QCD Wigner distribution

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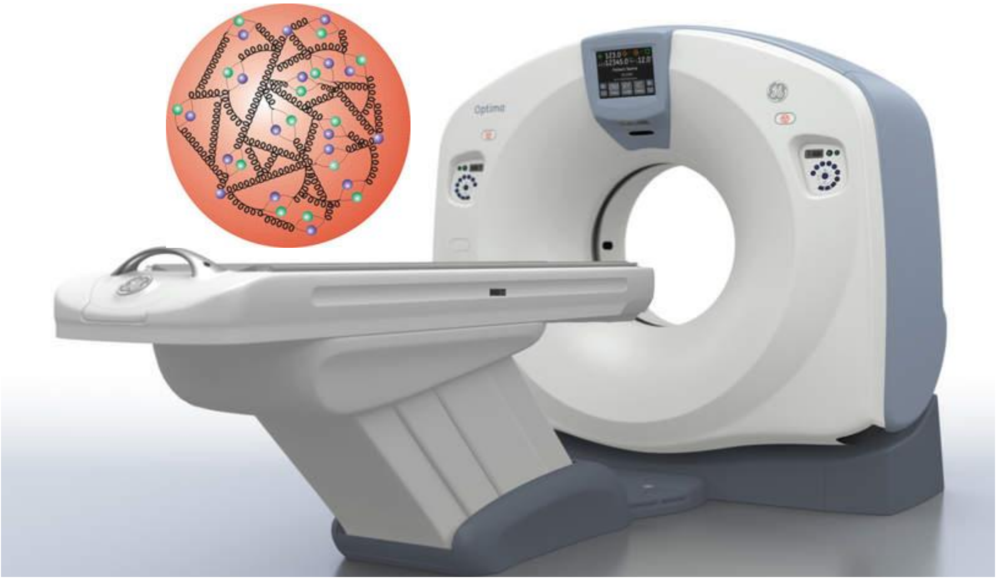
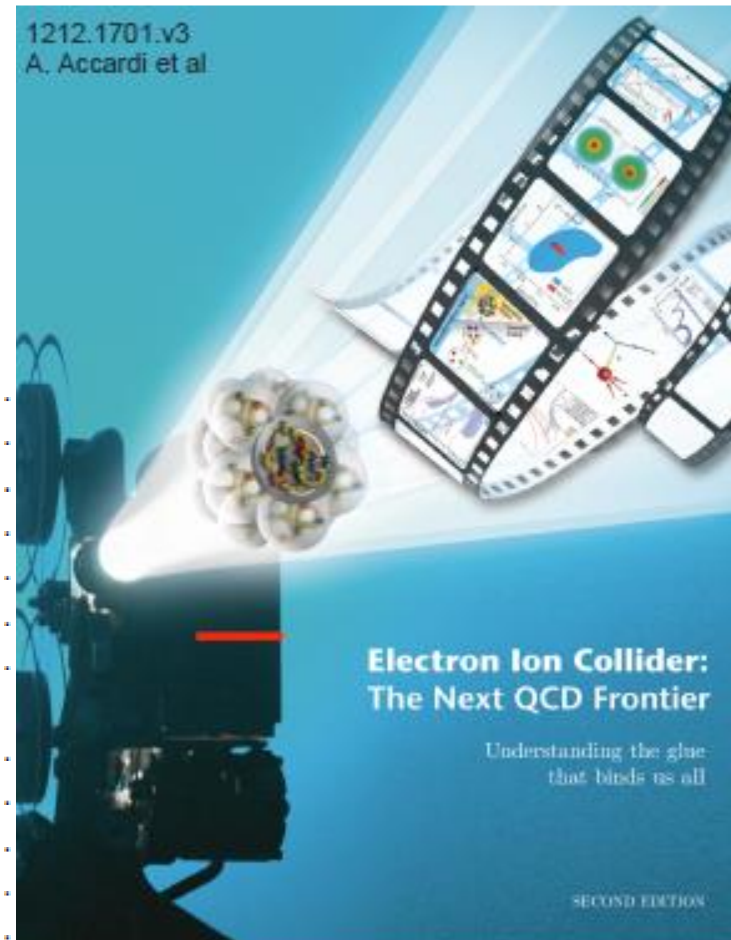
- Nucleon tomography
- Wigner distribution in QM and QCD
- Formal aspects, miscellaneous topics
- Experimental probes

Nucleon tomography



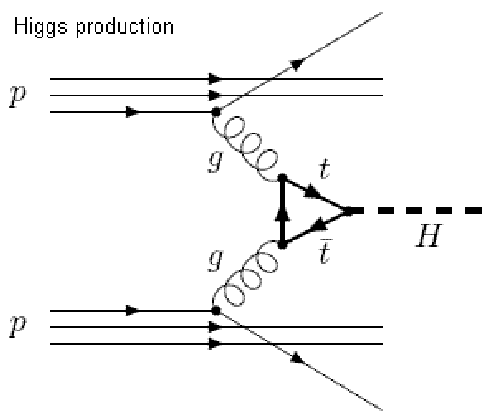


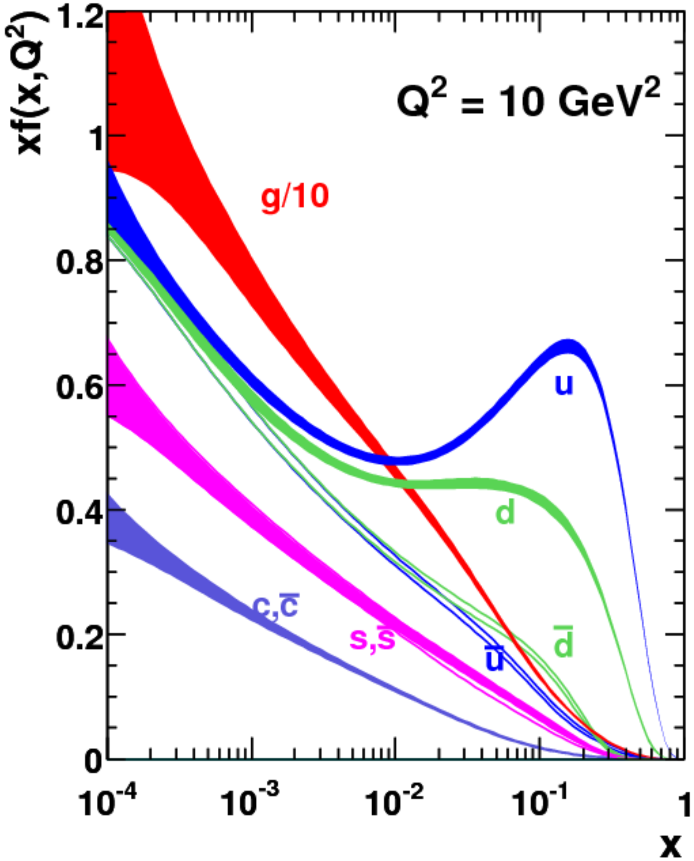
Parton distribution function

$$u(x) = \int \frac{dz^-}{4\pi} e^{ixP^+z^-} \langle P|\bar{u}(-z^-/2)\gamma^+ u(z^-/2)|P\rangle$$

Number distribution of up quarks
with momentum fraction x inside the proton

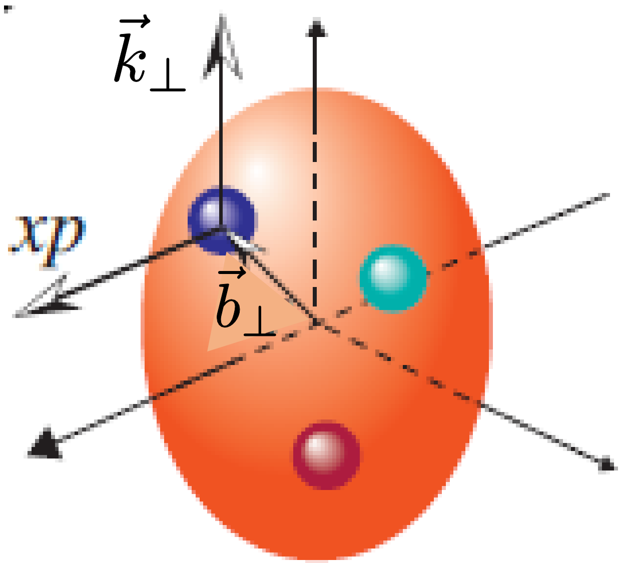
QCD factorization $\sigma = \sigma_0 \otimes g(x_1) \otimes g(x_2)$





