(Quantum) Field Theory
and
the Electroweak Standard Model
Lecture II

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Outline

Lecture I
- What is the Standard Model?
- Introducing Quantum Fields
- Interactions and Perturbation Theory
- Renormalizable or Non-Renormalizable?

Lecture II
- An Ode to Symmetry
- Global Symmetries (and Conserved Quantities)
- Local Symmetries (and Gauge interactions)
- From Fermi Model to EW theory

Lecture III
- Finalizing the EW SM (a bit of Higgsing)
- “Features” of the SM
- Experimental tests of the EW SM
- Issues and Prospects of the EW SM
An Ode to Symmetry...

With Action you can also study Symmetries...

The latter are intimately connected with transformations, which leave something invariant...

Symmetries are not only beautiful but also very useful:

An architect can design only half of the building \((\text{parity } x \rightarrow -x)\)

And winter decoration will take much less time (rotation by a finite angle)
Field Theory: Symmetries

- Transformations can be discrete, e.g.,

\[
\text{Parity : } \phi'(x, t) = P\phi(x, t) = \phi(-x, t), \\
\text{Time-reversal : } \phi'(x, t) = T\phi(x, t) = \phi(x, -t), \\
\text{Charge-conjugation : } \phi'(x, t) = C\phi(x, t) = \phi^\dagger(x, t),
\]

or depend on continuous parameters, e.g.,

\[
\phi'(x + a) = \phi(x)
\]

We can distinguish space-time and internal symmetries.

For \(x\)-dependent parameters we have local (gauge) transformations.

\[
\phi'(x) = e^{i\alpha}\phi(x)
\]
Field Theory: Symmetries

- Transformations can be **discrete**, e.g.,

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\begin{align*}
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\end{align*}
\]

or depend on **continuous** parameters, e.g.,

\[
\begin{align*}
\phi(x) & = \phi(x) \\
\phi'(x + a) & = \phi(x)
\end{align*}
\]

- We can distinguish **space-time** and **internal** symmetries.
Field Theory: Symmetries

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  \[ \text{Time-reversal: } \phi'(x, t) = T\phi(x, t) = \phi(x, -t), \]
  \[ \text{Charge-conjugation: } \phi'(x, t) = C\phi(x, t) = \phi^\dagger(x, t), \]

or depend on continuous parameters, e.g.,

\[ \Im \phi(x) \]

\[ \phi'(x + a) = \phi(x) \]

\[ \phi'(x) = e^{i\alpha} \phi(x) \]

- We can distinguish space-time and internal symmetries.
- For \( x \)-dependent parameters we have local (gauge) transformations.
Global Continuous Symmetries: Noether Theorem

Given $S[\phi]$ one can find its symmetries, i.e., particular infinitesimal variations $\delta \phi(x)$ that for any $\phi$ leave $S[\phi]$ invariant up to a surface term

$S[\phi'(x)] - S[\phi(x)] = \int d^4x \partial_\mu K_\mu, \quad \phi'(x) \equiv \phi(x) + \delta \phi(x)$.

We compare this with

$S[\phi'(x)] - S[\phi(x)] = \int d^4x \left[ \left( \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} - \frac{\partial L}{\partial \phi} \right) \delta \phi + \partial_\mu \left( \frac{\partial L}{\partial \partial_\mu \phi} \delta \phi \right) \right]$. 

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and require $\phi(x)$ to satisfy EOMs. This results in a local conservation law:

$$\partial_\mu J_\mu = 0, \quad J_\mu \equiv K_\mu - \frac{\partial L}{\partial \partial_\mu \phi} \delta \phi.$$

Integration over space leads to the conserved charge

$$\frac{d}{dt} Q = 0, \quad Q = \int d\mathbf{x} J_0.$$

NB: If $\delta \phi = \rho_i \delta_i \phi$ depends on parameters $\rho_i$, we have a conservation law for every $\rho_i$. For Global symmetries $\rho_i$ do not depend on coordinates.
The Noether Theorem: Space-time symmetries

Consider space-time translations

\[ \phi'(x + a) = \phi(x) \]

expand in \( a \Rightarrow \delta \phi(x) = -a_\nu \partial_\nu \phi(x), \]

\[ \delta \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \partial_\nu (-a_\nu \mathcal{L}) \]

Local conservation of Energy-Momentum Tensor \( T_{\mu\nu} \):

\[ J_\mu = -a_\mu \mathcal{L} + a_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi = a_\nu T_{\mu\nu}, \quad \partial_\mu T_{\mu\nu} = 0 \]

leads to time-independent “charges”

\[ P_\nu = \int d\mathbf{x} T_{0\nu} \]

