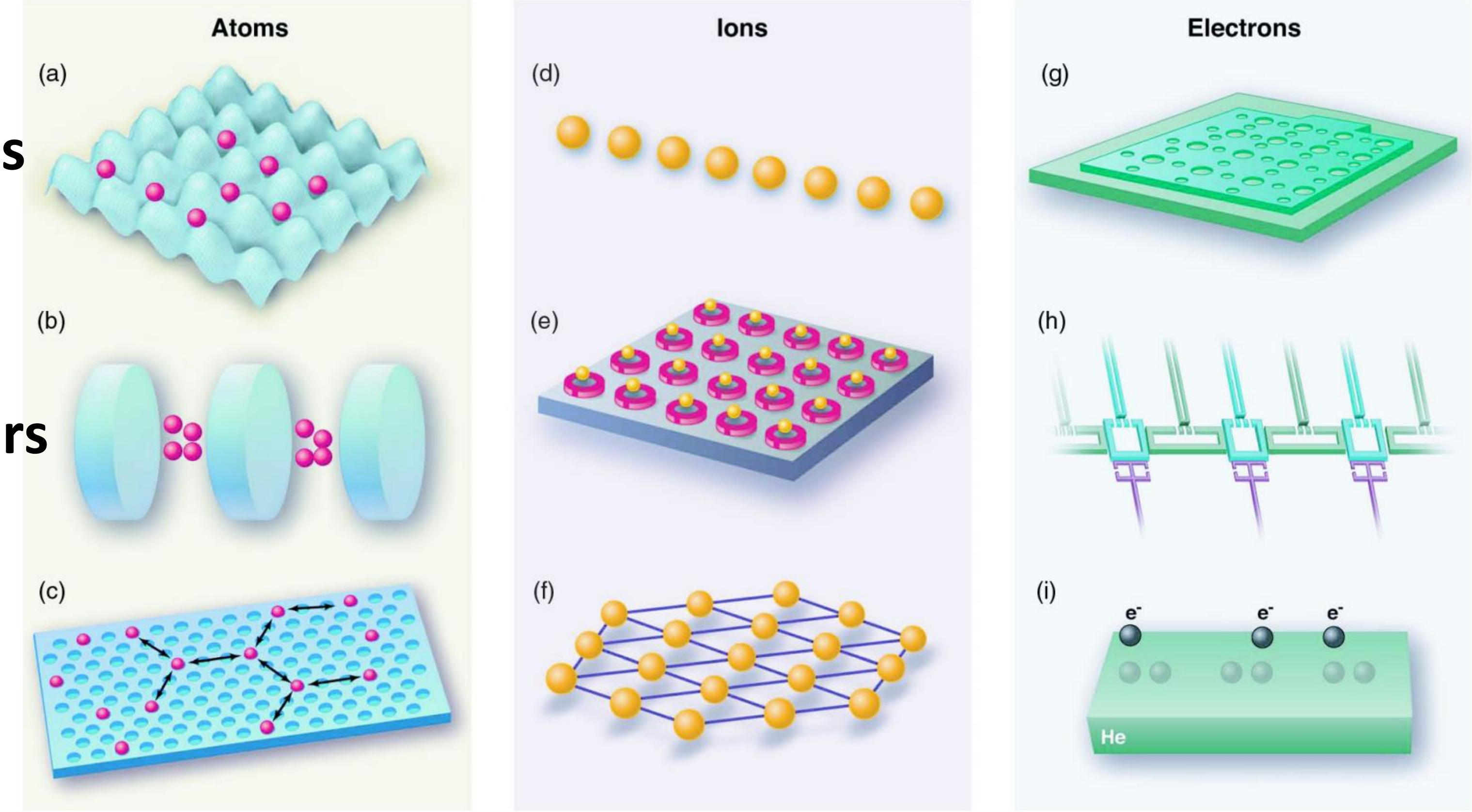
“controllable quantum systems” “quantum simulators”



Georgescu, Ashhab, Nori, Rev. Mod. Phys. 86 (2014) 153

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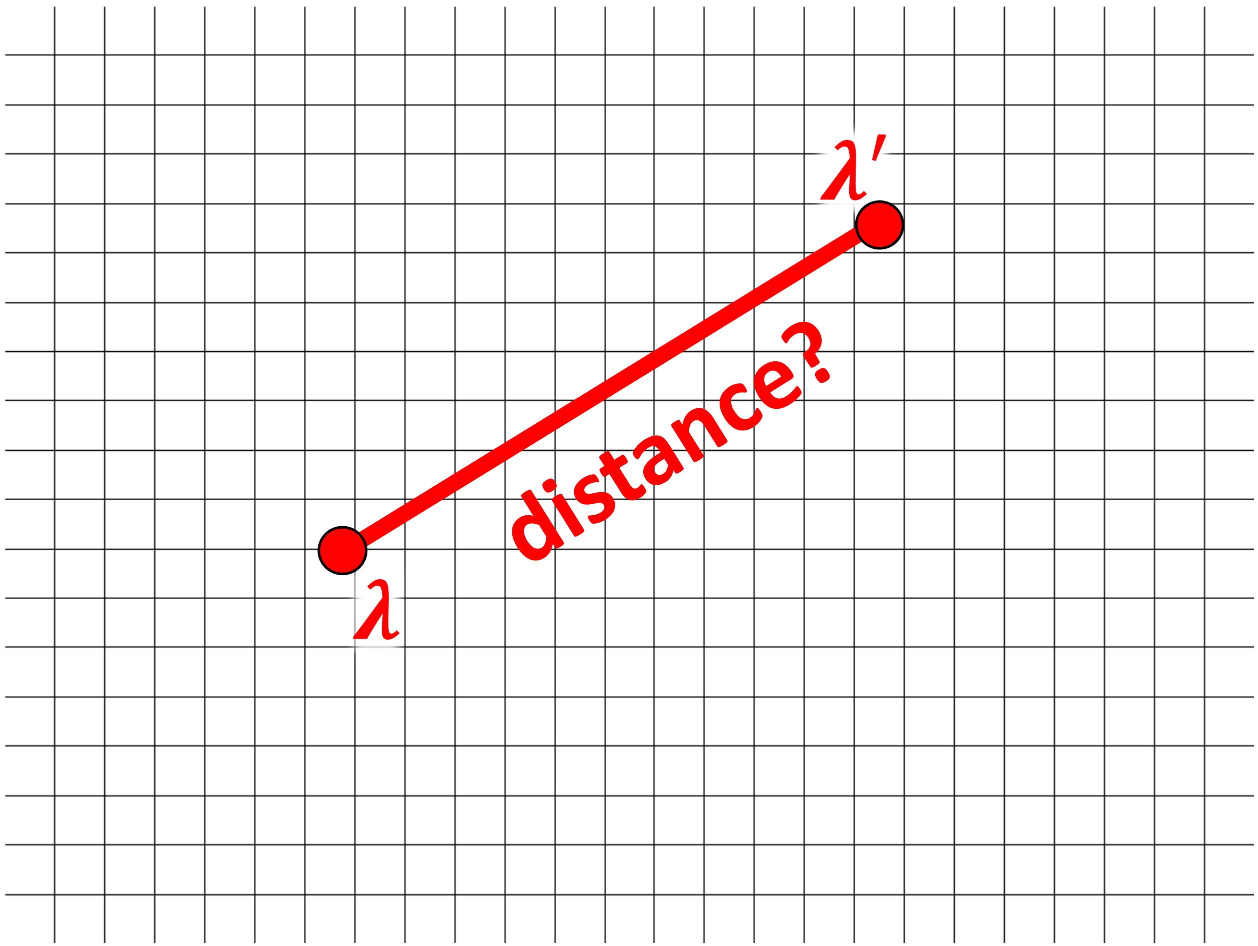


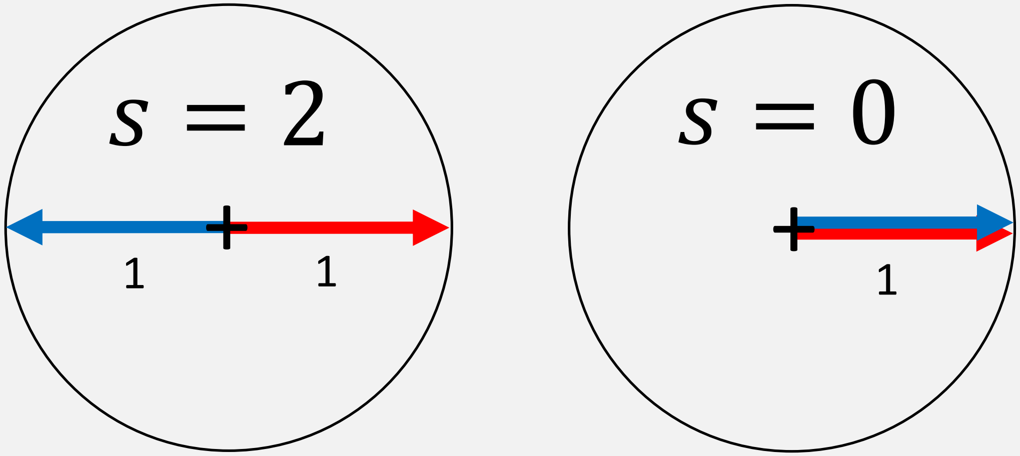
Eigenvector distance: Hilbert space distance

$$\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$$

$$\left(s^{(i)}\right)^2 = \langle\, i(\boldsymbol{\lambda}') - i(\boldsymbol{\lambda}) | i(\boldsymbol{\lambda}') - i(\boldsymbol{\lambda})\, \rangle$$

gauge dependent!

$$= 2 - 2\mathrm{Re}\langle i(\boldsymbol{\lambda}') | i(\boldsymbol{\lambda}) \rangle$$



$$|i(\boldsymbol{\lambda})\rangle$$

Eigenvector distance: fidelity based distance

$$\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$$

$$|i(\boldsymbol{\lambda})\rangle$$

$$\left(ds^{(i)}\right)^2 = 1 - \overbrace{|\langle\, i(\boldsymbol{\lambda} + d\boldsymbol{\lambda}) | i(\boldsymbol{\lambda})\,\rangle|^2}^{\text{fidelity}}$$