Universality of PDF—the same function can be used for different processes.
Fundamental to the predictive power of pQCD

Multi-dimensional tomography



The nucleon is much more complicated!
Partons also have transverse momentum $\vec{k}_\perp$
and are spread in impact parameter space $\vec{b}_\perp$

$u(x)$

$u(x, \vec{k}_\perp)$ Transverse momentum dependent distribution (TMD)
3D tomography

$u(x, \vec{b}_\perp)$ Generalized parton distribution (GPD)
3D tomography

$u(x, \vec{b}_\perp, \vec{k}_\perp)$ Wigner distribution
5D tomography

5D tomography:
Wigner distribution— the "mother distribution"

Belitsky, Ji, Yuan (2003)

$$W(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$=\int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{\Delta}_\perp} \int \frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp} \langle P-\frac{\Delta}{2}|\bar{q}(-z/2)\gamma^+ q(z/2)|P+\frac{\Delta}{2}\rangle$$

$\int d\vec{b}_\perp$

$\int d\vec{k}_\perp$

TMD $f(x,\vec{k}_\perp)$

GPD $f(x,\vec{b}_\perp)$

$\int dx$

$\int d\vec{k}_\perp$

$\int d\vec{b}_\perp$

$f(x)$

PDF

$F(\vec{b}_\perp)$ Form factor

$\int dx$

Q

charge

$\int d\vec{b}_\perp$

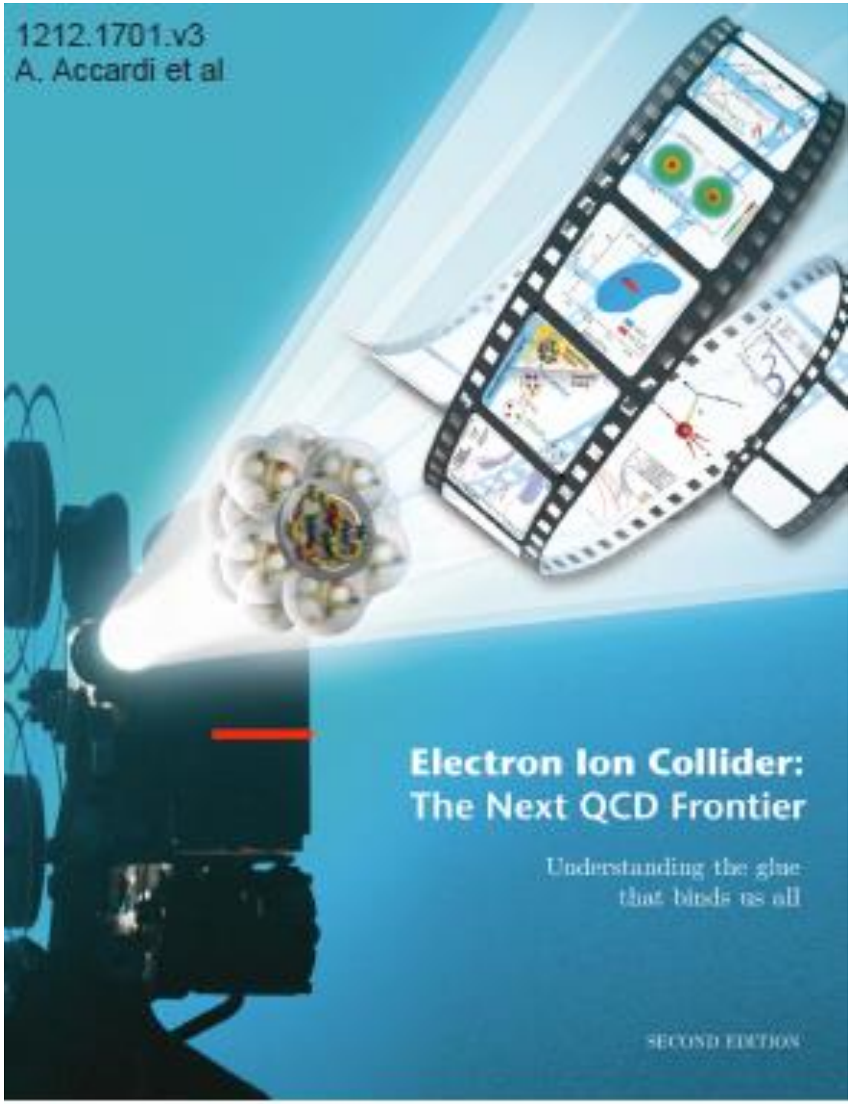
Wigner in the white paper?

Almost no account.
Only briefly mentioned in two places.

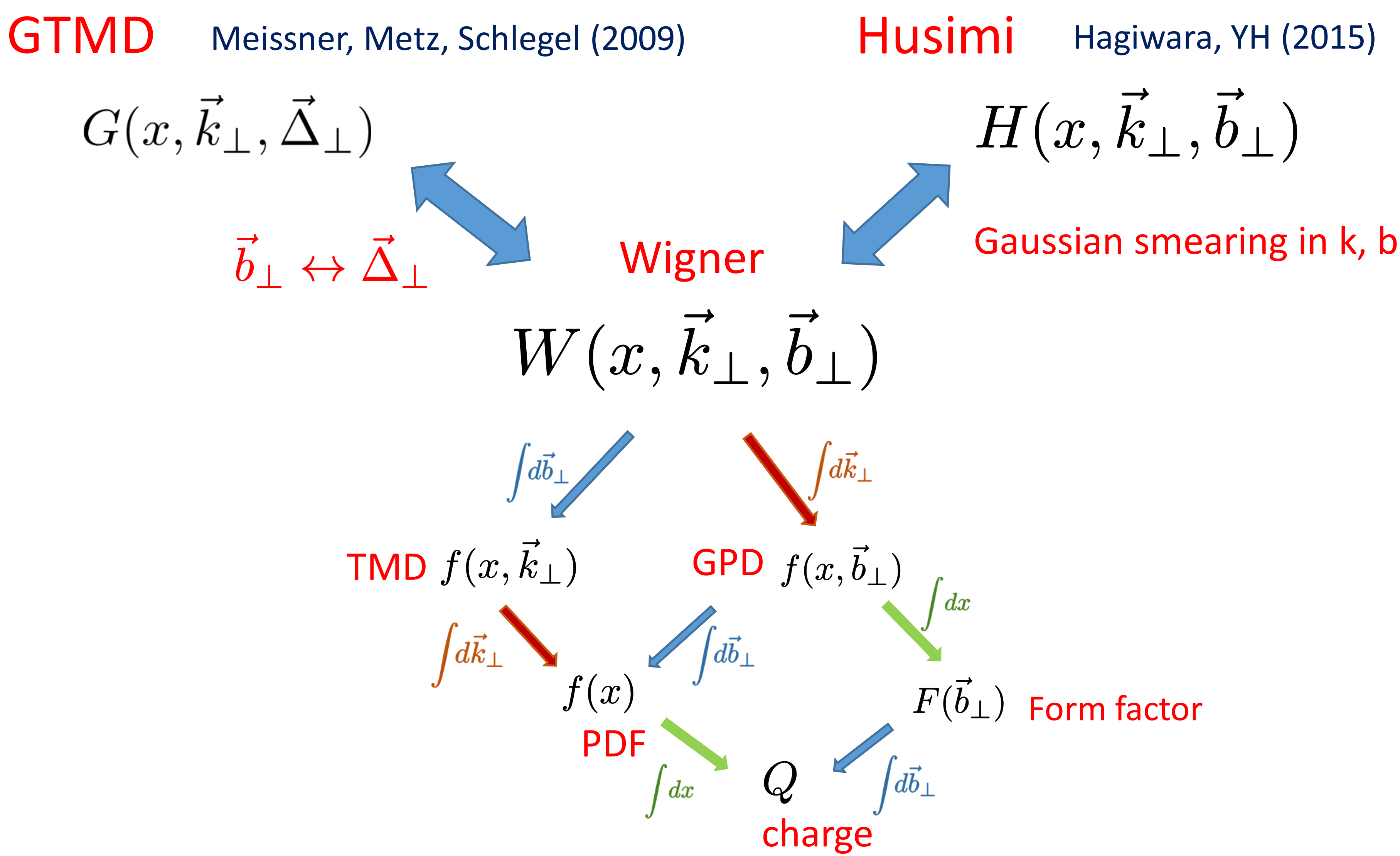
Although there is no known way to measure Wigner distributions for quarks and gluons, they provide a unifying theoretical framework for the different aspects of hadron structure.