Ex: For \( \mathcal{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2 \) find \( P_\mu = (\mathcal{H}, \mathbf{P}) \). Substitute \( \phi(x) \) by its expansion in terms of operators \( a_\mu^\pm \) and \( b_\mu^\pm \) and get the expressions (modulo operator ordering ambiguities) for the Hamiltonian \( \hat{\mathcal{H}} \) and 3-momentum \( \hat{\mathbf{P}} \) operators.
The Noether Theorem: Internal symmetries

There is an additional symmetry of

\[ \mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \]

\[ \phi'(x) = e^{i\alpha} \phi(x) \]
\[ \delta \phi(x) = i\alpha \phi(x) \]
\[ J_\mu = i(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) \]

It is a \textit{U(1)} symmetry:

- It acts in internal space ("rotates" complex number \( \phi(x) \) at every \( x \))
- It is a global symmetry (rotation angle \( \alpha \) does not depend on \( x \)).

\textbf{Ex:} Express the corresponding charge operator \( \hat{Q} \) in terms of \( a^\pm \) and \( b^\pm \).
Symmetries in Quantum Field Theory

After quantisation the operators corresponding to the conserved quantities can be used to define a convenient basis of states (quantum numbers), e.g.,

\[
|p\rangle \equiv |p, +1\rangle, \quad |\bar{p}\rangle \equiv |p, -1\rangle \Rightarrow \hat{Q}|p, q\rangle = q|p, q\rangle, \quad \hat{P}|p, q\rangle = p|p, q\rangle
\]

act as generators of symmetries, which are represented by unitary† operators, e.g., for the space-time translations:

\[
U(a) = \exp \left(i \hat{P}_\mu a_\mu \right), \quad \hat{\phi}(x + a) = U(a)\hat{\phi}(x)U^\dagger(a)
\]

NB: A symmetry \(S\) guarantees that transition probability between states do not change upon transformation

\[
|A_i\rangle \xrightarrow{S} |A'_i\rangle, \quad \mathcal{P}(A_i \rightarrow A_j) = \mathcal{P}(A'_i \rightarrow A'_j), \quad |\langle A_i|A_j\rangle|^2 = |\langle A'_i|A'_j\rangle|^2
\]

and represented by (anti) unitary operators:

\[
|A'_i\rangle = U|A_j\rangle, \quad \langle A'_i|A'_j\rangle = \langle A_i|U^\dagger U|A_j\rangle
\]

† or anti-unitary (time-reversal)
Local (Gauge) Symmetries

Let's consider the free Dirac Lagrangian:

$$\mathcal{L}_0 = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi$$

and make global $U(1)$-symmetry

$$\psi \rightarrow \psi' = e^{ie\omega} \psi$$

global, i.e., $\omega \rightarrow \omega(x)$

$$\delta \mathcal{L}_0 = \partial_\mu \omega \cdot J_\mu, \quad J_\mu = -e\bar{\psi} \gamma_\mu \psi,$$

a short-cut to Noether current

To compensate $\delta \mathcal{L}_0$ we add an interaction of the current $J_\mu$ with field $A_\mu$:

$$\mathcal{L}_0 \rightarrow \mathcal{L} = \mathcal{L}_0 + A_\mu J_\mu = \bar{\psi} \left[ i(\gamma^\mu \partial_\mu + ieA_\mu) - m \right] \psi, \quad A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \omega$$

NB: Symmetries not only restrict possible interactions, but can also force interactions to be introduced.
QED: Local U(1) Symmetry

We constructed Quantum ElectroDynamics:

\[
\mathcal{L}_{QED} = \bar{\psi} \left( i\hat{D} - m \right) \psi - \frac{1}{4} F_{\mu\nu}^2
\]

\[ D_\mu = \partial_\mu + ieA_\mu, \quad \text{covariant derivative,} \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \text{field strength tensor.} \]

\[
\psi \rightarrow \psi' = e^{ie\omega(x)} \psi \\
A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \omega \\
D_\mu \psi \rightarrow D'_\mu \psi' = e^{ie\omega(x)} D_\mu \psi,
\]

Covariant derivative makes the combination \( \psi_\alpha^\dagger D_\mu \psi_\beta \) gauge-invariant. (Ex: make it also Lorentz-invariant, \( \alpha, \beta \) are Dirac indices:)}
QED: Quantization Issues

The second Noether theorem says that theories possessing gauge symmetries are redundant (some degrees of freedom are not physical).