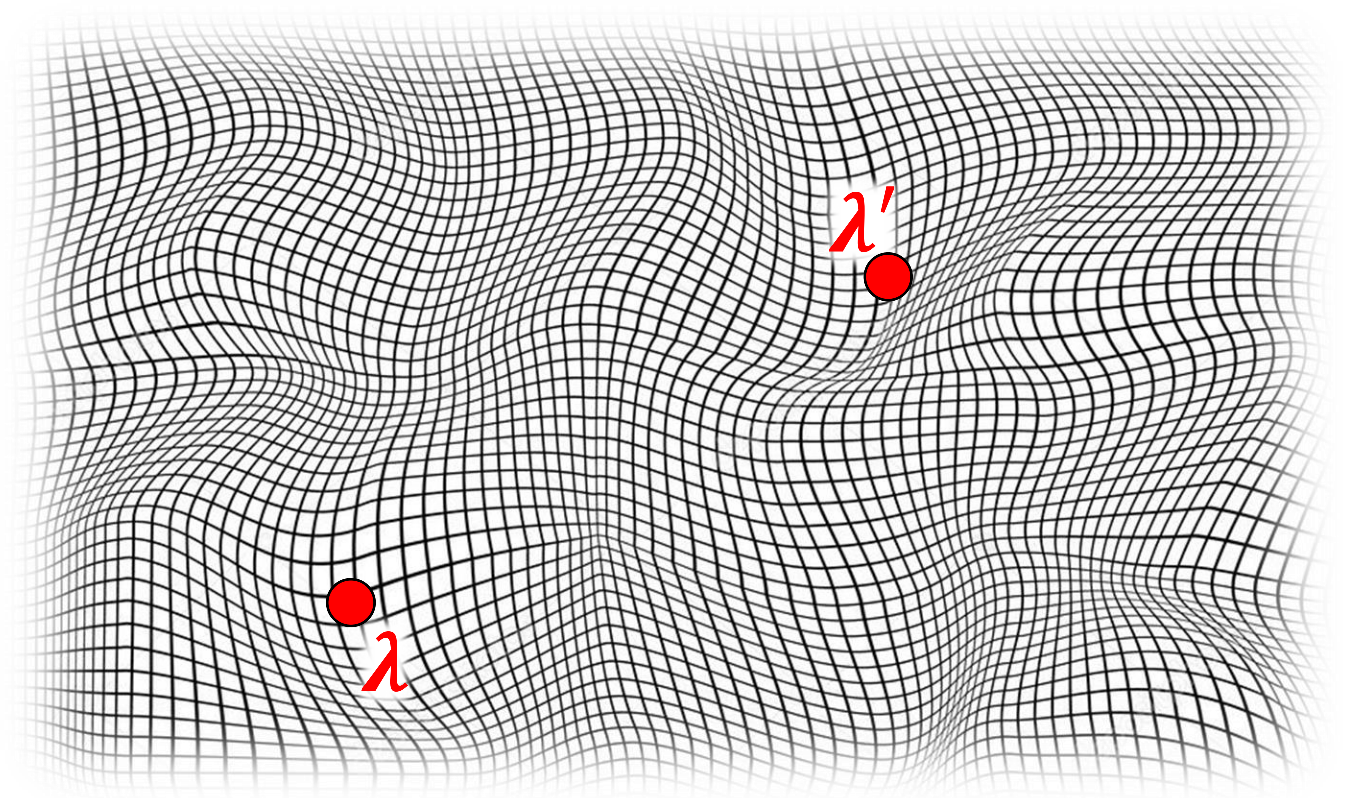
gauge independent!

$$g^{(i)}_{\alpha\beta}(\boldsymbol{\lambda}) = \frac{1}{2}\left(\left\langle \frac{\partial}{\partial\lambda_\alpha} i(\boldsymbol{\lambda}) \middle| \frac{\partial}{\partial\lambda_\beta} i(\boldsymbol{\lambda}) \right\rangle \right.$$
$$\left. - \left\langle \frac{\partial}{\partial\lambda_\alpha} i(\boldsymbol{\lambda}) \middle| i(\boldsymbol{\lambda}) \right\rangle \left\langle i(\boldsymbol{\lambda}) \middle| \frac{\partial}{\partial\lambda_\beta} i(\boldsymbol{\lambda}) \right\rangle + \text{c. c.} \right)$$

$$\left(ds^{(i)}\right)^2 = g^{(i)}_{\alpha\beta}(\boldsymbol{\lambda})\, d\lambda_\alpha d\lambda_\beta$$

Eigenvector parameter manifold as a curved space

$$\boldsymbol{\lambda} \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$$



$$|i(\boldsymbol{\lambda})\rangle$$

$$\boxed{\left(ds^{(i)}\right)^2 = g_{\alpha\beta}^{(i)}(\boldsymbol{\lambda}) d\lambda_\alpha d\lambda_\beta}$$

Eigenvector parameter manifold as a <u>curved space</u>

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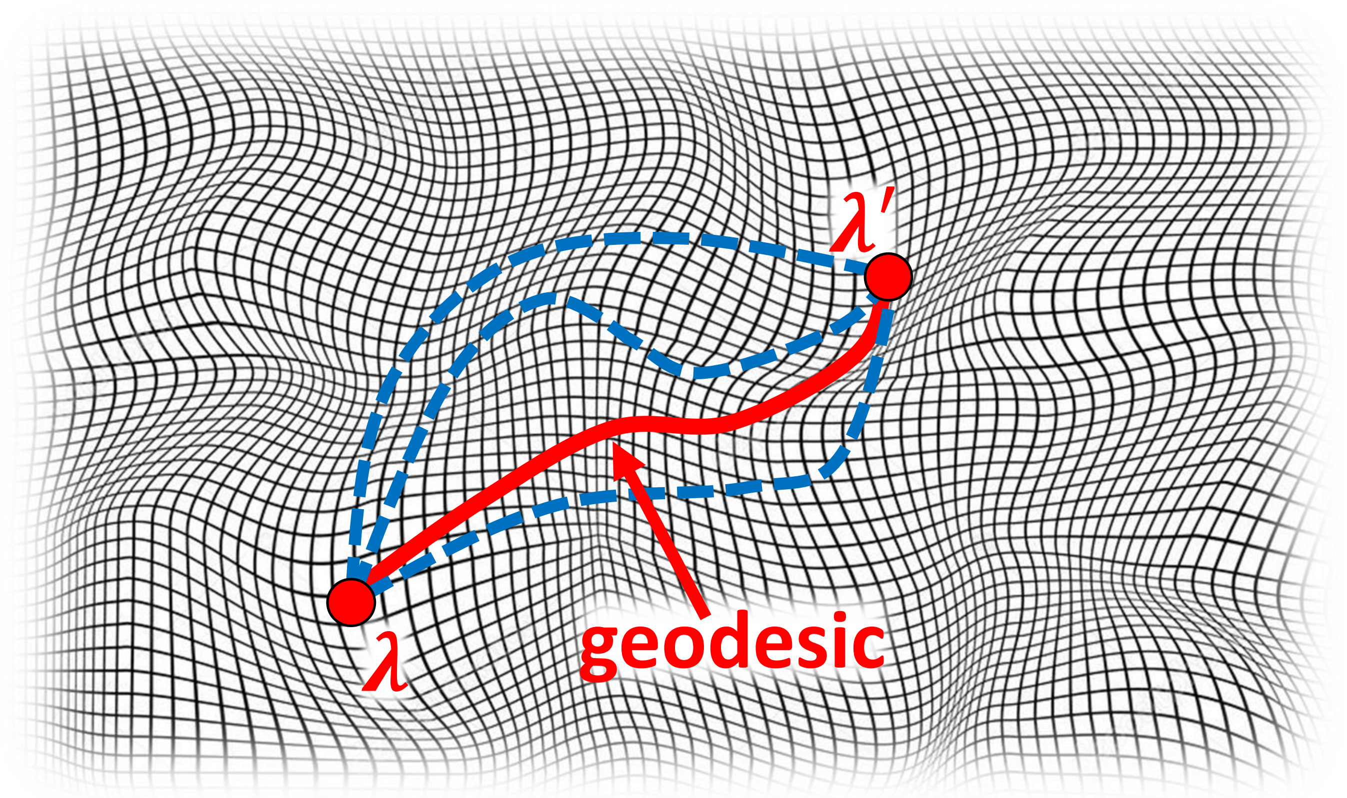


$$|i(\boldsymbol{\lambda})\rangle$$

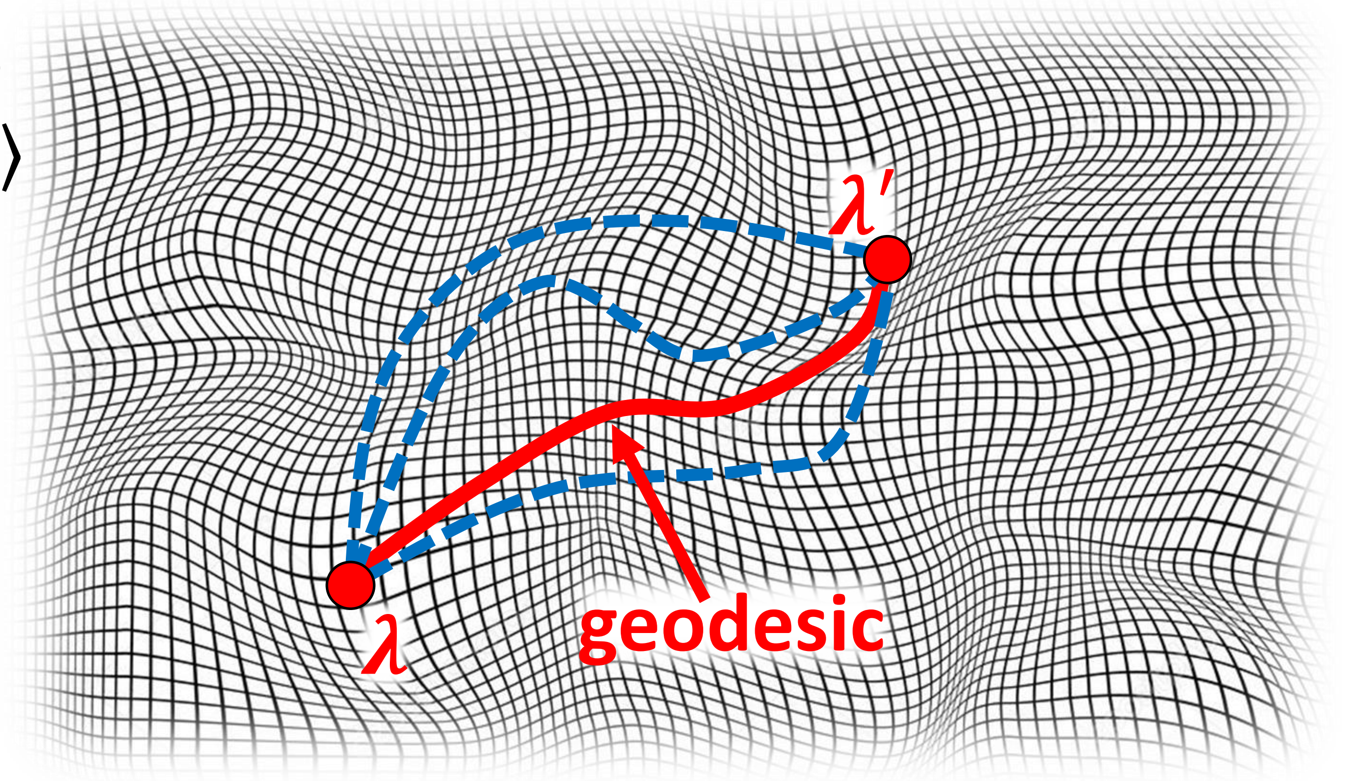
$$\left(ds^{(i)}\right)^2 = g^{(i)}_{\alpha\beta}(\boldsymbol{\lambda})\, d\lambda_\alpha d\lambda_\beta$$