A lot of progress since then!

Wigner $\neq$ TMD+GPD



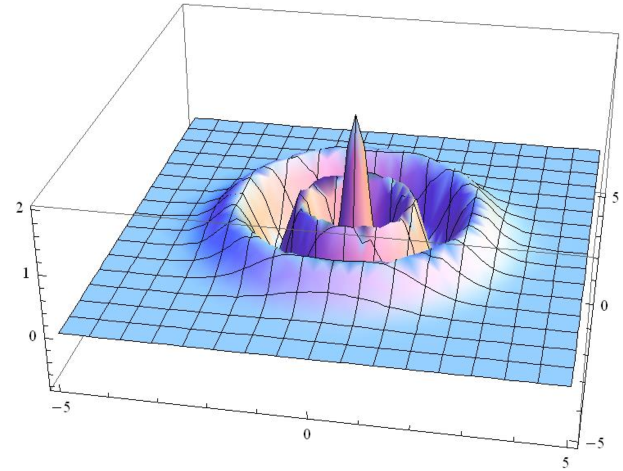
5D tomography: GTMD and Husimi



Wigner vs Husimi in QM

Wigner

$$f_W(q,p,t) = \int_{-\infty}^{\infty} dx e^{-ipx/\hbar} \langle \psi(t) | q - x/2 \rangle \langle q + x/2 | \psi(t) \rangle$$



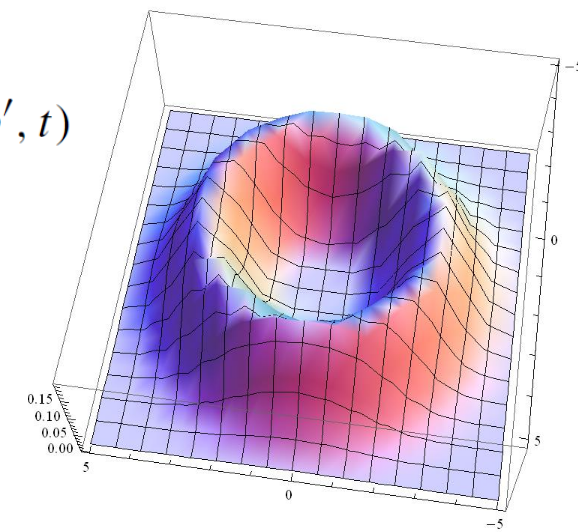
Not positive definite, no probabilistic interpretation
Reduces to q(p)-distribution upon p(q)-integration

$$\int \frac{dq}{2\pi\hbar} f_W(q,p,t) = |\langle \psi(t) | p \rangle|^2$$

4th exited state of 1D harmonic oscillator

Husimi

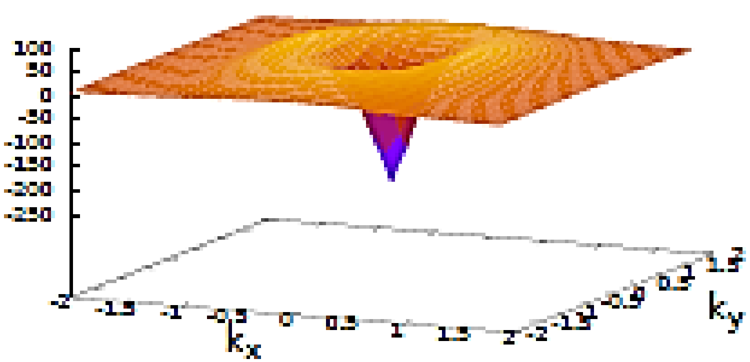
$$f_H(q,p,t) = \frac{1}{\pi\hbar} \int dq' dp' e^{-m\omega(q'-q)^2/\hbar - (p'-p)^2/m\omega\hbar} f_W(q',p',t)$$



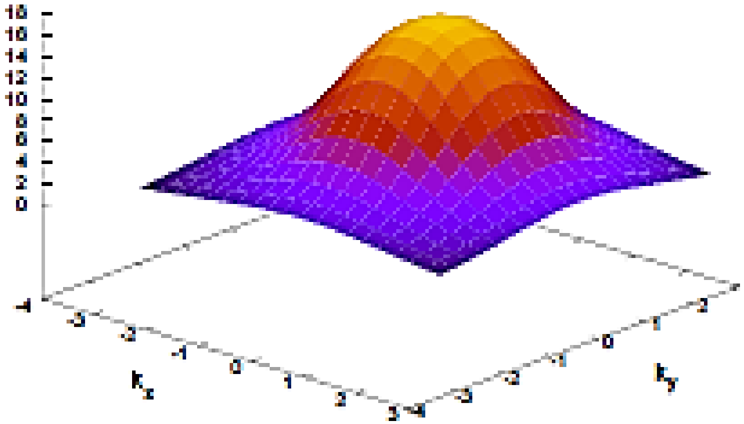
Gaussian smearing of Wigner in phase space
Positive definite→ probabilistic interpretation
Does not reduce to q(p)-distribution upon p(q)-integration

Sample 1-loop calculations

Quark Wigner, quark target

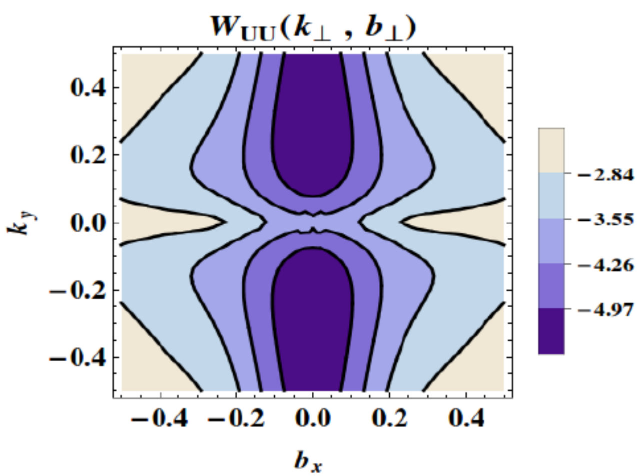


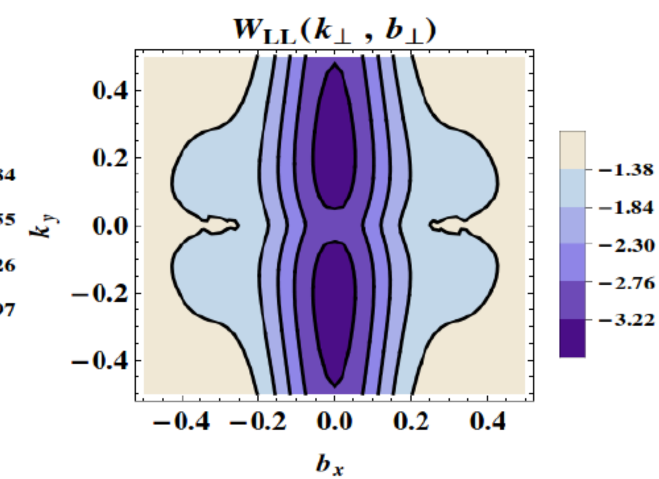
Husimi



Gluon Wigner, quark target

More, Mukherjee, Nair (2017)





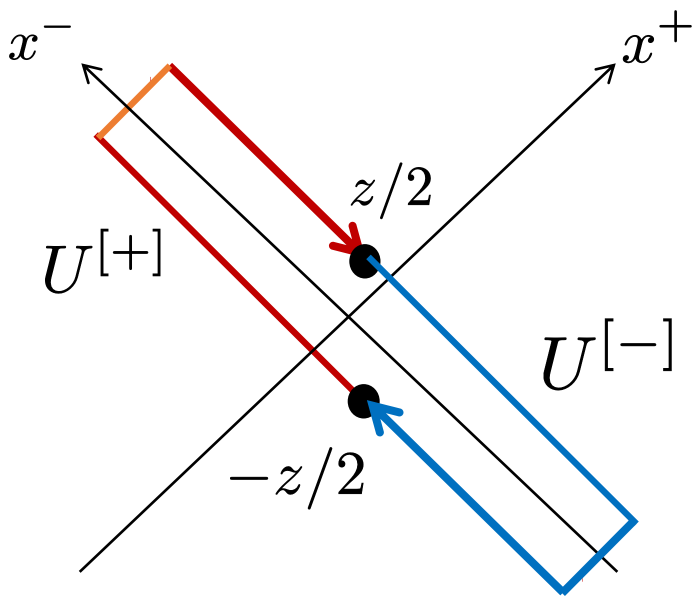
Gluon Wigner distribution—there are two of them

$$xW(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$= \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{\Delta}_\perp} \int \frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp} \langle P-\Delta/2|F^{+i}(-z/2)F^{+}_{\ i}(z/2)|P+\Delta/2\rangle$$