Let’s add a gauge-fixing term to the free vector-field Lagrangian:

$$\mathcal{L}_0(A) = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\xi} (\partial_\mu A_\mu)^2 \equiv -\frac{1}{2} A_\mu K_{\mu\nu} A_\nu.$$ 

This term allows one to obtain the photon propagator by inverting $K_{\mu\nu}$:

$$\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{-i \left[ g_{\mu\nu} - (1 - \xi) p_\mu p_\nu / p^2 \right]}{p^2 + i\epsilon} e^{-ip(x-y)}$$

The propagator now involves an auxiliary parameter $\xi$. It controls propagation of unphysical longitudinal polarization $\epsilon^L_\mu \propto p_\mu$. But the corresponding terms drop out of physical quantities*:

$$\epsilon^L_\mu J_\mu \propto p_\mu J_\mu = 0,$$

we have no source for unphysical $\gamma$.

NB: The propagator drops down for large $p$ as $1/p^2$.

* serves as a good cross-check of lengthy computations
**QED: Quantization Issues**

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So far we managed to describe half of the EW theory - QED...What about the Weak part?
Fermi Model: Harbinger of the EW theory

In 1957 R. Marshak & G. Sudarshan, R. Feynman & M. Gell-Mann modified the original Fermi theory of beta-decay to incorporate 100% violation of Parity discovered by C.S. Wu in 1956.

\[ \mathcal{L}_{\text{Fermi}} = \frac{G_F}{2\sqrt{2}} (J_\mu^+ J_\mu^- + \text{h.c.}), \]

\[ J_\rho^- = (V - A)_\rho^{\text{nucleons}} + \bar{\Psi}_e \gamma_\rho (1 - \gamma_5) \Psi_{\nu_e} + \bar{\Psi}_\mu \gamma_\rho (1 - \gamma_5) \Psi_{\nu_\mu} + \ldots \]

This is current-current interactions with \( G_F \approx 10^{-5} \) GeV\(^{-1}\).

Under Parity:

\[ V^0 \xrightarrow{P} V^0, \quad V \xrightarrow{P} -V, \]
\[ A^0 \xrightarrow{P} -A^0, \quad A \xrightarrow{P} A. \]

NB: Parity \( P \) is conserved for both a pure vector or axial interactions! Only \( V_\mu A_\mu \) plays a role!

NB: Charge-conjugation \( C \) is also violated!

Ex: Having in mind that \( \psi \xrightarrow{P} \gamma_0 \psi \), prove that \( \bar{\psi} \gamma_\mu \psi \) (\( \bar{\psi} \gamma_\mu \gamma_5 \psi \)) transform as \( V \) (\( A \)).
From Fermi Model to EW theory

From dimensional grounds we can estimate

\[ \sigma(\nu_e e \rightarrow \nu_e e) \propto G_F^2 s, \quad s = (p_e + p_\nu)^2. \]

This is another manifestation of self-inconsistency of Non-Renormalizable models: eventually they violate of unitarity!

- The modern view on the Fermi model treats it as an effective theory with certain limits of applicability:

Warning!

around \( G_F^{-1/2} \sim 10^2 - 10^3 \) GeV there should be some “New Physics”.

- QED is renormalizable. By analogy we introduce mediators - electrically charged vector fields \( W^{\pm}_\mu \):

\[
\mathcal{L}_{\text{Fermi}} = \frac{G_F}{2\sqrt{2}} (J^+_\mu J^-_\mu + \text{h.c.})
\]

\[ \rightarrow \mathcal{L}_I = -\frac{g}{2\sqrt{2}} (W^{+}_\mu J^-_\mu + \text{h.c.}). \]
Fermi Model vs QED: Chiral structure

- The QED vertex conserves chirality and treat $\psi_L$ and $\psi_R$ on equal footing:

$$\mathcal{L}_I \ni -eA_\mu \cdot \bar{\psi} \gamma_\mu \psi = -eA_\mu \left[ \bar{\psi}_L \gamma_\mu \psi_L + \bar{\psi}_R \gamma_\mu \psi_R + \bar{\psi}_L \gamma_\mu \psi_R + \bar{\psi}_R \gamma_\mu \psi_L \right]$$

In the limit ($m \to 0$) we have two non-zero helicity combinations:

- The Weak vertex* also conserve chirality but involve only $\psi_L$:

$$\mathcal{L}_I \ni -\frac{g}{2\sqrt{2}} W^+_\mu \cdot \bar{\psi}_e \gamma_\mu (1 - \gamma_5) \psi_{\nu_e} = -\frac{g}{\sqrt{2}} W^+_\mu \left[ \bar{\psi}_e \gamma_\mu \psi_{\nu_L} \right]$$
From Fermi Model to EW theory

- The field $W_\mu$ in

$$\mathcal{L}_{\text{int}} = -\frac{g}{2\sqrt{2}}(J^+_{\mu}W^-_{\mu} + \text{h.c.}).$$

should be massive to account for short-range weak interactions.