Driven dynamics: adiabatic driving

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}\left(\boldsymbol{\lambda}(t)\right)|\psi(t)\rangle$$

$$|\psi(0)\rangle = |i(\boldsymbol{\lambda})\rangle$$

$$|\psi(T)\rangle = |i(\boldsymbol{\lambda}')\rangle$$



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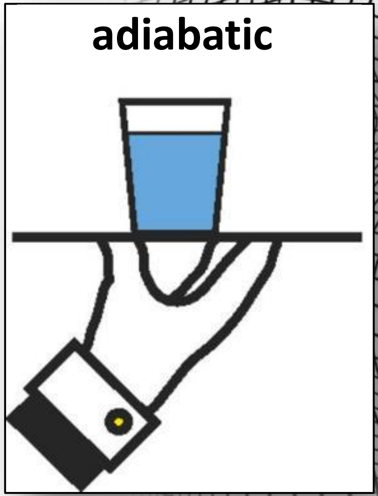
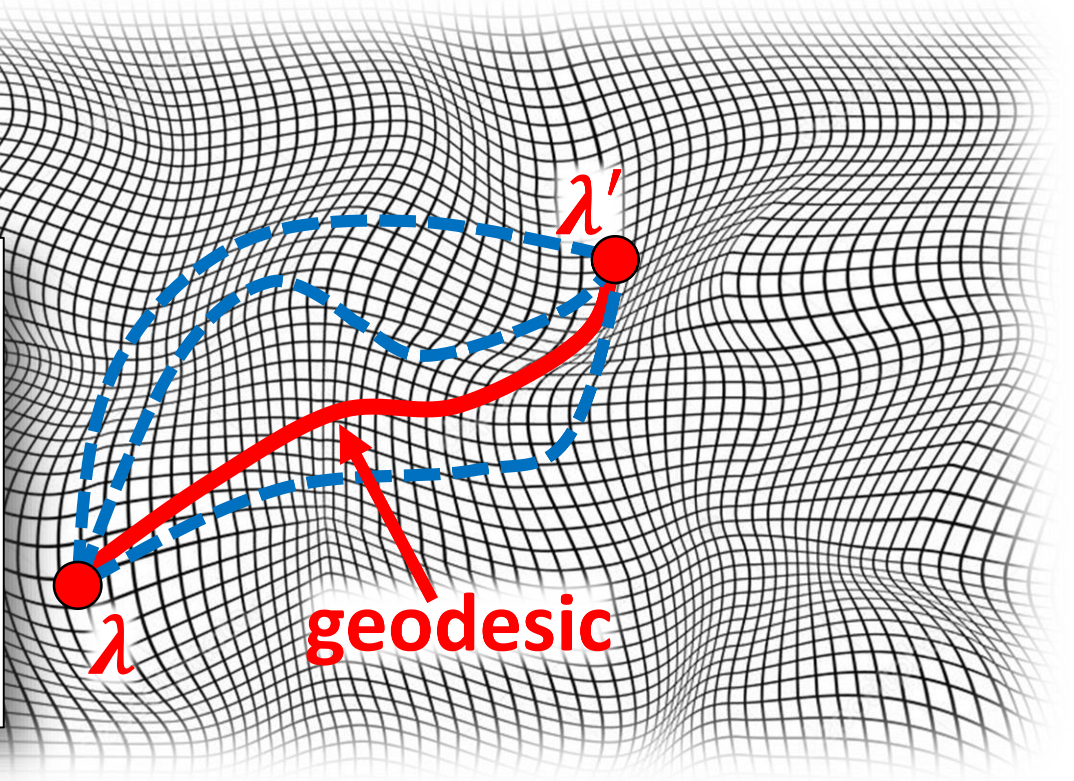
D.Sels, A.Polkovnikov
PNAS 114, E3909 (2017)





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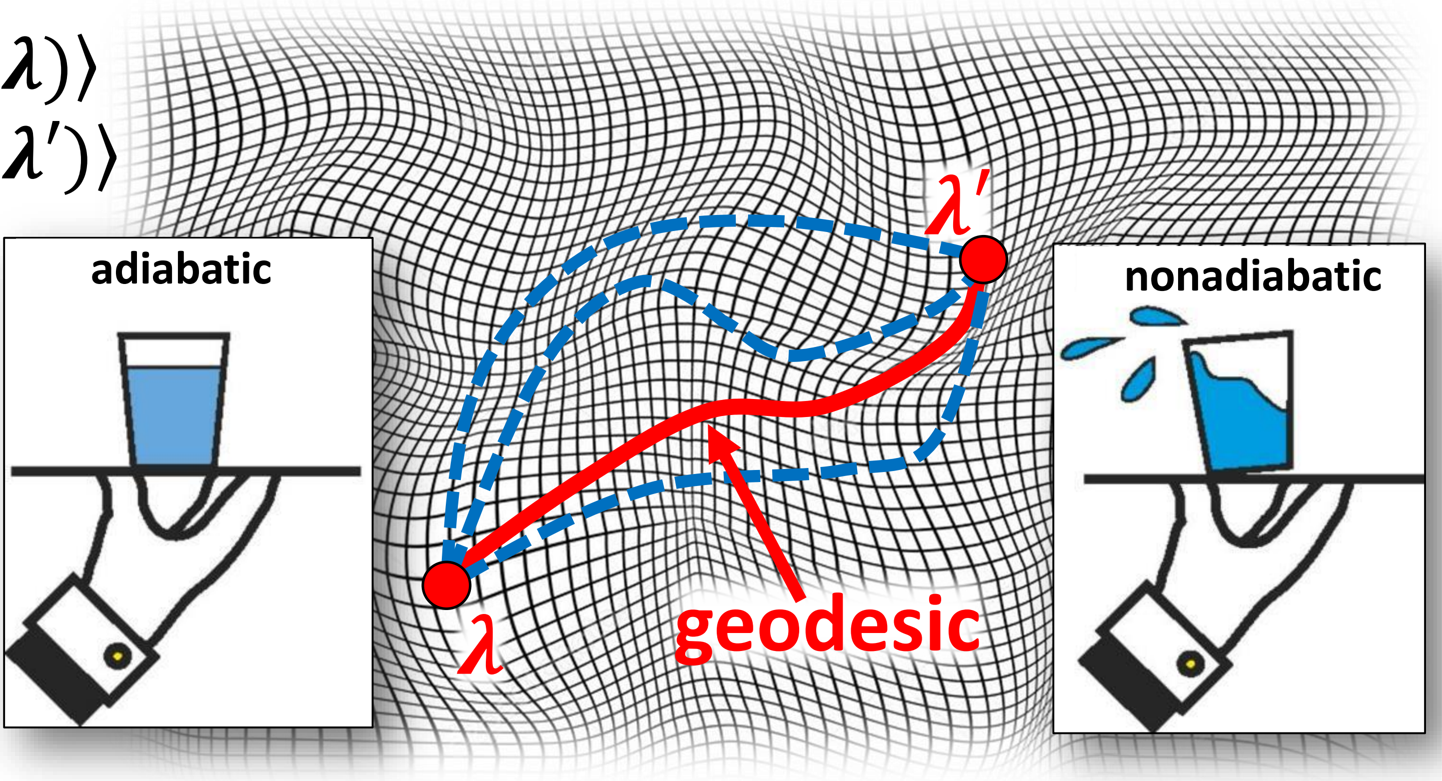


$$|i(\boldsymbol{\lambda})\rangle$$

$$\left(ds^{(i)}\right)^2 = g^{(i)}_{\alpha\beta}(\boldsymbol{\lambda})\,d\lambda_\alpha d\lambda_\beta$$

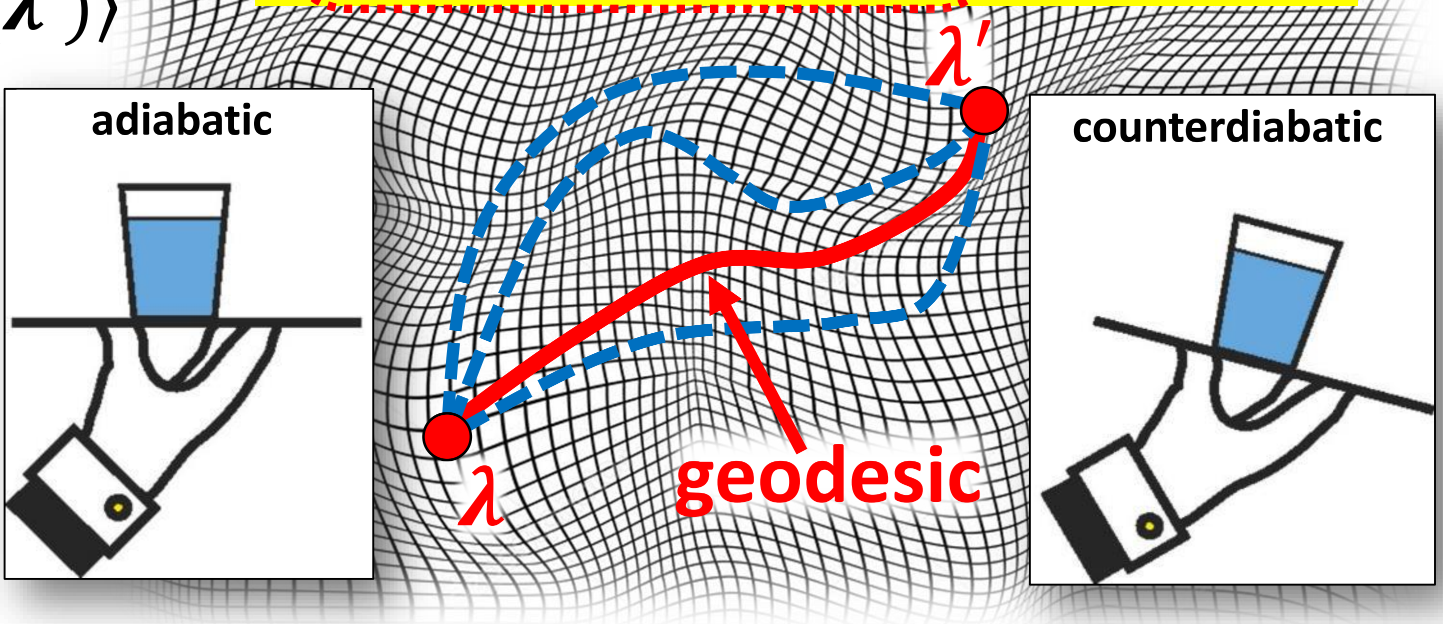
Driven dynamics: counterdiabatic driving

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \left[\hat{H}(\boldsymbol{\lambda}(t)) \overbrace{+\dot{\boldsymbol{\lambda}}(t)\cdot\hat{\boldsymbol{A}}(\boldsymbol{\lambda}(t))}^{\text{balancing term}}\right]|\psi(t)\rangle$$

$$|\psi(0)\rangle = |i(\boldsymbol{\lambda})\rangle$$

$$|\psi(T)\rangle = |i(\boldsymbol{\lambda}')\rangle$$

D.Sels, A.Polkovnikov
PNAS 114, E3909 (2017)



$$|i(\boldsymbol{\lambda})\rangle$$

$$\left(ds^{(i)}\right)^2 = g^{(i)}_{\alpha\beta}(\boldsymbol{\lambda})d\lambda_\alpha d\lambda_\beta$$

Piš, barde, střádej …

(Write, bard, collect …)