There are two ways to make it gauge invariant

Bomhof, Mulders (2008)

Dominguez, Marquet, Xiao, Yuan (2011)



Weizsacker-Williams (WW) distribution

$$\mathrm{Tr}[F(-z/2)U^{[+]}F(z/2)U^{[+]}]$$

Dipole distribution

$$\mathrm{Tr}[F(-z/2)U^{[+]}F(z/2)U^{[-]}]$$

Proper definition of Wigner/GTMD

Being a generalization of TMD, all the complications involved in defining and evolving TMD are still there. Soft factor, rapidity divergence,…

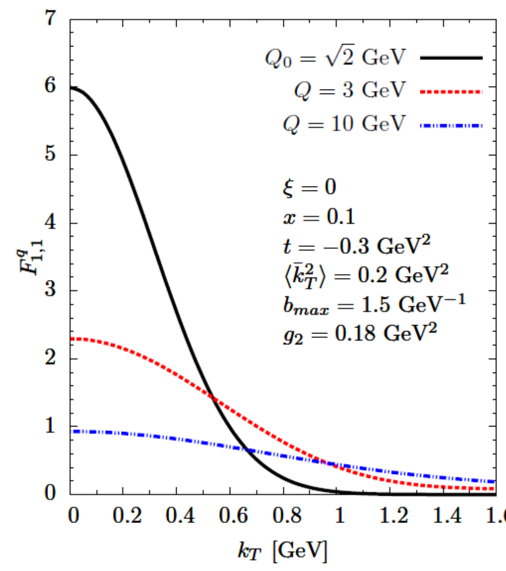
Simpler at vanishing skewness $\xi \propto \Delta^+ = 0$, identical to the TMD case

Echevarria, et al. (2016)

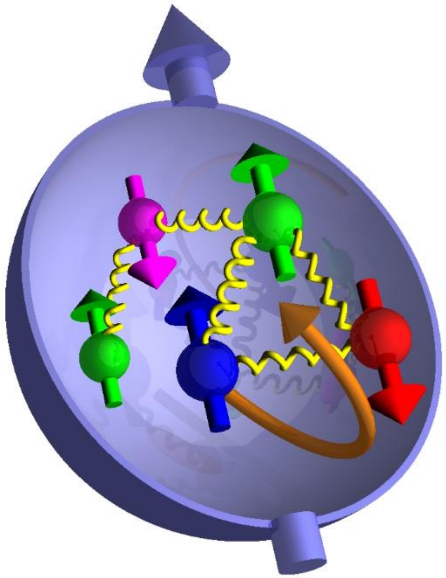
$$W^{[\Gamma],q}_{\lambda\lambda'} = \frac{1}{2}\int \frac{dz^- d^2 z_\perp}{(2\pi)^3}\, e^{+i\left(\frac{1}{2}z^- \bar{k}^+ - \boldsymbol{z}_\perp\cdot\bar{\boldsymbol{k}}_\perp\right)}\, \phi^{[\Gamma],q}_{\lambda\lambda'}(0, z^-, \boldsymbol{z}_\perp)\, S^{\frac{1}{2}}(z_T)\,,$$

Generalization to $\xi \neq 0$? Open question.

Factorization theorems yet to be established.



Wigner distribution and orbital angular momentum



Jaffe-Manohar decomposition

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g$$

Quarks' helicity

Gluons' helicity

Canonical Orbital angular momentum (OAM)

$$L^{q,g} = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^{q,g}(x, \vec{b}_\perp, \vec{k}_\perp)$$

Lorce, Pasquini, (2011);
YH (2011)
Ji, Xiong, Yuan (2012)

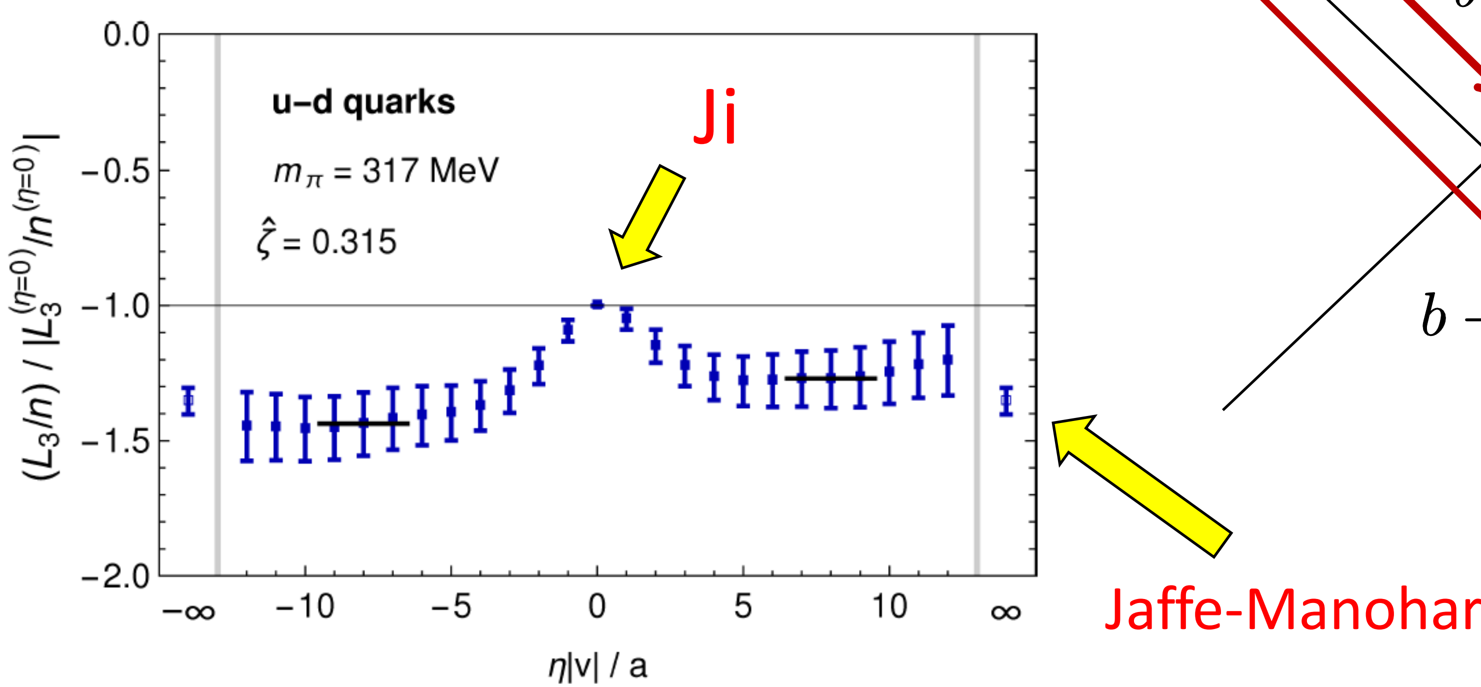
`PDF' for OAM $$L^{q,g}(x) = \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^{q,g}(x, \vec{b}_\perp, \vec{k}_\perp)$$

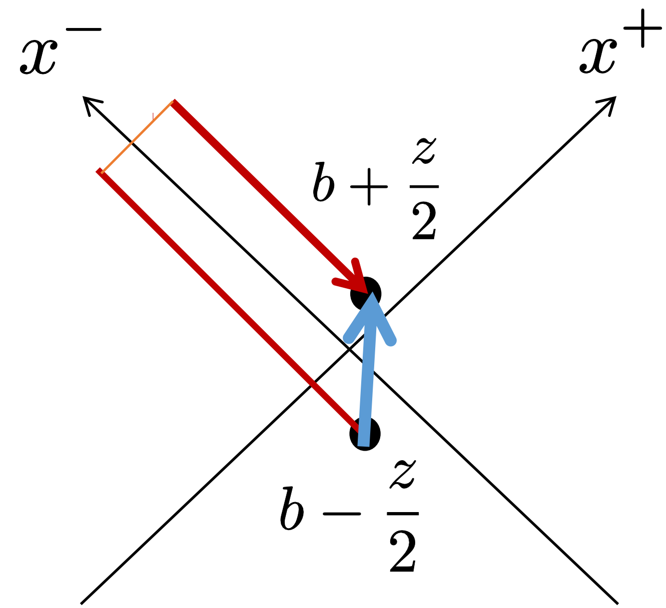
OAM and Wilson line

Jaffe-Manohar OAM from staple Wilson line YH (2011)