- The scattering amplitude

$$T = i(2\pi)^4 \frac{g^2}{8} J^+_{\alpha} \left[ \frac{g_{\alpha\beta} - p_\alpha p_\beta / M_W^2}{p^2 - M_W^2} \right] J^-_{\beta}$$

reproduces the result due to the current-current interaction in the limit $|p| \ll M_W$ if we identify (“match”)

$$(\text{effective theory}) \quad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \quad (\text{more fundamental theory})$$

- However, we have to be more clever, since the behavior of the amplitude in the opposite limit ($p \gg M_W$) is still the same.
On the WW-production

Another manifestation of the same problem can be observed in the process of $W$-production:

- We know that $W^\pm$ interact with fermions and are electrically charged.
- We can consider $e^+ e^- \rightarrow W^+ W^-$ process at a lepton-antilepton collider (e.g., LEP):

\[
e^+ + e^- \rightarrow W^+ + W^- \rightarrow \nu_e + e^+ + e^- + W^+ + W^- + \gamma
\]

and make predictions...

- The calculated cross-section for longitudinal $W^-$ is found to increase with center-of-mass energy $s$ without limits (scales with $s$).

The solution to these problems is to utilize gauge symmetry...
Abelian vs Non-Abelian

- The U(1) internal symmetry is **Abelian** (the order of two transformations (rotations in 2d plane) is not relevant):
  \[ \text{Re} \phi(x), \text{Im} \phi(x) \]

  \[ e^{i\alpha_1} \cdot e^{i\alpha_2} = e^{i\alpha_2} \cdot e^{i\alpha_1} \]

- U(1) transformations **commute** with each other.

**NB:** Abelian symmetry is not sufficient to account for \( W^\pm_\mu \). Why?
Abelian vs Non-Abelian

- We can generalize $U(1)$ to the Non-Abelian case. Consider $SU(n)$ group, i.e., unitary $n \times n$ matrices $U_{ij}$ depending on $n^2 - 1$ parameters $\omega^a$ and having $\det U = 1$:

$$
\psi_i \rightarrow \psi_i' = U_{ij}(\omega) \psi_j, \quad U(\omega) = e^{igt^a\omega^a}
$$

- Generators $t^a$ of the group obey $su(n)$-algebra

$$
[t^a, t^b] = if^{abc} t^c, \quad f^{abc} - \text{structure constants}
$$

- For constant $\omega^a$ this transformation is a symmetry of

$$
\mathcal{L}_0 = \bar{\psi}_i \left( i\hat{\partial} - m \right) \psi_i, \quad i = 1, \ldots, n
$$

describing $n$ free fermions in fundamental representation of $SU(n)$.  

Abelian vs Non-Abelian

- For space-time dependent $\omega^a(x)$ we introduce (matrix) covariant derivative depending on $n^2 - 1$ gauge fields $W^a_\mu$:

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig t^a_{ij} W^a_\mu$$

- The transformation properties of $W^a_\mu$ can be deduced from

$$D'_\mu \psi' = U(\omega)(D_\mu \psi), \quad U(\omega) = e^{igt^a \omega^a}$$

$$\Rightarrow W'_\mu^a = W^a_\mu + \partial_\mu \omega^a + gf^{abc} W^b_\mu \omega^c$$

$$= W^a_\mu + (D_\mu)^{ab} \omega^b, \quad (D_\mu)^{ab} \equiv \partial_\mu \delta^{ab} - ig(-if^{abc}) W^c_\mu$$

- The field-strength tensor is the commutator of the covariant derivatives (Ex: check this also for $U(1)$ symmetry)

$$[D_\mu, D_\nu] = -igt^a F^a_{\mu \nu},$$

$$F^a_{\mu \nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + gf^{abc} W^a_\mu W^b_\nu$$
Abelian vs Non-Abelian

- Non-Abelian gauge symmetry predicts not only interactions between fermions $\psi$ (more generally, fields in fundamental representation of the gauge group) and gauge fields $W^a_\mu$, but also self-interactions of the latter (gauge-fields are also “charged” under the group):

$$
\mathcal{L} = \bar{\psi} \left( i \hat{D} - m \right) \psi - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} = \mathcal{L}_0 + \mathcal{L}_I
$$

$$
\mathcal{L}_0 = \bar{\psi} \left( i \hat{\partial} - m \right) \psi - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu
$$

$$
\mathcal{L}_I = g \bar{\psi}^i \gamma^\alpha \gamma^\beta t^a_{ij} \psi^j W^a_\mu
$$

$$
- \frac{g}{2} f^{abc} W^b_\mu W^c_\nu F^a_{\mu\nu} - \frac{g^2}{4} f^{abc} f^{ade} W^a_\mu W^b_\nu W^d_\mu W^e_\nu
$$

NB1: The strength of interaction is governed by a single coupling $g$.

