A systematic procedure to build the beyond generalized Proca field theory

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MAIN RESULTS

1. The procedure naturally yields the beyond generalized Proca field theory terms.
2. The procedure constitutes a systematic method to build general theories for multiple vector fields with or without internal symmetries.
3. The procedure extends previously proposed covariantization processes.

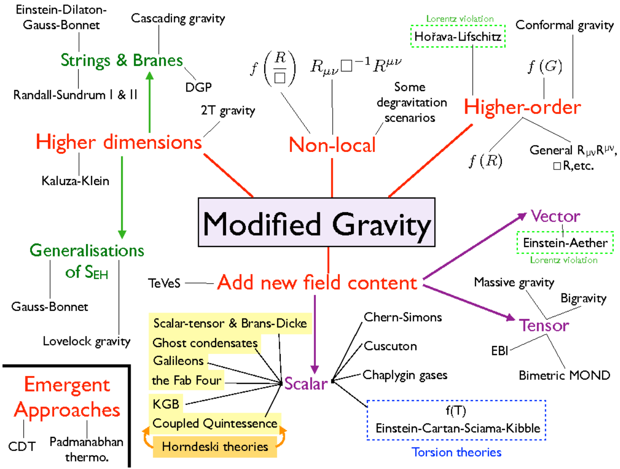

Figure: Tree diagram of modified theories of gravity. Image Credit: Tessa Baker.

Motivation

- GR is still considered to be incomplete since it cannot be reconciled with a quantum theory in order to produce a unified theory of quantum gravity.
- Some cosmological observations might point to modifications of GR.
- The procedure to obtain some theories of gravity has not been systematic or exhaustive.

Write down all possible test $\mathcal{L}$'s in FST.

number of A^μ	1	2	3	4	5	6
number of Lorentz scalars	0	1	0	4	0	25

Table: Number of Lorentz scalars that can be constructed with multiple copies of A_μ.

number of $\partial^\mu A^\nu$ / number of $A^\rho A^\sigma$	0	1	2
1	1	2	2
2	4	10	11

Table: Number of Lorentz scalars that can be built for a given product of A_μ and $\partial_\mu A_\nu$.

Using group theory we assure that all possible terms are written down.

Only three DoF for A_μ propagate.

(Primary) Hessian condition

$$\mathcal{H}^{\mu\nu}_{\mathcal{L}_{\rm test}} = \frac{\partial^2 \mathcal{L}_{\rm test}}{\partial \dot{A}_\mu \partial \dot{A}_\nu} = 0. \tag{1}$$

When going to CST, the Hessian condition is not sufficient to count for the ghost and Laplacian instabilities $\rightarrow$ Hamiltonian analysis must be performed.

Find constraints among the test $\mathcal{L}$'s

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To this end, it is handy to use the identity

$$A^{\mu\alpha}\tilde{B}_{\nu\alpha} + B^{\mu\alpha}\tilde{A}_{\nu\alpha} = \frac{1}{2}(B^{\alpha\beta}\tilde{A}_{\alpha\beta})\delta^{\mu}_{\nu}, \tag{2}$$

valid for all antisymmetric tensors A and B.

Identify the test $\mathcal{L}$'s related by $\partial_\mu J^\mu$.

Define currents of the form

$$J^\mu_\delta \equiv \delta^{\mu\mu_2}_{\nu_1\nu_2} A^{\nu_1}\partial_{\mu_2}A^{\nu_2}, \quad J^\mu_\epsilon \equiv \epsilon^{\mu\nu\rho\sigma}A_\nu(\partial_\rho A_\sigma), \quad J^\mu_D \equiv D^{\mu\nu}A_\nu \,. \tag{3}$$

In a FST, we would use them to eliminate one or several test $\mathcal{L}$'s in terms of others.

However, in general, what in FST were total derivatives, in CST are not.

Hence the test $\mathcal{L}$'s that yield the same field equations in FST *do not* yield the same field equations anymore.

Covariantize the resulting FST Abelian theory.

We could simply follow the minimal coupling principle.

Include possible direct coupling terms between A_μ and the curvature tensors.

$A_\mu \to \nabla_\mu \phi$

When gravity is turned on, it could excite the temporal polarization of A_μ, introducing new propagating DoF.

Verify that the field equations for all physical DoF are at most 2nd-order in their derivatives.

Split A^μ into the pure scalar and vector modes

$$A^\mu = \nabla^\mu \phi + \hat{A}^\mu, \tag{4}$$

where ϕ is the Stuckelberg field and $\hat{A}^\mu$ is the divergence-free contribution.

$A_\mu \rightarrow \nabla_\mu \phi$

For a theory built out of 1st-order derivatives in A_μ, $\hat{A}^\mu$ cannot lead to any derivative of order higher than two.

As for the scalar part, derivatives of order three or more could appear when covariantazing $\rightarrow$ add appropriate counterterms.