Ji's OAM from straight Wilson line Ji, Xiong, Yuan (2012)

First lattice calculation Engelhardt (2017)





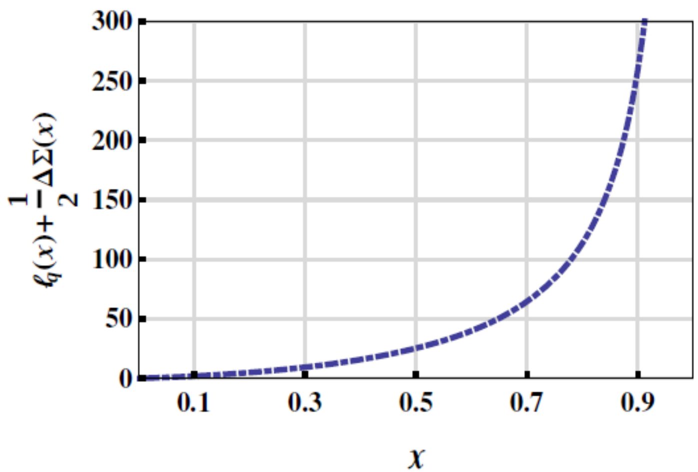
OAM at small-x

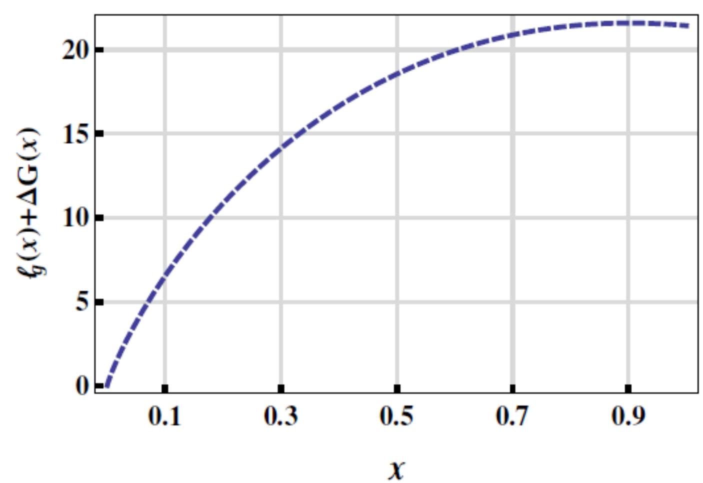
Relation between OAM and GTMD $$L^{q,g}(x) = \int d^2k \frac{k^2}{m^2} F_{14}^{q,g}(x,k)$$

Significant cancellation between helicity and OAM at small-x

YH, Nakagawa, Xiao, Yuan, Zhao (2016)
YH, Yang (2018)

Dressed quark model calculation More, Mukherjee, Nair (2017)





All-order result Boussarie, YH, Yuan (2019) $$L_g(x) \approx -\frac{2}{1+\alpha}\Delta G(x) \sim \frac{1}{x^\alpha}$$

`Entropy' of partons

Hagiwara, YH, Xiao, Yuan (2018)

Phase space distribution naturally defines an entropy.
Use the QCD Husimi distribution

$$S(x) \equiv -\int d^2b_\perp d^2k_\perp xH(x,b_\perp,k_\perp)\ln xH(x,b_\perp,k_\perp)$$

$$S(x) \sim \frac{N_c}{\alpha_s}Q_s^2(x)S_\perp \propto A\left(\frac{1}{x}\right)^{\#\alpha_s}$$

$x \to 0$

cf. Kutak (2011)
Kovner-Lublinsky (2015)

Measure of `complexity' of the multiparton system.
Saturation of entropy due to the Pomeron loop effect?
Connection to the `jet entropy' in the final state? Neill, Waalewijn (2018)

Gluon Wigner/GTMD at small-x

$$xW(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$=\int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{i\vec{b}_\perp\cdot\vec{\Delta}_\perp}\int\frac{dz^-d^2z_\perp}{16\pi^3}e^{ixP^+z^- -i\vec{k}_\perp\cdot\vec{z}_\perp}\langle P-\Delta/2|F^{+i}(-z/2)F^{+}_{i}(z/2)|P+\Delta/2\rangle$$

At small-x, approximate $e^{ixP^+z^-}\approx 1$ YH, Xiao, Yuan (2016)

$$\boxed{xW(x,\vec{k}_\perp,\vec{b}_\perp)\approx\frac{2N_c}{\alpha_s}\int\frac{d^2\vec{r}_\perp}{(2\pi)^2}e^{i\vec{k}_\perp\cdot\vec{r}_\perp}\left(\frac{1}{4}\vec{\nabla}_b^2-\vec{\nabla}_r^2\right)S_x(\vec{b}_\perp,\vec{r}_\perp)}$$

``Dipole S-matrix'' $S_x(\vec{b}_\perp,\vec{r}_\perp)=\left\langle\frac{1}{N_c}\mathrm{Tr}\,U\left(\vec{b}_\perp-\frac{\vec{r}_\perp}{2}\right)U^\dagger\left(\vec{b}_\perp+\frac{\vec{r}_\perp}{2}\right)\right\rangle_x$

$\cos 2\phi$ correlation expected

$$W(x,\vec{k}_\perp,\vec{b}_\perp)=W_0(x,k_\perp,b_\perp)+2\cos 2(\phi_k-\phi_b)W_1(x,k_\perp,b_\perp)+\cdots$$

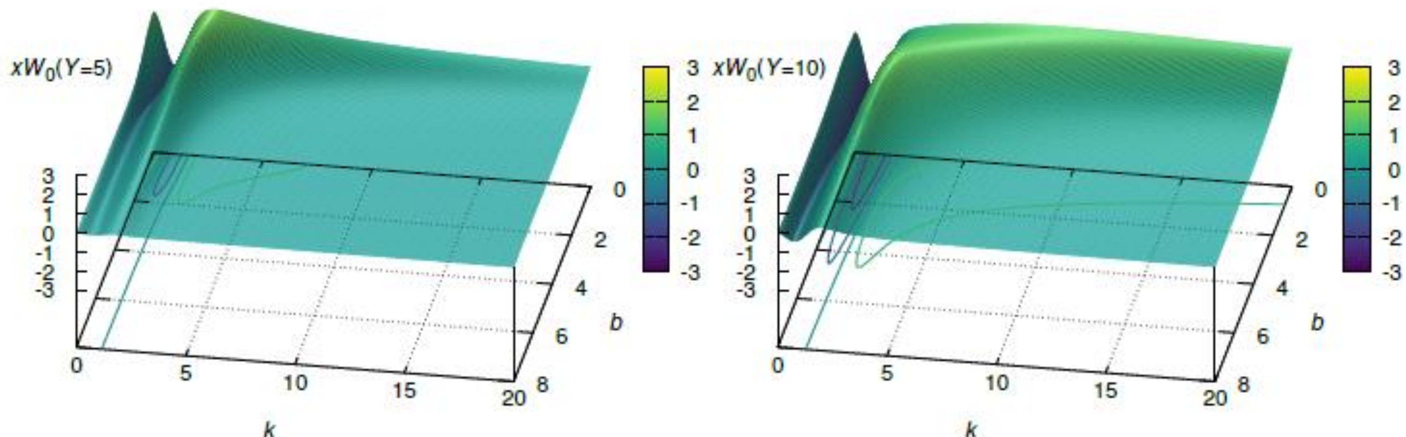
`Elliptic Wigner' distribution

Dipole Wigner from Balitsky-Kovchegov equation

Hagiwara, YH, Ueda (2016)