$15 million

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Metric tensor and curvature

$$g_{\alpha\beta}^{(i)}(\boldsymbol{\lambda}) = \mathbf{Re}\left(\left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle - \left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| i \right\rangle \left\langle i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle\right)$$

$$= \mathbf{Re} \sum_{j(\neq i)} \frac{\left\langle i \middle| \frac{\partial\hat{H}}{\partial\lambda_\alpha} \middle| j \right\rangle \left\langle j \middle| \frac{\partial\hat{H}}{\partial\lambda_\beta} \middle| i \right\rangle}{\left(E_i - E_j\right)^2}$$

Metric tensor

Metric tensor and curvature

$$g_{\alpha\beta}^{(i)}(\boldsymbol{\lambda}) = \mathbf{Re}\left(\left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle - \left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| i \right\rangle \left\langle i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle\right)$$

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Metric tensor

$$v_{\alpha\beta}^{(i)}(\boldsymbol{\lambda}) = -2\ \mathbf{Im} \sum_{j(\neq i)} \frac{\left\langle i \middle| \frac{\partial\hat{H}}{\partial\lambda_\alpha} \middle| j \right\rangle \left\langle j \middle| \frac{\partial\hat{H}}{\partial\lambda_\beta} \middle| i \right\rangle}{\left(E_i - E_j\right)^2}$$

Curvature tensor

$$= -2\ \mathbf{Im}\left(\left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle - \left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| i \right\rangle \left\langle i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle\right)$$

Metric tensor and curvature

$$g^{(i)}_{\alpha\beta}(\boldsymbol{\lambda}) = \mathbf{Re}\left(\left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle - \left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| i \right\rangle \left\langle i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle\right)$$

$$= \mathbf{Re} \sum_{j(\neq i)} \frac{\left\langle i \middle| \frac{\partial \hat{H}}{\partial \lambda_\alpha} \middle| j \right\rangle \left\langle j \middle| \frac{\partial \hat{H}}{\partial \lambda_\beta} \middle| i \right\rangle}{\left(E_i - E_j\right)^2}$$

Metric tensor

Geometric tensor

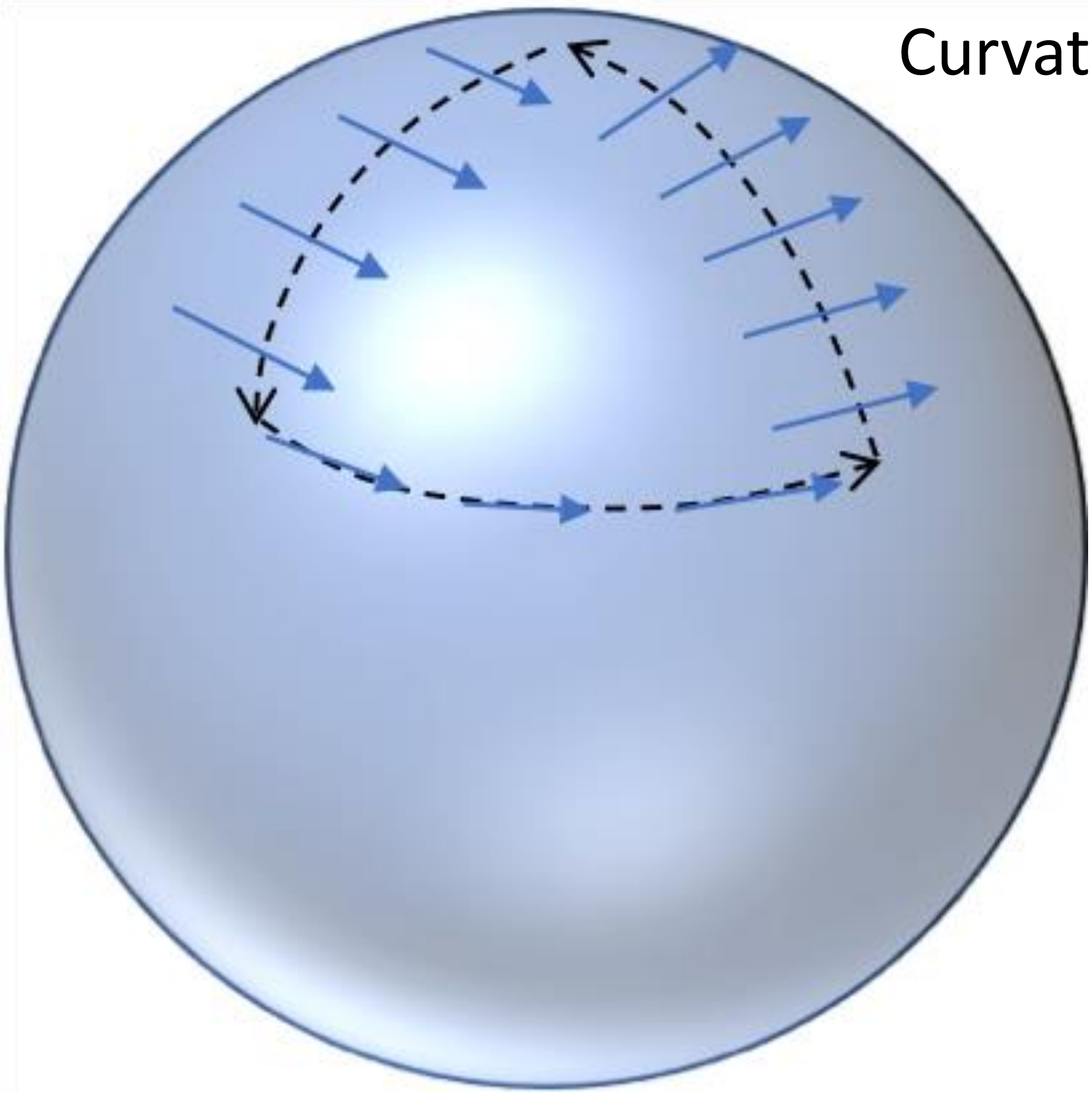
$$\boldsymbol{\chi}^{(i)}_{\alpha\beta}(\boldsymbol{\lambda}) = \boldsymbol{g}^{(i)}_{\alpha\beta}(\boldsymbol{\lambda}) - i\, 2\boldsymbol{v}^{(i)}_{\alpha\beta}(\boldsymbol{\lambda})$$

Curvature tensor

$$v^{(i)}_{\alpha\beta}(\boldsymbol{\lambda}) = -2\ \mathbf{Im} \sum_{j(\neq i)} \frac{\left\langle i \middle| \frac{\partial \hat{H}}{\partial \lambda_\alpha} \middle| j \right\rangle \left\langle j \middle| \frac{\partial \hat{H}}{\partial \lambda_\beta} \middle| i \right\rangle}{\left(E_i - E_j\right)^2}$$

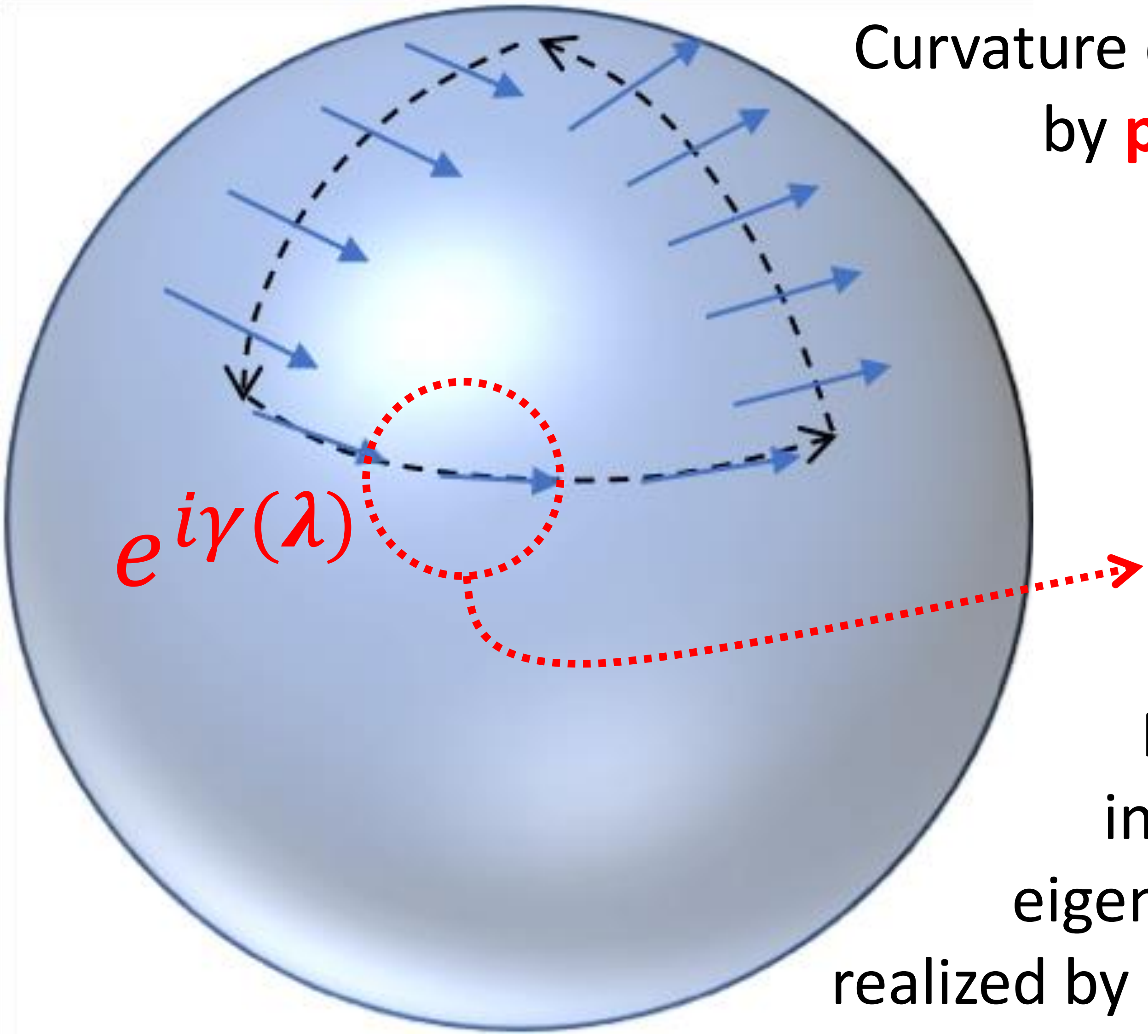
$$= -2\ \mathbf{Im}\left(\left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle - \left\langle \frac{\partial}{\partial\lambda_\alpha} i \middle| i \right\rangle \left\langle i \middle| \frac{\partial}{\partial\lambda_\beta} i \right\rangle\right)$$