Some Lagrangians could vanish in the scalar limit $\rightarrow$ terms corresponding to purely intrinsic vector modes.

In a FST we might have relations of the form [1]

$$\partial_\mu J^\mu = \mathcal{L}_i + \mathcal{L}_j \,, \tag{5}$$

which allows us to remove $\mathcal{L}_j$ in favour of $\mathcal{L}_i$ or viceversa.

When promoting this expression to a CST we have

$$\nabla_\mu J^\mu = \mathcal{L}_i + \mathcal{L}_j + \mathcal{F}(A^\mu, \nabla^\mu A^\nu) \,. \tag{6}$$

In a CST, the field equations for $\mathcal{L}_i$ and $\mathcal{L}_j$ will no longer be the same due to $\mathcal{F}$.

It could also be the case that $\mathcal{F}$ vanishes identically, or that it is a total derivative.

Now we implement the procedure in the case of the $\mathcal{L}_4^{\rm P}$ Proca Lagrangian.

Paying attention to the covariantization of the FST currents, we will arrive to the BGP theory which, as will be shown, is equivalent to the beyond Horndeski theory in the longitudinal limit.

For a term of the form $(\partial_\mu A_\nu)(\partial_\rho A_\sigma)$

$$\mathcal{L}_1 = (\partial \cdot A)^2, \qquad \mathcal{L}_2 = (\partial_\mu A_\nu)(\partial^\mu A^\nu)\,, \tag{7}$$

$$\mathcal{L}_3 = (\partial_\mu A_\nu)(\partial^\nu A^\mu), \qquad \mathcal{L}_4 = \epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)(\partial_\rho A_\sigma)\,. \tag{8}$$

For a term of the form $A_\alpha A_\beta(\partial_\mu A_\nu)(\partial_\rho A_\sigma)$

$$\mathcal{L}_5 = (\partial \cdot A)(\partial_\rho A_\sigma)A^\rho A^\sigma, \qquad \mathcal{L}_6 = (\partial_\mu A_\nu)(\partial^\mu A_\sigma)A^\nu A^\sigma, \tag{9}$$

$$\mathcal{L}_7 = (\partial_\mu A_\nu)(\partial_\rho A^\mu)A^\nu A^\rho, \qquad \mathcal{L}_8 = (\partial_\mu A_\nu)(\partial_\rho A^\nu)A^\mu A^\rho, \tag{10}$$

$$\mathcal{L}_9 = \epsilon^{\mu\rho\sigma\beta}A_\beta(\partial_\nu A_\mu)(\partial_\rho A_\sigma)A^\nu, \quad \mathcal{L}_{10} = \epsilon^{\mu\rho\sigma\beta}A_\beta(\partial_\mu A_\nu)(\partial_\rho A_\sigma)A^\nu. \tag{11}$$

For a term of the form $A_\alpha A_\beta A_\gamma A_\delta (\partial_\mu A_\nu)(\partial_\rho A_\sigma)$

$$\mathcal{L}_{11} = (A^\mu (\partial_\mu A_\nu) A^\nu)^2 \,, \tag{12}$$

where the other are equivalent to the previous $\mathcal{L}$'s multiplied by A^2.

One can conclude that *all the possible $\mathcal{L}_{\rm test}$'s that involve two $\partial_\mu A_\nu$'s are expressed as the ones in Eqs. (7)-(12) multiplied each one of them by an arbitrary function of A^2.*

Using the test Lagrangian

$$\mathcal{L}_{\text{test}} = \sum_{i=1}^{11} f_i(X)\mathcal{L}_i , \tag{13}$$

where $X \equiv -A^2/2$ and $f_i(X)$ are arbitrary functions.

The Hessian condition

$$\mathcal{H}^{\mu 0}_{\mathcal{L}_{\text{test}}} = 0 \, , \tag{14}$$

leads to four independent algebraic equations which we solve for

$$f_1 = -f_2 - f_3 \, , \quad f_5 = -2f_6 - f_7 \, , \quad f_6 = f_8 \, , \text{ and } \quad f_{11} = 0 \, . \tag{15}$$

Thus, our test Lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_{\rm test} \; &= \; f_2(X)(\mathcal{L}_2 - \mathcal{L}_1) + f_3(X)(\mathcal{L}_3 - \mathcal{L}_1) + f_4(X)\mathcal{L}_4 \\
&\quad + f_6(X)(\mathcal{L}_6 - 2\mathcal{L}_5 + \mathcal{L}_8) + f_7(X)(\mathcal{L}_7 - \mathcal{L}_5) \\
&\quad + f_9(X)\mathcal{L}_9 + f_{10}(X)\mathcal{L}_{10} \, .
\end{aligned} \tag{16}
$$

For the Faraday and Hodge dual tensors $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$, we can write

$$F^{\mu\alpha}\tilde{F}_{\nu\alpha}A_\mu A^\nu = \frac{1}{4}(A\cdot A)\,F^{\alpha\beta}\tilde{F}_{\alpha\beta}\,. \tag{17}$$

From which we obtain, also valid in a CST,

$$\mathcal{L}_9 - \mathcal{L}_{10} = -X\mathcal{L}_4 = -X\frac{1}{2}F^{\alpha\beta}\tilde{F}_{\alpha\beta}\,, \tag{18}$$

and means that $\mathcal{L}_4$ actually belongs to $\mathcal{L}_2^{\mathrm{P}}$, thus we can remove $f_9(X)\mathcal{L}_9$ and $[f_4(X) - Xf_9(X)]\,\mathcal{L}_4$ from $\mathcal{L}_4^{\mathrm{P}}$.