Peak at the saturation momentum

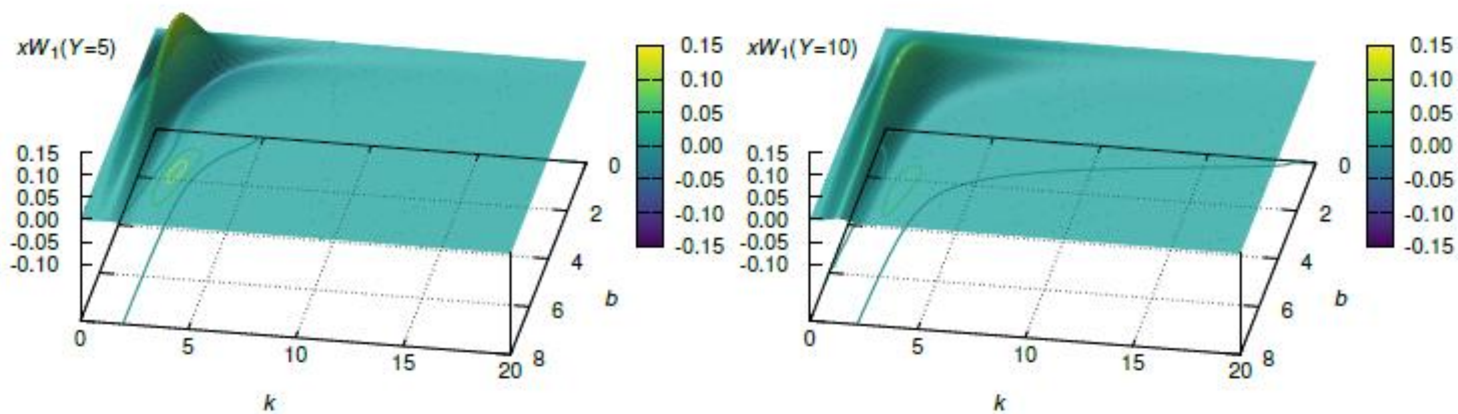
$W_0(k_\perp, b_\perp)$



$Y = \ln \frac{1}{x} = 5$

$Y = 10$

$W_1(k_\perp, b_\perp)$



Elliptic part small in magnitude (a few percent effect). No geometric scaling.

Observables for Wigner/GTMD

Can we measure Wigner/GTMD $G(x, k_\perp, \Delta_\perp)$ in experiments?

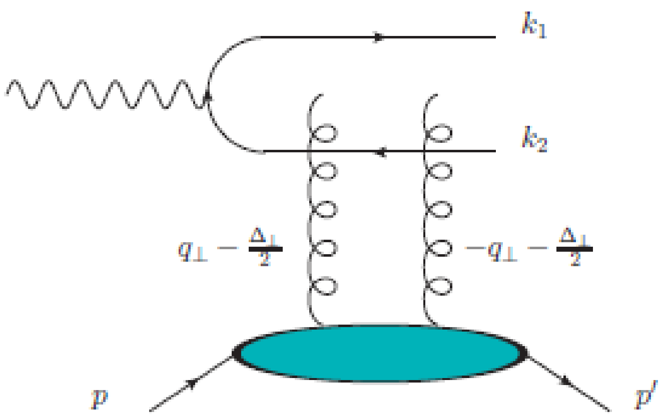
Distributions with more variables → more exclusive processes

Must be diffractive (proton remains intact), $\Delta_\perp$ is the proton recoil momentum.

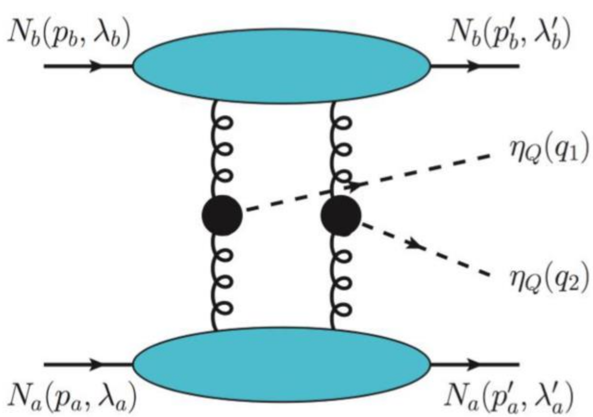
$k_\perp$ is the relative momentum of quark and antiquark $\gamma^* \to q\bar{q}$.

Vector meson production $\gamma^* \to q\bar{q} \to V$ → indirect probe (integrated over $k_\perp$)

Dijet (di-hadron) production $\gamma^* \to q\bar{q} \to jj$ → direct probe (differential in $k_\perp$)



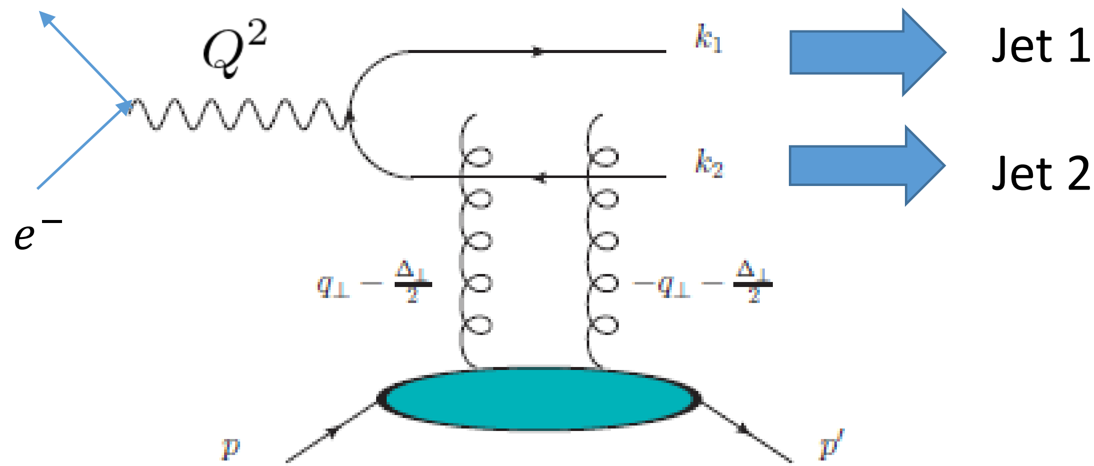
YH, Xiao, Yuan (2016)



Bhattacharya, Metz, Ojha, Tsai, Zhou (2018)

Probing dipole Wigner (GTMD) in diffractive dijet production

YH, Xiao, Yuan (2016)
Altinoluk, Armesto, Beuf, Rezaeian (2015)



$$\vec{\Delta}_\perp = -(\vec{k}_{1\perp} + \vec{k}_{2\perp})$$
$$\vec{P}_\perp = \frac{1}{2}(\vec{k}_{2\perp} - \vec{k}_{1\perp})$$

GTMD

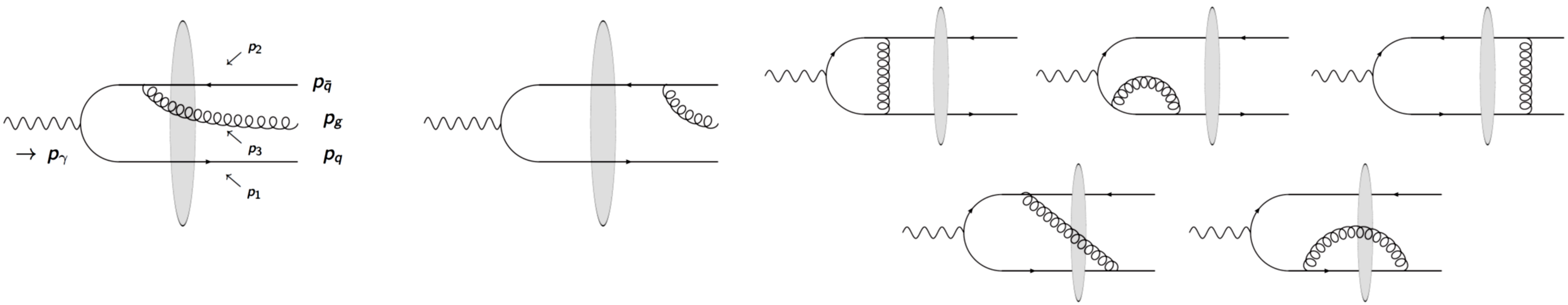
$$\frac{d\sigma}{dy_1 dy_2 d^2\vec{\Delta}_\perp d^2\vec{P}_\perp} \propto z(1-z)[z^2+(1-z)^2]\int d^2q_\perp d^2q'_\perp S(q_\perp,\Delta_\perp)S(q'_\perp,\Delta_\perp)$$
$$\times\left[\frac{\vec{P}_\perp}{P_\perp^2+\epsilon^2} - \frac{\vec{P}_\perp-\vec{q}_\perp}{(P_\perp-q_\perp)^2+\epsilon^2}\right]\cdot\left[\frac{\vec{P}_\perp}{P_\perp^2+\epsilon^2} - \frac{\vec{P}_\perp-\vec{q}\,'_\perp}{(P_\perp-q'_\perp)^2+\epsilon^2}\right]$$

$$\sim d\sigma_0 + 2\cos 2(\phi_P - \phi_\Delta)d\tilde{\sigma}$$

$$\epsilon^2 = z(1-z)Q^2$$

Factorization at NLO

Boussarie, Grabovsky, Szymanowski, Wallon (2016)



$$\Phi_L^{(0)} = \frac{2x\bar{x}p_V^+ Q}{(\bar{x}\vec{p}_1 - x\vec{p}_2)^2 + x\bar{x}Q^2},$$