NB2: The mass term $m^2 W^a_\mu W^a_\mu / 2$ violates gauge symmetry...

NB3: Non-Abelian gauge theory is also called Yang-Mills theory.
To quantize YM theory we add

\[ \mathcal{L}_{gf} = -\frac{1}{2\xi} (F^a)^2, \quad F^a = \partial_\mu W^a_\mu \]

\[ \Leftarrow \text{gauge-fixing function} \]

Contrary to the case of QED, fermionic current \( J^a_\mu = g \bar{\psi} t^a \gamma_\mu \psi \) is not conserved and can produce longitudinal \( W^a_\mu \).

Self-interactions of \( W^a_\mu \) guarantee that at the tree-level all amplitudes involving unphysical polarizations (for external states) vanish.
Fadeev-Popov Ghosts

- The cancellation does not work for loops.
- Add Faddev-Popov ghosts $\bar{c}_a$ and $c_a$ — anticommuting “scalars”.
- The Lagrangian for ghosts is related to the gauge-fixing function $F_a(= \partial_\mu W_\mu)$

\[
\mathcal{L}_{\text{ghosts}} = -\bar{c}^a \frac{\partial F_a(W^\omega)}{\partial \omega_b} c^b = -\bar{c}^a \partial_\mu D^a_{\mu b} c^b
\]

\[
= -\bar{c}^a \partial^2 c^a - g f^{abc} (\partial_\mu \bar{c}^a) c^b A^b_\mu
\]

NB: This is the price to pay for explicitly covariant formalism.
Gauge Theory of Electroweak Interactions

To describe both electromagnetic and weak interactions in the SM we use 

\[ SU(2)_L \otimes U(1)_Y \]

- The group has four generators = gauge bosons. Three (\(W_\mu\)) belongs to \(SU(2)_L\), while photon-like \(B_\mu\) mediates \(U(1)_Y\).

NB: To explain beta-decay and electromagnetism we need only 3 gauge bosons \(W^\pm\) and \(\gamma\). But we predict an additional one! Where is it hiding?

From J. Maldacena, “The symmetry and simplicity of the laws of physics and the Higgs boson”.
Gauge Theory of Electroweak Interactions

The SM fermions are “charged” under $SU(2) \otimes U(1)$.

Leptons

\[
\begin{pmatrix}
\nu_e \\
e \\
\mu \\
\tau
\end{pmatrix}_L,
\begin{pmatrix}
\nu_\mu \\
\mu \\
\tau
\end{pmatrix}_L,
\begin{pmatrix}
\nu_\tau \\
\tau
\end{pmatrix}_L
\]

Quarks

\[
\begin{pmatrix}
u_e \\
e_R
\end{pmatrix},
\begin{pmatrix}
\nu_\mu \\
\mu_R
\end{pmatrix},
\begin{pmatrix}
\nu_\tau \\
\tau_R
\end{pmatrix},
\begin{pmatrix}
u_\tau \\
\tau_R
\end{pmatrix}
\]

\[
\begin{pmatrix}
u_e \\
e_R
\end{pmatrix},
\begin{pmatrix}
\nu_\mu \\
\mu_R
\end{pmatrix},
\begin{pmatrix}
\nu_\tau \\
\tau_R
\end{pmatrix},
\begin{pmatrix}
u_\tau \\
\tau_R
\end{pmatrix}
\]

Quarks

\[
\begin{pmatrix}
u_e \\
e_R
\end{pmatrix},
\begin{pmatrix}
\nu_\mu \\
\mu_R
\end{pmatrix},
\begin{pmatrix}
\nu_\tau \\
\tau_R
\end{pmatrix},
\begin{pmatrix}
u_\tau \\
\tau_R
\end{pmatrix}
\]