Berry's phase



Curvature can be measured by **parallel transport**

Berry’s phase

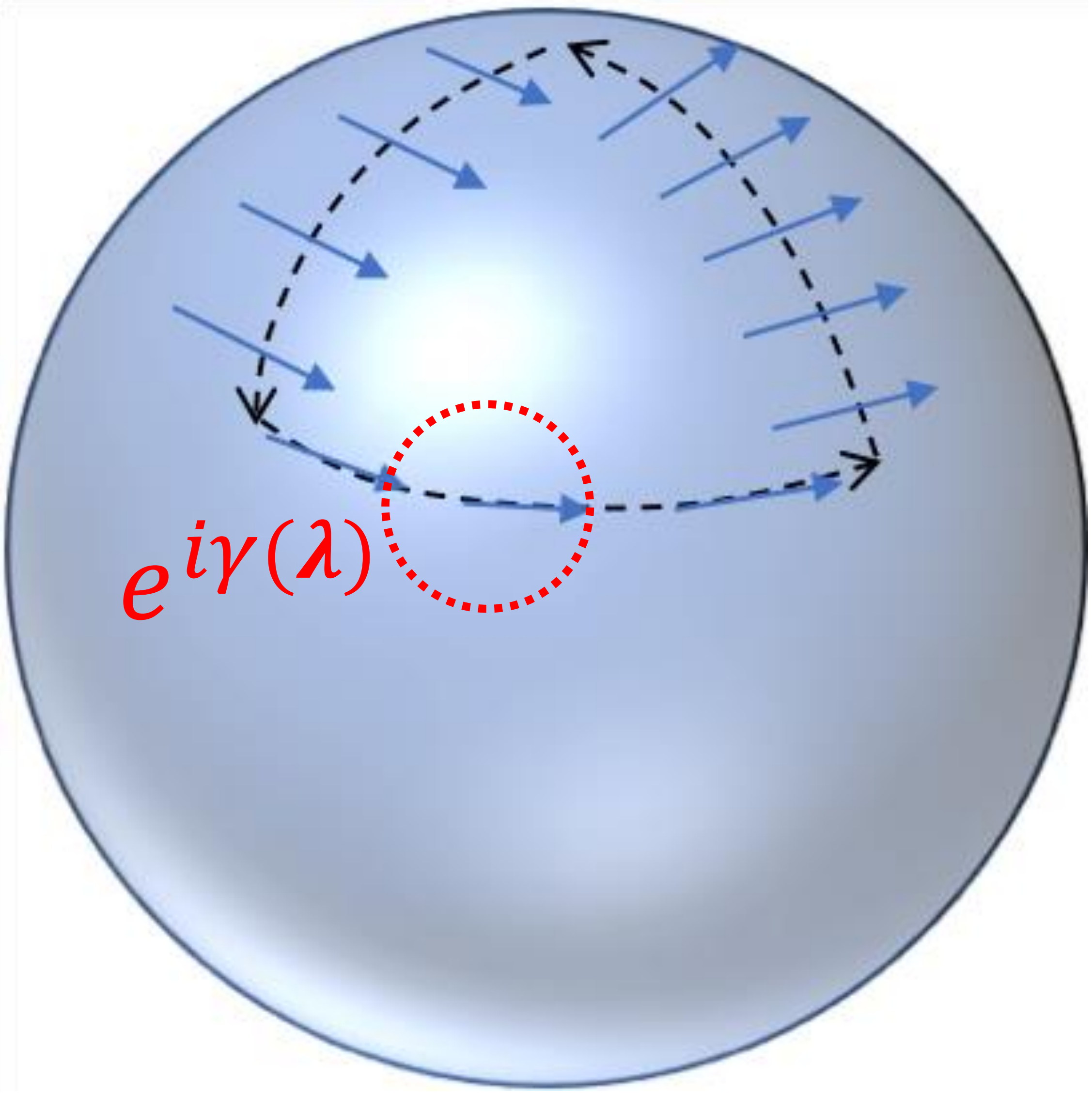


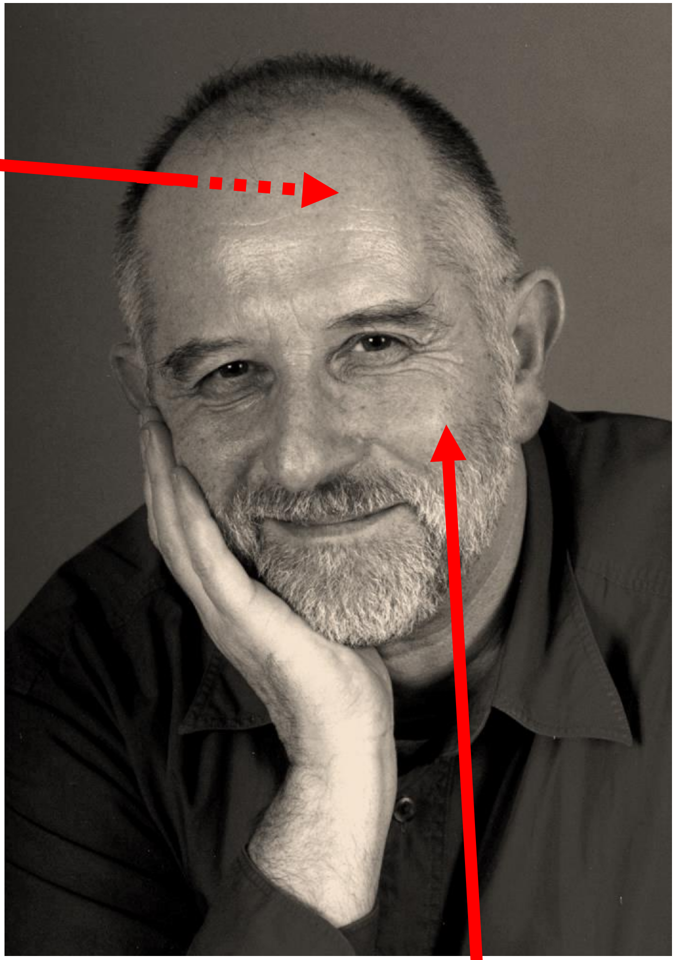
Curvature can be measured by **parallel transport**

Local phase (gauge) of the eigenvector

Parallel transport in the Hamiltonian eigenstate manifold is realized by **adiabatic driving**

Berry’s phase *1983





Michael Berry *1941

Berry’s face

Berry’s phase

Ig Nobel Prize 2000

Levitation without Meditation

Michael Berry & Andrey Geim

https://michaelberryphysics.wordpress.com/

Degeneracies & avoided crossings

$$\chi_{\alpha\beta}^{(i)}(\boldsymbol{\lambda}) = \sum_{j(\neq i)} \frac{\left\langle i \left| \frac{\partial \hat{H}}{\partial \lambda_\alpha} \right| j \right\rangle \left\langle j \left| \frac{\partial \hat{H}}{\partial \lambda_\beta} \right| i \right\rangle}{\boxed{\left(E_i - E_j\right)^2}}$$

!!!

Singularities $E_i = E_j$

Degeneracies & avoided crossings

$$\chi^{(i)}_{\alpha\beta}(\boldsymbol{\lambda}) = \sum_{j(\neq i)} \frac{\left\langle i\left|\frac{\partial\hat{H}}{\partial\lambda_\alpha}\right|j\right\rangle \left\langle j\left|\frac{\partial\hat{H}}{\partial\lambda_\beta}\right|i\right\rangle}{\boxed{\left(E_i - E_j\right)^2}}$$

!!!

Singularities $E_i = E_j$

Hamiltonian 2 x 2

$$\hat{H}(\boldsymbol{\lambda}) = \begin{pmatrix} H_{11}(\boldsymbol{\lambda}) & H_{12}(\boldsymbol{\lambda}) \\ H^*_{12}(\boldsymbol{\lambda}) & H_{22}(\boldsymbol{\lambda}) \end{pmatrix}$$

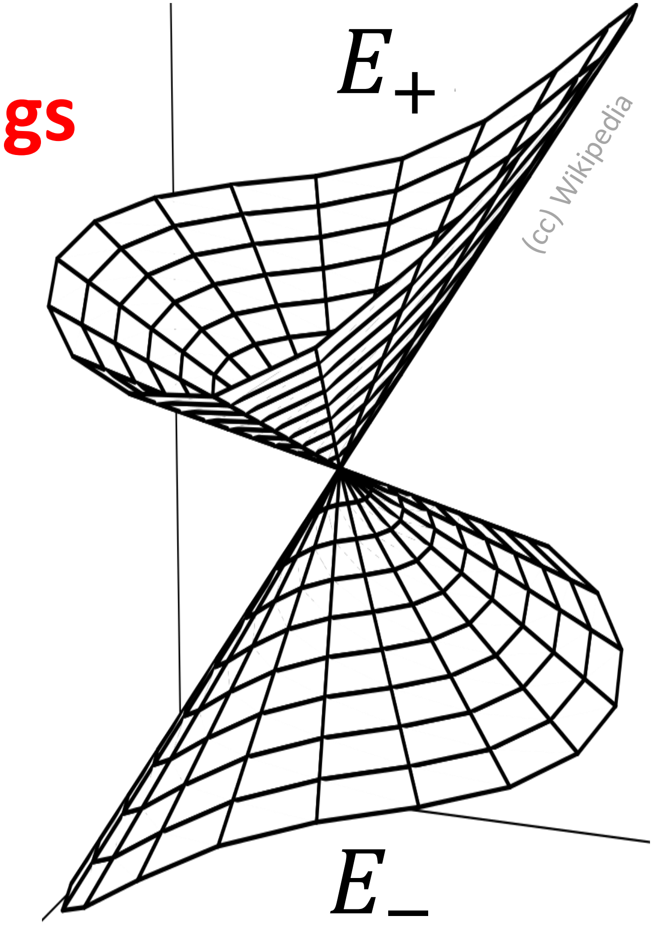
eigenvalue equation

$$\mathrm{Det}\left[\,\hat{H}(\boldsymbol{\lambda}) - E\,\hat{\mathbb{I}}\,\right] = 0$$

solutions

$$E_\pm(\boldsymbol{\lambda}) = \frac{H_{11}(\boldsymbol{\lambda})+H_{22}(\boldsymbol{\lambda})}{2} \pm \frac{1}{2}\sqrt{\left(\frac{H_{11}(\boldsymbol{\lambda})-H_{22}(\boldsymbol{\lambda})}{2}\right)^2+\left(\mathrm{Re}\,H_{12}(\boldsymbol{\lambda})\right)^2+\left(\mathrm{Im}\,H_{12}(\boldsymbol{\lambda})\right)^2}$$

Degeneracies & avoided crossings



$$\chi^{(i)}_{\alpha\beta}(\lambda) = \sum_{j(\neq i)} \frac{\left\langle i\left|\frac{\partial \hat{H}}{\partial \lambda_\alpha}\right|j\right\rangle \left\langle j\left|\frac{\partial \hat{H}}{\partial \lambda_\beta}\right|i\right\rangle}{\boxed{\left(E_i - E_j\right)^2}} \quad !!!$$

Singularities $E_i = E_j$

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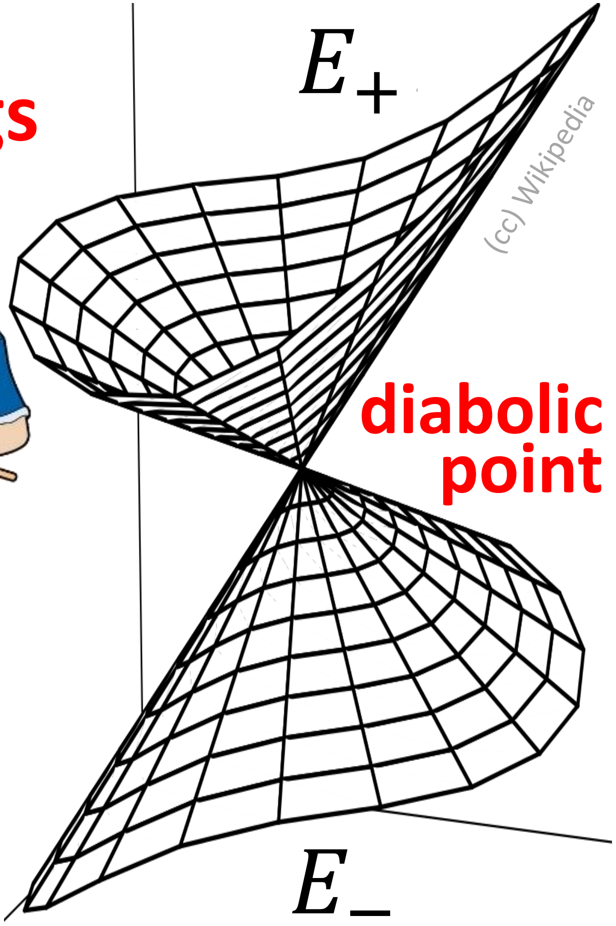
$$\mathbf{0} \overset{!}{=}$$

conical intersection $\propto |\lambda - \lambda_0|$

Degeneracies & avoided crossings

$$\chi^{(i)}_{\alpha\beta}(\boldsymbol{\lambda}) = \sum_{j(\neq i)} \frac{\left\langle i\left|\frac{\partial \hat{H}}{\partial \lambda_\alpha}\right|j\right\rangle \left\langle j\left|\frac{\partial \hat{H}}{\partial \lambda_\beta}\right|i\right\rangle}{\boxed{(E_i - E_j)^2}}$$!!!