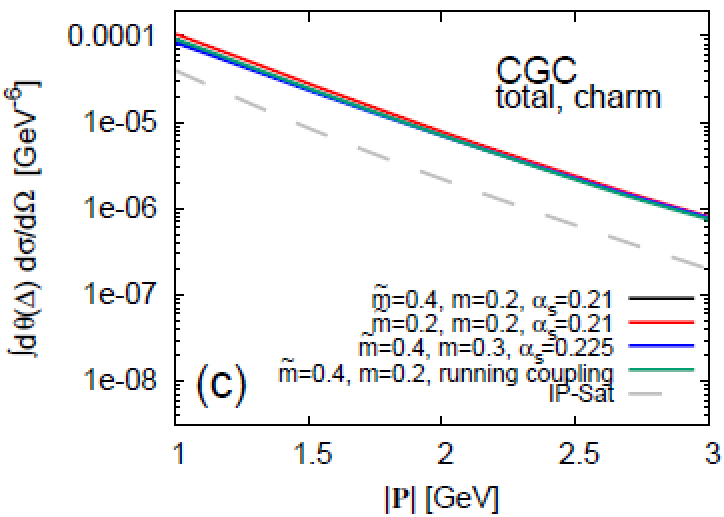
$$\Phi_T^{(0)} = -\frac{(x-\bar{x})p_V^+(\bar{x}\vec{p}_{1\perp} - x\vec{p}_{2\perp})\cdot\vec{\varepsilon}_{\gamma_T}}{(\bar{x}\vec{p}_1 - x\vec{p}_2)^2 + x\bar{x}Q^2}$$

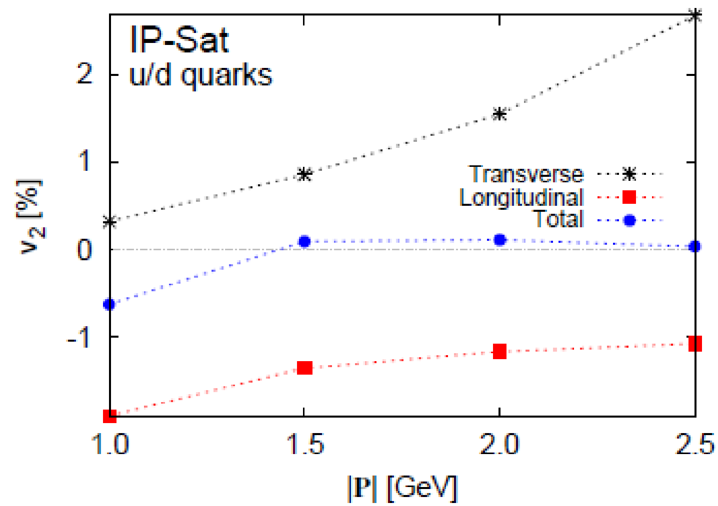
No end point singularity, even for a transverse photon and even in the photoproduction limit and even at NLO.
With null transverse momenta in the t channel, one could encounter $x \in \{0, 1\}$ end point singularities as $\frac{1}{x\bar{x}Q^2}$ thus breaking collinear factorization.

Diffractive dijet: Numerical results

Mantysaari, Mueller, Schenke (2019)

First realistic calculation from b-dependent JIMWLK





$$\frac{d\sigma}{dy_1 dy_2 d^2\vec{\Delta}_\perp d^2\vec{P}_\perp} \sim d\sigma_0 + 2\cos 2(\phi_P - \phi_\Delta) d\tilde{\sigma} + \cos(\phi_P - \phi_\Delta) d\sigma'$$

surprise!

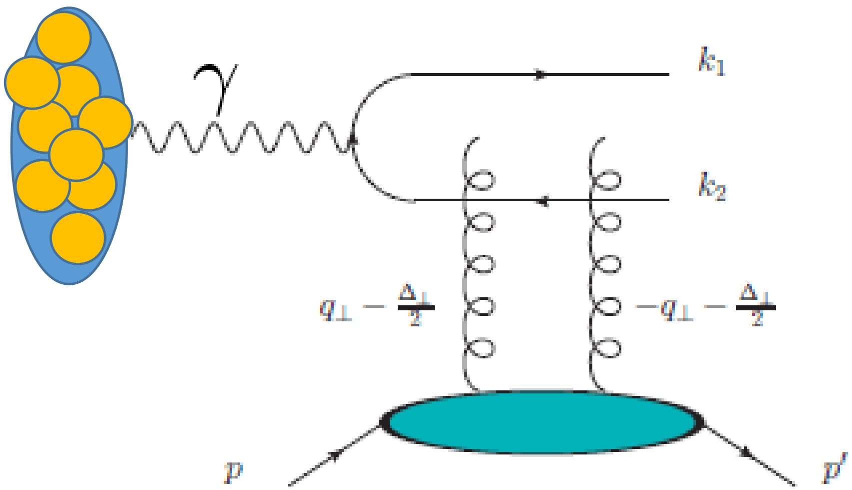
Soft gluon resummation YH, Mueller, Ueda, Yuan (2019)

Towards NLO numerics Boussarie, Grabovsky, Szymanowski, Wallon (2019) + more

Beyond the correlation limit Mantysaari, Mueller, Salazar, Schenke (2019)

Diffractive dijet in ultra-peripheral pA collisions

Hagiwara, YH, Pasechnik, Tasevsky, Teryaev (2017)



Q^2 preferably small

Use the Weiszacker-Williams photons in UPC!

$$\frac{d\sigma^{pA}}{dy_1 dy_2 d^2\vec{k}_{1\perp} d^2\vec{k}_{2\perp}} = \omega \frac{dN}{d\omega} \frac{N_c \alpha_{em} (2\pi)^4}{P_\perp^2} \sum_f e_f^2 2z(1-z)(z^2+(1-z)^2)\left(A^2 + 2\cos 2(\phi_P - \phi_\Delta) AB\right)$$

photon flux $\propto Z^2$

Inversion can be done analytically.

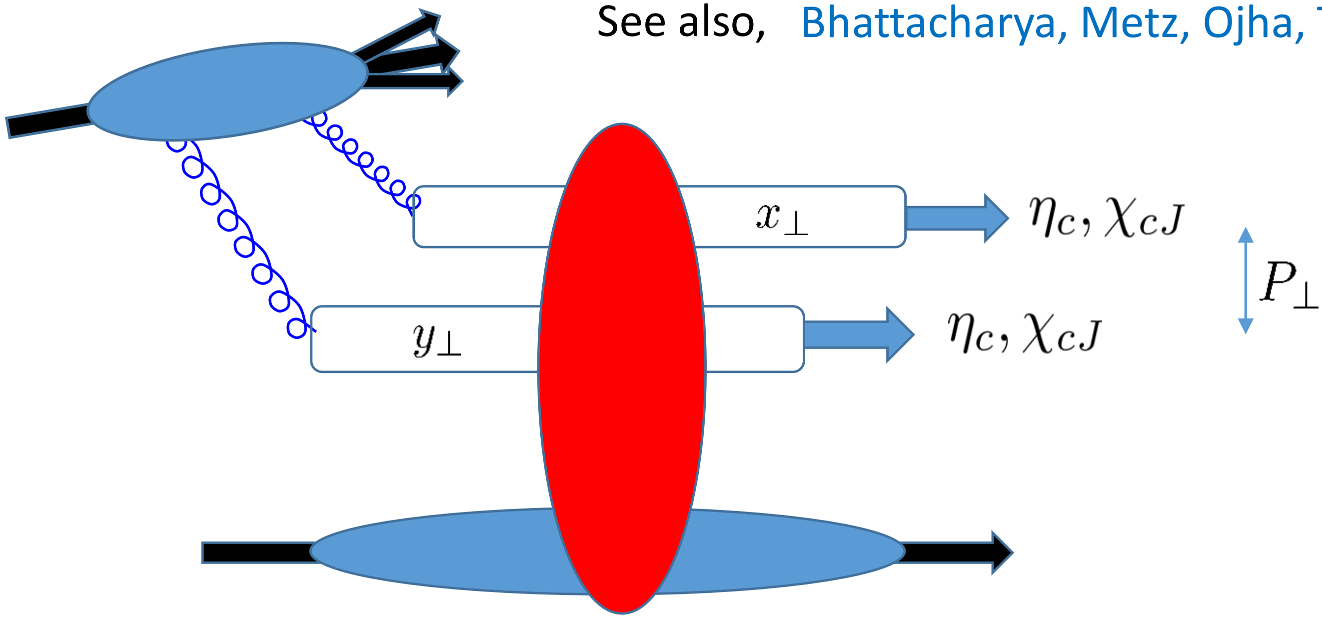
$$S_0(P_\perp, \Delta_\perp) = \frac{1}{P_\perp} \frac{\partial}{\partial P_\perp} A(P_\perp, \Delta_\perp).$$