To account for $(V - A)$ pattern only left fermions interact with $W_\mu$. The covariant derivative for left $SU(2)$ doublets

\[
D^L_\mu = \begin{pmatrix}
\partial_\mu - \frac{i}{2} (g W^3_{\mu} + g' Y^f_L B_\mu) \\
-i \frac{g}{\sqrt{2}} W^-_\mu
\end{pmatrix} \begin{pmatrix}
\partial_\mu + \frac{i}{2} (g W^3_{\mu} - g' Y^f_L B_\mu)
\end{pmatrix},
\]

while for the right $SU(2)$ singlets we have

\[
D^R_\mu = \partial_\mu - ig' Y^f_R B_\mu
\]

Here $g$ and $g'$ are the $SU(2)$ and $U(1)$ couplings. But what is $Y^f_{L/R}$?
Gauge Theory of Electroweak Interactions

Let us consider the interactions mediated by the introduced gauge bosons in more details. We have “charged-current” interactions

\[ \mathcal{L}_{CC}^l = \frac{g}{\sqrt{2}} \bar{\nu}_L \gamma_\mu W^\mu_+ e_L + \text{h.c.} = \frac{g}{2\sqrt{2}} \bar{\nu}_e \gamma_\mu W^\mu_+ (1 - \gamma_5) e + \text{h.c.} \]

and “neutral-current” interactions

\[ \mathcal{L}_{NC}^l = \bar{\nu}_L \gamma_\mu \left( + \frac{1}{2} g W^3_\mu + \frac{Y^l}{2} g' B_\mu \right) \nu_L \\
+ \bar{e}_L \gamma_\mu \left( - \frac{1}{2} g W^3_\mu + \frac{Y^l}{2} g' B_\mu \right) e_L \\
+ g' \bar{e}_R \gamma_\mu \frac{Y^e_R}{2} B_\mu e_R \]

Q: Where is QED photon \( A_\mu \)?

**Weinberg angle**

\[ W^3_\mu = Z_\mu \cos \theta_W + A_\mu \sin \theta_W \]
\[ B_\mu = - Z_\mu \sin \theta_W + A_\mu \cos \theta_W \]

The photon \( A_\mu \) should not interact with \( \nu_L \), but couple to \( e_L \) and \( e_R \) with the same strength!

**Ex:** Derive similar expressions for quarks.
Gauge Theory of Electroweak Interactions

It is a simple exercise to get the following relations between fermion electric charges $Q_f$ and weak hypercharges $Y^f$:

$$g \sin \theta_W = e(Q_\nu - Q_e) = e(Q_u - Q_d)$$
$$g' Y^e_L \cos \theta_W = e(Q_\nu + Q_e) = -e$$
$$g' Y^Q_L \cos \theta_W = e(Q_u + Q_d) = \frac{1}{3} e$$
$$g' Y^f_R \cos \theta_W = 2eQ_f, \quad f = e, u, d$$

and the elementary charge $e$.

From these relations we obtain $e = g \sin \theta_W$ and, e.g., $e = -g' Y^e_L \cos \theta_W$, so that

$$Y^Q_L = -\frac{1}{3} Y^e_L, \quad Y^e_R = 2 Y^e_L, \quad Y^u_R = -\frac{4}{3} Y^e_L, \quad Y^d_R = +\frac{2}{3} Y^e_L,$$

are fixed in terms of $Y^e_L$. It is convenient to normalize $g'$ so that $Y^e_L = -1$. 

Sorry, too many e's.
The neutral-current Lagrangian $\mathcal{L}_{NC}$ takes the form:

$$\mathcal{L}_{NC} = e J^A_\mu A^\mu + \frac{g}{\cos \theta_W} J^Z_\mu Z^\mu,$$

where

$$J^A_\mu = \sum_f Q_f \bar{f} \gamma_\mu f,$$

$$J^Z_\mu = \frac{1}{4} \sum_f \bar{f} \gamma_\mu (v_f - a_f \gamma_5) f,$$

$$v_f = 2 T_3^f - 4 Q_f \sin^2 \theta_W,$$

$$a_f = 2 T_3^f$$

and $T_3^f = \pm \frac{1}{2}$ for up-type/down-type fermions.

Example: $Q_u = 2/3$, $T_3^u = 1/2$, so

$$v_u = 1 - \frac{8}{3} \sin^2 \theta_W,$$

$$a_u = 1$$

Ex: Deduce $v_f/a_f$ for other fermions.