Singularities $E_i = E_j$



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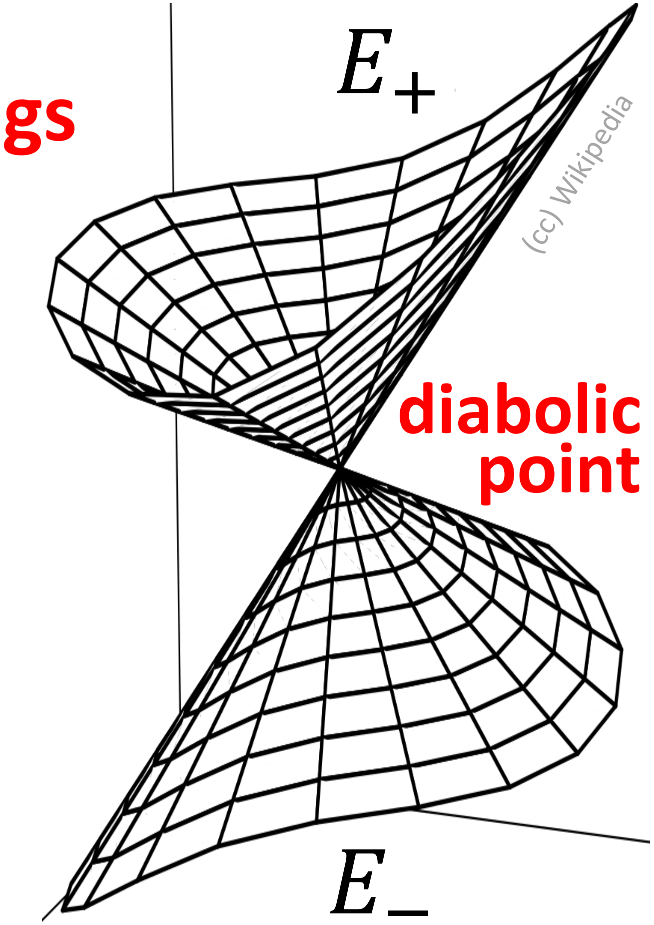
$$\mathbf{0} \overset{!}{=}$$

conical intersection

Degeneracies & avoided crossings

$$\chi^{(i)}_{\alpha\beta}(\lambda) = \sum_{j(\neq i)} \frac{\left\langle i\left|\frac{\partial \hat{H}}{\partial \lambda_\alpha}\right|j\right\rangle \left\langle j\left|\frac{\partial \hat{H}}{\partial \lambda_\beta}\right|i\right\rangle}{\boxed{\left(E_i - E_j\right)^2}} \quad !!!$$



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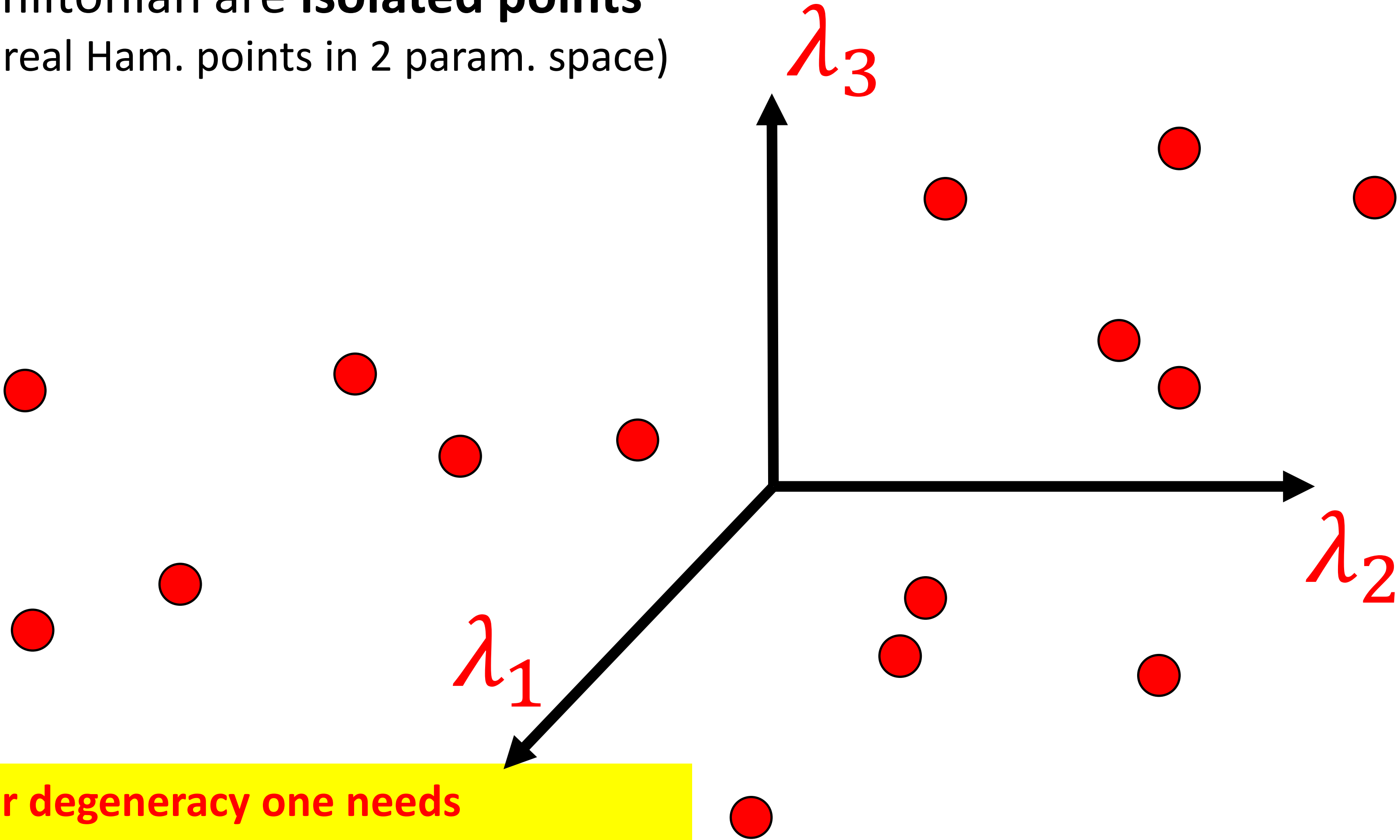
For degeneracy one needs
3 independent functions to vanish!

Degeneracies & avoided crossings

For systems with 3 parameters degeneracies of a complex Hamiltonian are **isolated points**

(for real Ham. points in 2 param. space)



For degeneracy one needs
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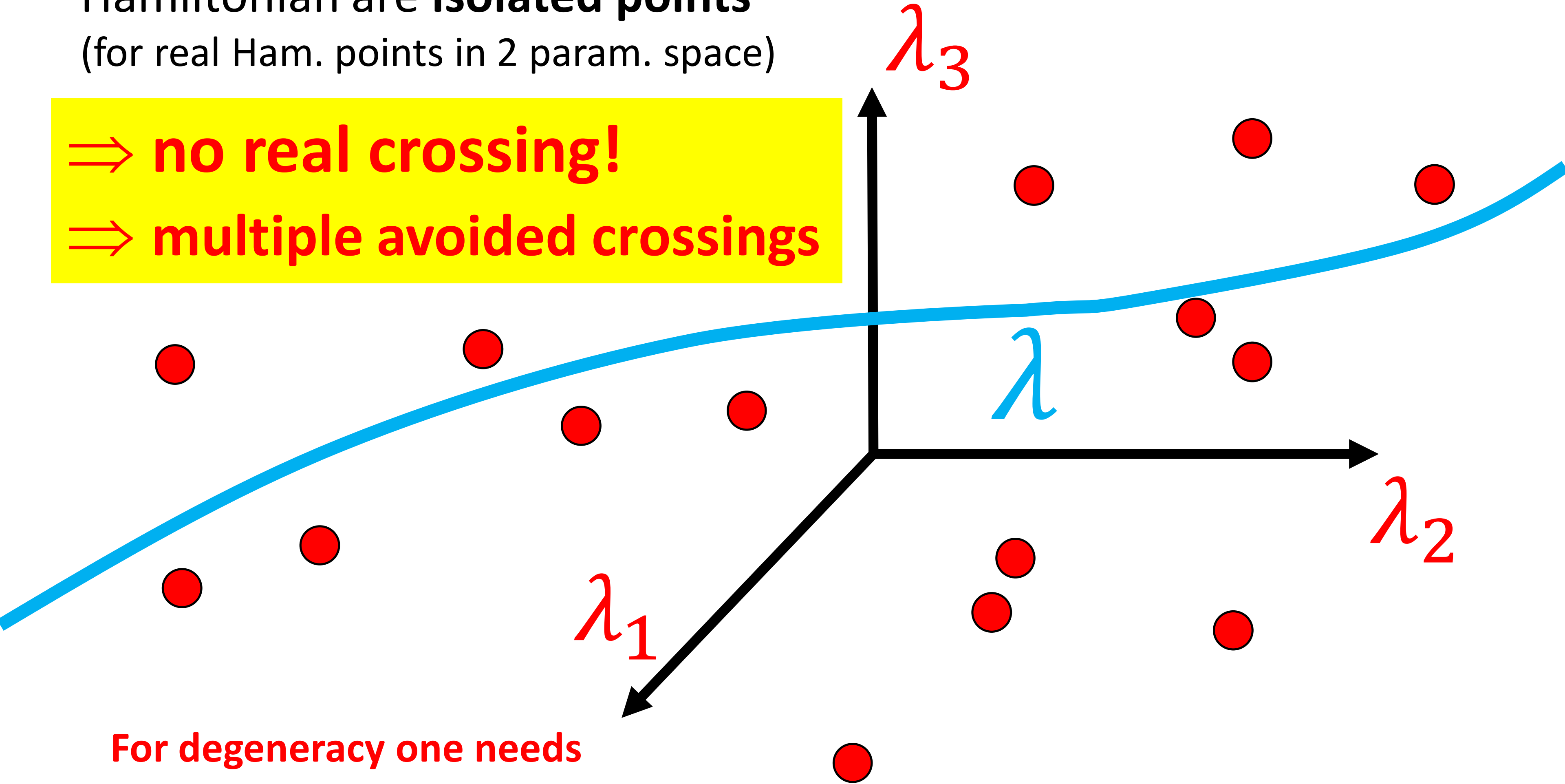
Degeneracies & avoided crossings