$$S_1(P_\perp, \Delta_\perp) = \frac{\partial B(P_\perp, \Delta_\perp)}{\partial P_\perp^2} - \frac{2}{P_\perp^2} \int^{P_\perp^2} \frac{dP_\perp'^2}{P_\perp'^2} B(P_\perp', \Delta_\perp)$$

Probing the Weiszacker-Williams gluon GTMD

Boussarie, YH, Xiao, Yuan (2018)

See also, Bhattacharya, Metz, Ojha, Tsai, Zhou (2018)



$$\frac{d\sigma^{\chi_c}}{dY_1 dY_2 d^2\Delta_\perp d^2P_\perp} = \frac{x_1 x_2 F(x_1, x_2)}{64 m^{18} N_c^4 (N_c^2 - 1)^2} \alpha_s^4 \langle \mathcal{O}_{\chi_1} \rangle^2 P_\perp^4 \left(G(P_\perp, \Delta_\perp) + \frac{P_\perp^2}{2M^2} G_2 \right)^2$$

gluon double PDF

Long distance matrix element (LDME)

WW gluon GTMD

Linearly polarized WW gluon GTMD

Caveat: Only color-singlet production included.

Spin-dependent GTMDs at small-x

$$\int d^4v\delta(v^+)e^{ix\bar{P}^+v^- - i(\boldsymbol{k}\cdot\boldsymbol{v})}\langle P', S'|F^{i-}(-\frac{v}{2})\mathcal{U}^{[+]}_{\frac{v}{2},-\frac{v}{2}}F^{i-}(\frac{v}{2})\mathcal{U}^{[-]}_{-\frac{v}{2},\frac{v}{2}}|P, S\rangle$$

$$= (2\pi)^3\frac{\bar{P}^+}{2M}\bar{u}_{P',S'}\left[F^g_{1,1} + i\frac{\sigma^{i+}}{\bar{P}^+}(\boldsymbol{k}^iF^g_{1,2} + \boldsymbol{\Delta}^iF^g_{1,3}) + i\frac{\sigma^{ij}\boldsymbol{k}^i\boldsymbol{\Delta}^j}{M^2}F^g_{1,4}\right]u_{P,S}.$$

Nucleon spinor

Small-x
(eikonal approximation)

$$\int d^2\boldsymbol{v}e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})}\langle P'S'|\frac{1}{N_c}\mathrm{Tr}\left[U_{\frac{\boldsymbol{v}}{2}}U^\dagger_{-\frac{\boldsymbol{v}}{2}}\right]|PS\rangle$$

$$= (2\pi)^4\delta(P^+ - P'^+)\frac{\bar{P}^+}{2M}\frac{g_s^2}{N_c\left(\boldsymbol{k}^2 - \frac{\boldsymbol{\Delta}^2}{4}\right)}\bar{u}_{P'S'}\left[F^g_{1,1} + i\frac{\sigma^{i+}}{\bar{P}^+}(\boldsymbol{k}^iF^g_{1,2} + \boldsymbol{\Delta}^iF^g_{1,3})\right]u_{PS}$$

$F^g_{1,n}$ are all complex functions.

Real part → Pomeron
Imaginary part → Odderon

Odderon as GTMDs

Boussarie, YH, Szymanowski, Wallon (2019)

Focus on the imaginary part. Most general coupling of the Odderon to nucleon

$$\frac{N_c}{g^2}\left(\boldsymbol{k}^2-\frac{\boldsymbol{\Delta}^2}{4}\right)\int d^2\boldsymbol{v}e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})}\langle P'S'|\mathcal{O}(\boldsymbol{v})|PS\rangle$$

Spin-dependent odderon

Zhou (2013)

$$\sim \frac{\bar{P}^+}{2M}\bar{u}_{P'S'}\left[i\frac{(\boldsymbol{k}\cdot\boldsymbol{\Delta})}{M^2}g_{1,1}+\frac{\sigma^{+i}}{\bar{P}^+}(\boldsymbol{k}^i g_{1,2}+\boldsymbol{\Delta}^i\frac{(\boldsymbol{k}\cdot\boldsymbol{\Delta})}{M^2}g_{1,3})\right]u_{PS}$$

Spin-independent odderon

For transverse polarization, $\bar{u}(PS)\sigma^{+i}u(PS)\sim\epsilon^{ij}S_\perp^j$

→ $g_{1,2}$ reduces to the gluon Sivers in the forward limit

At small-x, **gluon Sivers = Odderon** Zhou (2013)

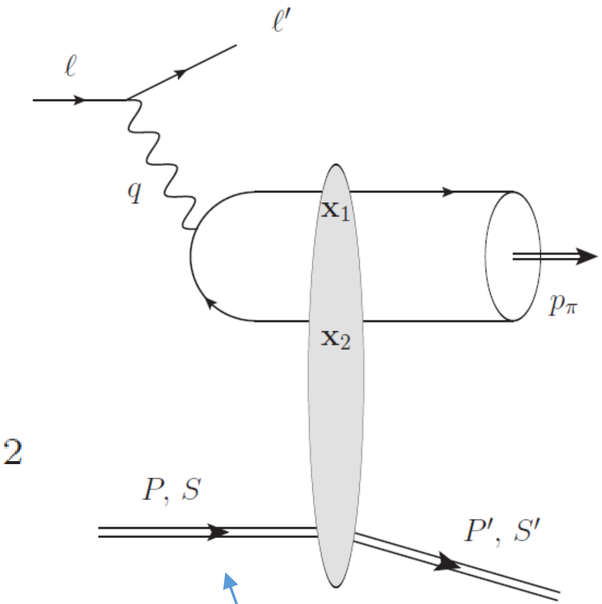
Note that the same spinor product is nonvanishing also for longitudinal polarization, but with helicity flip

$$\bar{u}(PS)\sigma^{+i}u(P,-S)\sim\epsilon^{ij}S_L^j \qquad \vec{S}_L=(1,ih)$$

Probing Sivers with unpolarized beams

Boussarie, YH, Szymanowski, Wallon (2019)

Diffractive π^0 production in unpolarized DIS in the forward limit $t \to 0$.



$$\frac{d\sigma}{dx_B dQ^2 d\,|t|} = \frac{\pi^5 \alpha_{\rm em}^2 \alpha_s^2 f_\pi^2}{2^3 x_B N_c^2 M^2 Q^6} (1 - y + \frac{y^2}{2})$$
$$\times \left[\int_0^1 dz \frac{\phi_\pi(z)}{z\bar{z}} \int d\boldsymbol{k}^2 \frac{\boldsymbol{k}^2}{\boldsymbol{k}^2 + z\bar{z}Q^2} x f_{1T}^{\perp g}(x, \boldsymbol{k}^2) \right]^2$$

Sum over initial and final proton helicities.
Helicity flip automatically included

Leading contribution coming from gluon Sivers!

Conclusions

- Let’s get 5 dimensional. Even richer physics than TMD and GPD combined. Not discussed in the white paper.
- Wigner/GTMD measurable in ep, pp, pA, including the elliptic part and spin-dependent part (connection to OAM). Many interesting applications especially at small-x.
- Need more foundational works. Proper definition, factorization, evolution…