NB: The couplings does not depend on generation (Universality).
On Axial Anomalies and Charge Assignment in the SM

Anomalies correspond to situations when a symmetry of the classical Lagrangian is violated at the quantum level. A well-known example is Axial/Chiral Anomaly, when classical conservation law for the current is modified at quantum level:

\[ J^A_\mu = \bar{\Psi} \gamma_\mu \gamma_5 \Psi, \quad \partial_\mu J^A_\mu = 2im\Psi \gamma_5 \Psi + \frac{\alpha}{2\pi} F_{\mu\nu} \tilde{F}_{\mu\nu}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}} \]

If \( J^A_\mu \) couples to a gauge field, anomalies render the model inconsistent!

The EW interactions in the SM do distinguish left from right.

\[ Anom \sim \text{Tr}[t^a, \{t^b, t^c\}]_L - \text{Tr}[t^a, \{t^b, t^c\}]_R \]

In the SM contributions to anomalies miraculously cancel each other separately for each generation.

**Ex:** Check that all anomalies* are also zero.

*From ArXiv:0901.2208 by Kazakov D.I.*
The EW interactions: Lecture II summary

✓ We use gauge principle to introduce EW interactions.
✓ We account for \((V - A)\) structure for charged currents

\[
\mathcal{L}_{CC} = \frac{g}{2\sqrt{2}} \bar{\nu} e \gamma_\mu (1 - \gamma_5) W^+ \mu e + \bar{u} \gamma_\mu (1 - \gamma_5) W^+ \mu d + \text{h.c.}
\]

✓ We reproduce electromagnetic interactions:

\[
\mathcal{L}_{NC}^A = e J^A A^\mu
\]

✓ We predict additional neutral boson

\[
\mathcal{L}_{NC}^A = g_z J^Z Z^\mu, \quad g_z = \frac{g}{\cos \theta_W}
\]

- \(Z\)-boson should be \textbf{massive} and lead to Fermi-like current-current interactions \(J^Z_\mu J^Z_\mu\).

- Relative strength of \textbf{charged} and \textbf{neutral} current-current interactions is parametrized by

\[
\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2 \theta_W}, \quad \text{Up to now, no relation between } M_Z \text{ and } M_W...
\]
The EW interactions: Lecture II summary

✓ Non-Abelian $SU(2)_L$ predicts also gauge-field triple and quartic self-interactions!

\[ \gamma, Z \]
\[ W^+ \]
\[ W^- \]
\[ W^+ \]
\[ W^- \]
\[ W^+ \]
\[ W^- \]

NB: $ZWW$ coupling cures bad behavior of $ee \rightarrow WW$.

So far so good, but there is an inconsistency in our reasoning...

× All gauge bosons should be massless! A mass term like $M^2_W W^+_\mu W^-_\mu$ is forbidden by symmetry $W_\mu \rightarrow W_\mu + \partial_\mu \omega + ...$

× All fermions are massless! A mass term like $m_e \bar{e}e$ mixes left and right. But $e_L$ belongs to a $SU(2)$ doublet, while $e_R$ is an $SU(2)$ singlet!

A solution is to be provided via the Higgs mechanism...
Lecture III?

Question

Do you want to Continue?

Yes  No  Cancel
Break the symmetry?

Q: How to make $W$ and $Z$ massive (keeping the nice features of the gauge theory intact)?

- **Explicit** breaking via the mass terms
  
  $$\mathcal{L} \ni m_W^2 W^\mu_- W^{\mu}_+ + \frac{m_Z^2}{2} Z^\mu Z_\mu$$

  leads to inconsistencies...

- We have to do something more clever...

  **Hidden** symmetry?
Spontaneous Symmetry Breaking

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \rho)^2 + \frac{e^2 \rho^2}{2} B_\mu B_\mu - V(\rho^2/2) - \frac{1}{4} F_{\mu\nu}^2 (B) \]

Here \( \rho(x) \) is a dynamical field. We get mass term if \( \rho(x) \rightarrow \nu = \text{const.} \)

\( V(\phi) \)

\[ V = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \]

It is invariant under global phase-shift symmetry \( \phi \rightarrow e^{i\alpha} \phi \).