For systems with 3 parameters degeneracies of a complex Hamiltonian are **isolated points**

(for real Ham. points in 2 param. space)

$\Rightarrow$ **no real crossing!**

$\Rightarrow$ **multiple avoided crossings**



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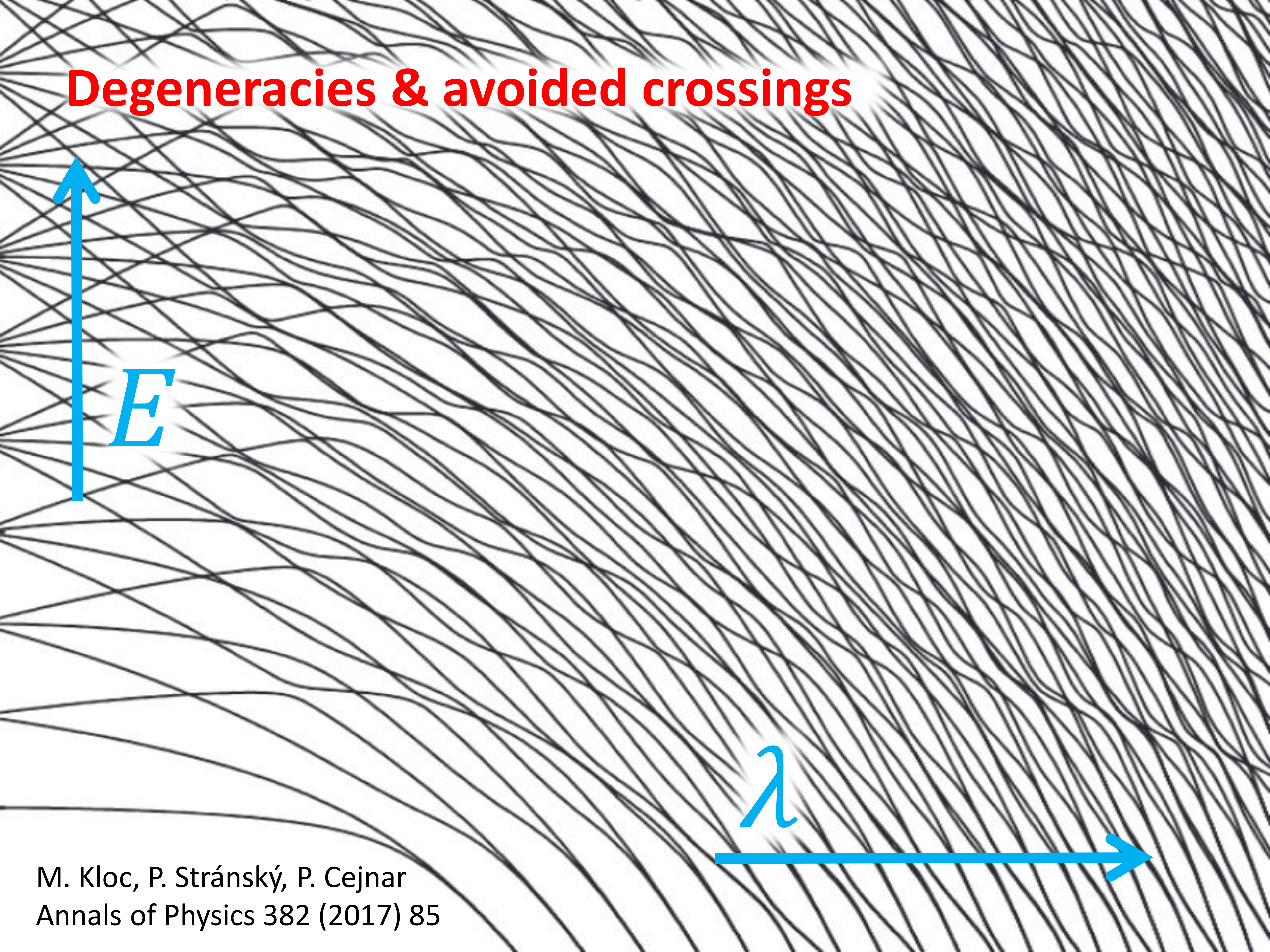
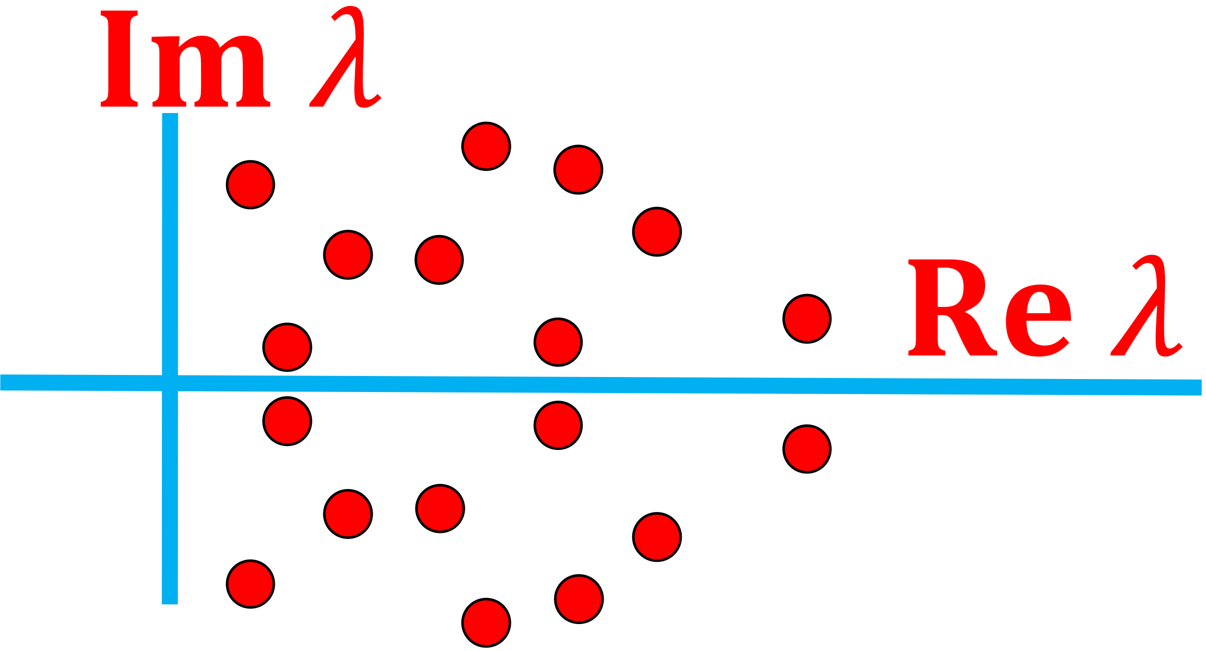


M. Kloc, P. Stránský, P. Cejnar
Annals of Physics 382 (2017) 85

Degeneracies & avoided crossings & complex extensions

For systems with 1 parameter we seek for degeneracies in $\lambda \in \mathbb{C}$



Degeneracies & avoided crossings & complex extensions

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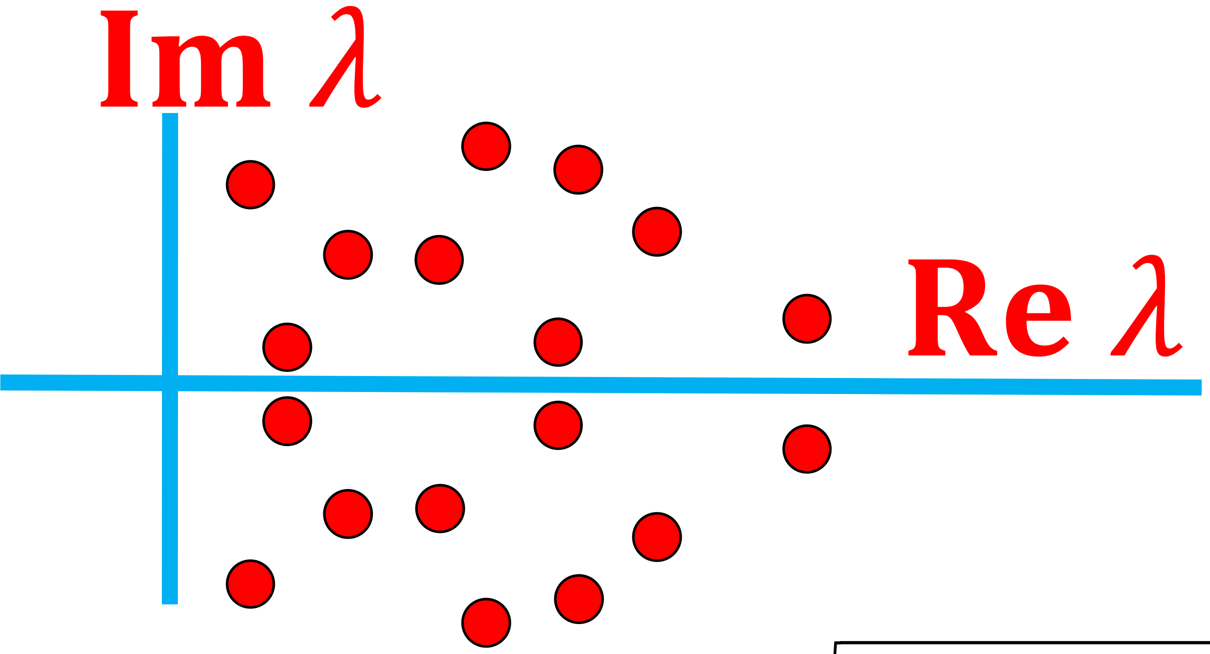
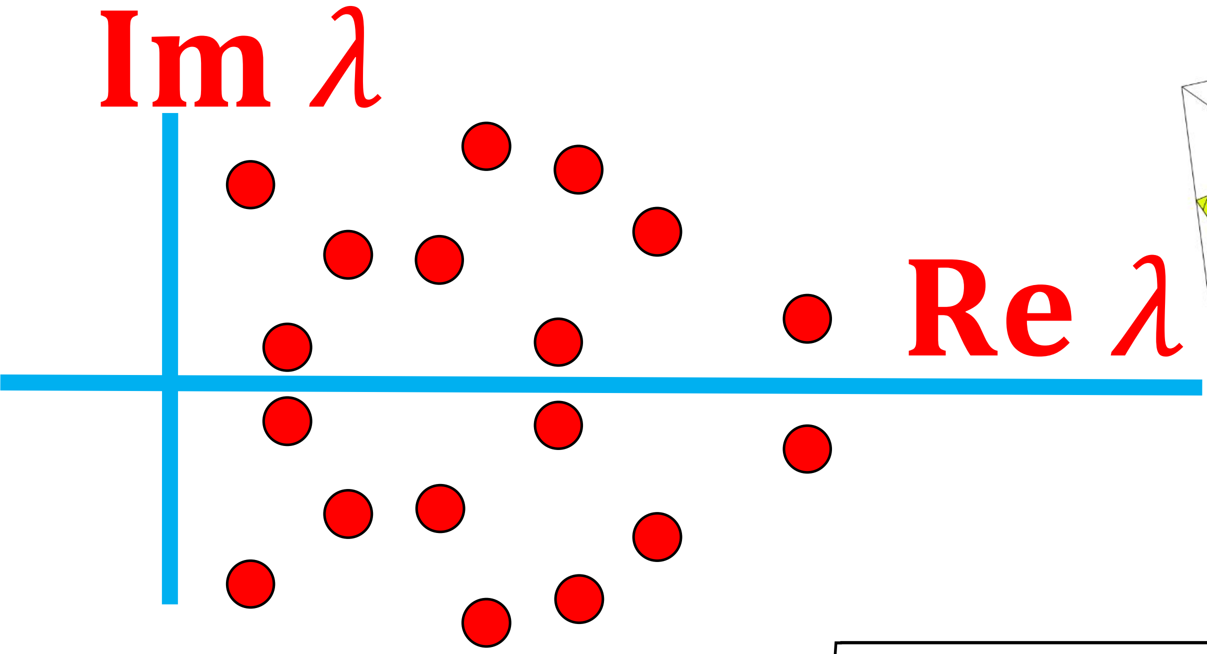


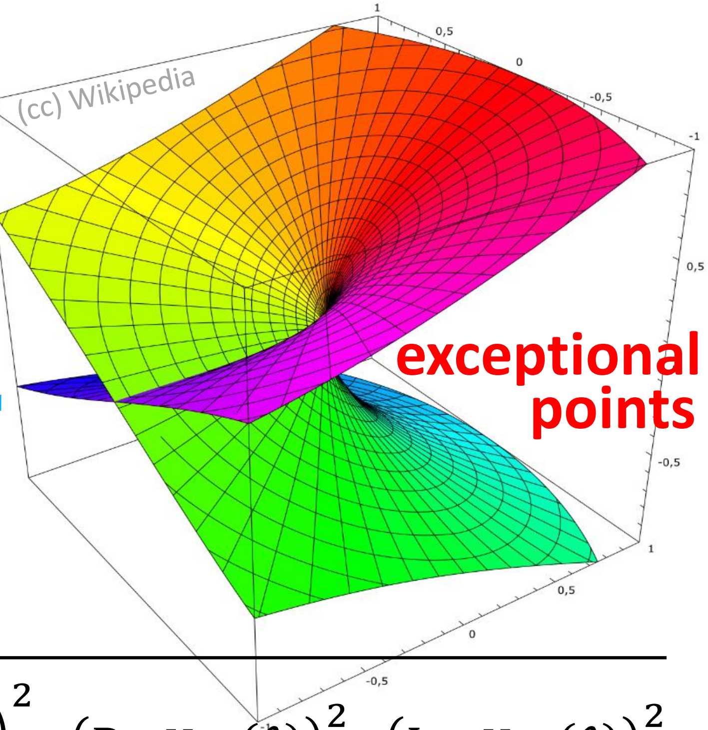
$$E_\pm(\lambda) = \frac{H_{11}(\lambda)+H_{22}(\lambda)}{2} \pm \frac{1}{2}\underbrace{\sqrt{\underbrace{\left(\frac{H_{11}(\lambda)-H_{22}(\lambda)}{2}\right)^2+(\mathrm{Re}\,H_{12}(\lambda))^2+(\mathrm{Im}\,H_{12}(\lambda))^2}_{\propto\,(\lambda-\lambda_0)}}}_{\propto\,\sqrt{\lambda-\lambda_0}}$$

$$0 \overset{!}{=}$$

Degeneracies & avoided crossings { & complex extensions

For systems with 1 parameter we seek for degeneracies in $\lambda \in \mathbb{C}$





(cc) Wikipedia

exceptional points

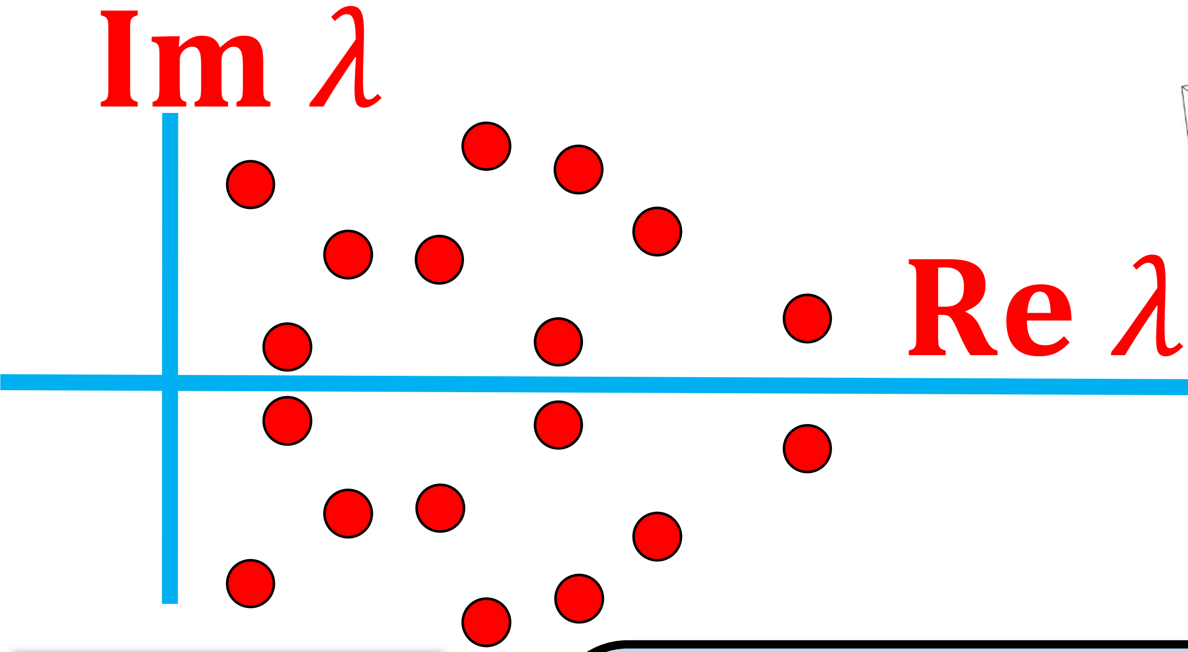
$$E_\pm(\lambda) = \frac{H_{11}(\lambda)+H_{22}(\lambda)}{2} \pm \frac{1}{2}\underbrace{\sqrt{\underbrace{\left(\frac{H_{11}(\lambda)-H_{22}(\lambda)}{2}\right)^2+(\mathrm{Re}\,H_{12}(\lambda))^2+(\mathrm{Im}\,H_{12}(\lambda))^2}_{\propto(\lambda-\lambda_0)}}}_{\propto\sqrt{\lambda-\lambda_0}}$$

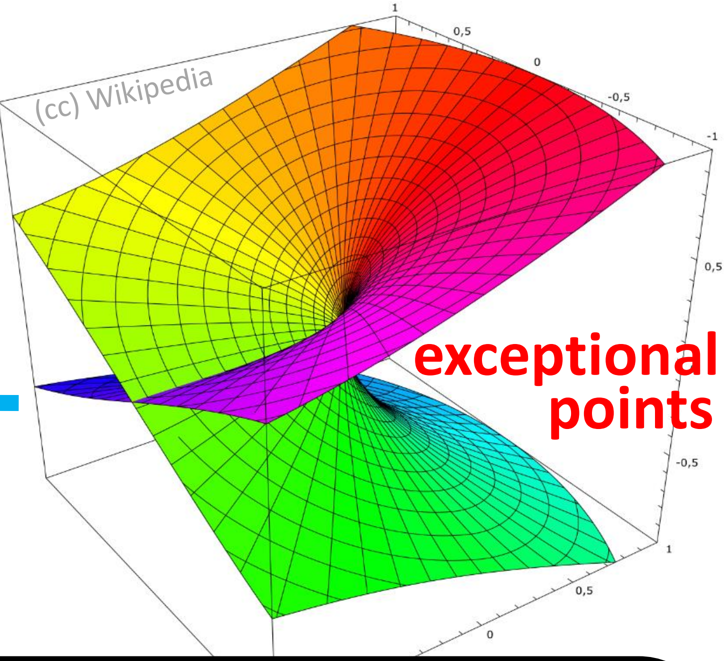
$$0 \overset{!}{=}$$

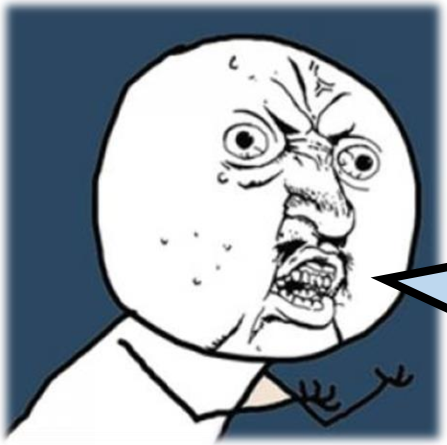
square root branching point

Degeneracies & avoided crossings & complex extensions

For systems with 1 parameter we seek for degeneracies in $\lambda \in \mathbb{C}$

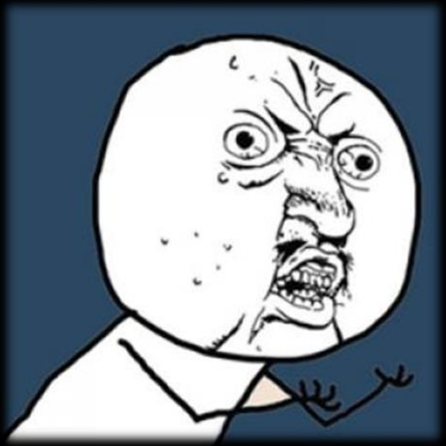






Accumulation of exceptional points near $(\mathrm{Re}\lambda, \mathrm{Im}\lambda) = (\lambda_c, 0)$ indicates a **quantum critical point** ...

This is where our work starts.



Thank you!

This is where our work starts.

Some home reading:

- M.V. Berry, in: *Geometric Phases in Physics*, edited by A. Shapere, F. Wilczek (World Scientific, Singapore, 1989)
  “Quantal phase factors accompanying adiabatic changes” (1984)
  “The quantum phase, five years after” (1988)
- M. Kolodrubetz, D. Sels, P. Mehta, A. Polkovnikov, *Physics Reports* 697 (2017) 1
  “Geometry and non-adiabatic response in quantum and classical systems”
- M. Tomka, T. Souza, S. Rosenberg, A. Polkovnikov, arXiv:1606.05890
  “Geodesic paths for quantum many-body systems”
- P. Stránský, M. Dvořák, P. Cejnar, *Physical Review* E 97 (2018) 012112
  “Exceptional points near first- and second-order quantum phase transitions”