- The case \( \mu^2 > 0 \) is trivial.
- For \( \mu^2 < 0 \) we have a valley of degenerate minima:

\[ \frac{\partial V}{\partial \phi^\dagger} = 0 \Rightarrow \phi_0^\dagger \phi_0 = -\frac{\mu^2}{2\lambda} = \frac{\nu^2}{2} > 0 \Rightarrow \phi_0 = \frac{\nu}{\sqrt{2}} e^{i\beta} \]
The Brout-Englert-Higgs mechanism

Non-zero $\phi$ in the minimum of the potential is interpreted as the vacuum expectation value (vev) of the quantum field:

$$\frac{v}{\sqrt{2}} = \langle 0 | \phi(x) | 0 \rangle, \quad \beta = 0 \text{ (Why?)}$$

To introduce particles as excitations we have to shift the field:

$$\phi(x) = \frac{v + h(x)}{\sqrt{2}} e^{i \zeta(x)/v}, \quad \langle 0 | h(x) | 0 \rangle = 0, \quad \langle 0 | \zeta(x) | 0 \rangle = 0$$

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu h \right)^2 + \frac{e^2 v^2}{2} B_\mu B_\mu + e v h B_\mu B_\mu + \frac{e^2}{2} B_\mu B_\mu h^2 - V - \frac{1}{4} F_{\mu \nu}^2(B)$$

$$V = -\frac{|\mu|^2}{2} (v + h)^2 + \frac{\lambda}{4} (v + h)^4 = \frac{2\lambda v^2}{2} h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4 - \frac{\lambda}{4} v^4$$

Massive field $B_\mu$ without explicit symmetry breaking! This is the essence of Brout-Englert-Higgs-Hagen-Guralnik-Kibble mechanism. The symmetry is hidden..
The Brout-Englert-Higgs mechanism: Counting DOFs

We start with
\[ L_1 = \partial_\mu \phi^\dagger \partial_\mu \phi + i e \left( \phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger \right) A_\mu + e^2 A_\mu A_\mu \phi^\dagger \phi - V - \frac{1}{4} F_{\mu\nu}^2 (A), \]
and end with
\[ \sqrt{2} \phi = (v + h) \exp(i \zeta(x)/v), \quad B_\mu = A_\mu - \partial_\mu \zeta/(ev) \]
\[ L_2 = \frac{1}{2} (\partial_\mu h)^2 - \frac{e^2 v^2}{2} \left( 1 + \frac{h^2}{v^2} \right) B_\mu B_\mu + ev h B_\mu B_\mu - V - \frac{1}{4} F_{\mu\nu}^2 (B). \]

\( L_1 \): 2 DOFs (complex scalar \( \phi, \phi^\dagger \)) + 2 DOFs (massless vector \( A_\mu \)).

\( L_2 \): 1 DOFs (real scalar \( h \)) + 3 DOFs (massive vector field \( B_\mu \)).

One scalar DOF was “eaten” by the gauge field to become massive.

Q: Which one?
A: The (would-be) Nambu-Goldstone (boson)!
SSB and Renormalizability

Both $\mathcal{L}_1$ and $\mathcal{L}_2$ has some issues:

$\mathcal{L}_1$ : Manifestly gauge-invariant, but not suitable for PT (imaginary mass);

$\mathcal{L}_2$ : Hidden gauge symmetry, only physical DOFs, but non-renormalizable by power counting.

There is another, explicitly renormalizable version of the Lagrangian with shifted $\phi$ written in cartesian coordinates: $\phi = \frac{1}{\sqrt{2}} (v + \eta + i\xi)$

$$
\mathcal{L}_3 = \frac{v^4\lambda}{4} - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{e^2 v^2}{2} A_\mu A_\mu + \frac{1}{2} \partial_\mu \xi \partial_\mu \xi - ev A_\mu \partial_\mu \xi \\
+ \frac{1}{2} \partial_\mu \eta \partial_\mu \eta - \frac{2v^2\lambda}{2} \eta^2 + eA_\mu \xi \partial_\mu \eta - eA_\mu \eta \partial_\mu \xi - v\lambda \eta(\eta^2 + \xi^2) \\
- \frac{\lambda}{4}(\eta^2 + \xi^2)^2 + \frac{e^2}{2} A_\mu A_\mu (2v\eta + \eta^2 + \xi^2).
$$

NB: Now we have massless unphysical field $\xi$ in the spectrum, but it mixes with longitudinal component of $A_\mu$ (“partially eaten”).
A Remark on the Goldstone Theorem

The Goldstone theorem states that if the vacuum breaks a global continuous symmetry there is a massless boson (Nambu-Goldstone) in the spectrum: any non-derivative interactions violates

$$
\zeta \rightarrow \zeta + e v \omega, \quad \omega = \text{const}
$$

Fortunately, we have local symmetry with hungry $A_\mu$...

- We need 3 massive bosons $W^\pm, Z_\mu$.
- 3 symmetries out of $SU(2)_L \times U(1)_Y$ has to be spontaneously broken to get 3 victims (would-be) Nambu-Goldstone bosons.

**NB:** The Higgs boson (see lect. by J. Ellis) was a by-product of the mass-generation mechanism (the main task was to “exorcise” massless fields).