Another constraint can be found by noticing that

$$(\mathcal{L}_2 - \mathcal{L}_1) - (\mathcal{L}_3 - \mathcal{L}_1) = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \,, \tag{19}$$

so that $f_3(X)(\mathcal{L}_3 - \mathcal{L}_1)$ can be removed in favour of $f_3(X)(\mathcal{L}_2 - \mathcal{L}_1)$ and a Lagrangian belonging to $\mathcal{L}_2^{\mathrm{P}}$ (which can also be removed).

The current $J^\mu_\delta \equiv f(X)\delta^{\mu\mu_2}_{\nu_1\nu_2}A^{\nu_1}\partial_{\mu_2}A^{\nu_2}$ leads to

$$\partial_\mu J^\mu_\delta = -f(X)(\mathcal{L}_3 - \mathcal{L}_1) + f_X(X)(\mathcal{L}_7 - \mathcal{L}_5)\,, \tag{20}$$

where $f_X(X) \equiv \partial f(X)/\partial X$, and we have used $A^\mu\partial_\mu\partial_\nu A^\nu - A^\nu\partial_\mu\partial_\nu A^\mu = 0$, since in a FST the partial derivatives of the Proca field commute.

This part is crucial since, in a CST, the ∇_μ's do not commute.

Thus the term $(\mathcal{L}_7 - \mathcal{L}_5)$ can be removed from $\mathcal{L}_4^{\mathrm{P}}$, *only in FST*, since

$$f_7(X)(\mathcal{L}_7 - \mathcal{L}_5) = \left(\int f_7(X)\, dX\right)(\mathcal{L}_3 - \mathcal{L}_1) + \partial_\mu J^\mu_\delta\,. \tag{21}$$

A symilar procedure leads to

$$\partial_\mu J^\mu_\epsilon = f(X)\mathcal{L}_4 - f_X(X)\mathcal{L}_{10}\,, \tag{22}$$

showing that $\mathcal{L}_{10}$ can be removed from $\mathcal{L}_4^{\mathrm{P}}$ since it gives the same field equations as a term belonging to $\mathcal{L}_2^{\mathrm{P}}$.

This formula is valid *even in CST* because the commutation of partial second-order derivatives has not been invoked.

We first rewrite our test Lagrangian having included the constraints

$$\mathcal{L}_{\text{test}} = F_4(X)(\mathcal{L}_2 - \mathcal{L}_1) + f_6(X)(\mathcal{L}_6 - 2\mathcal{L}_7 + \mathcal{L}_8) + G_N(X)\, A^\nu(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)A^\mu, \quad (23)$$

where $F_4(X)$ and $G_N(X)$ are arbitrary functions of X.

Now we promote all the partial derivatives to covariant ones

$$\mathcal{L}_{\text{test}} = -F_4(X)\delta^{\mu_1\mu_2}_{\nu_1\nu_2}(\nabla_{\mu_1}A^{\nu_1})(\nabla^{\nu_2}A_{\mu_2}) + f_6(X)\cancelto{\mathcal{L}_2^{\mathrm{P}}}{A_\mu A_\nu F^\mu{}_\alpha F^{\nu\alpha}} + G_N(X)\, R_{\mu\nu}A^\mu A^\nu, \quad (24)$$

where $R_{\mu\nu}$ is the Ricci tensor.

Our test Lagrangian can be written as

$$\mathcal{L}_{\text{test}} = -F_4(X)\delta^{\mu_1\mu_2}_{\nu_1\nu_2}(\nabla_{\mu_1}A^{\nu_1})(\nabla^{\nu_2}A_{\mu_2}) + G_N(X)R_{\mu\nu}A^{\mu}A^{\nu}\,. \tag{25}$$

The existence of the second term in the previous expression had not been recognized before because the covariantization was performed over the final FST Lagrangian.

Nobody had paid attention to the fact that new terms could be generated in CST, terms that simply vanish in FST.

Beyond generalized Proca theory

For for two fields and two field derivatives the BGP terms are given by [2]

$$\mathcal{L}_4^{\mathrm{N}} = f_4^{\mathrm{N}}(X)\delta^{\beta_1\beta_2\beta_3\gamma_4}_{\alpha_1\alpha_2\alpha_3\gamma_4}A^{\alpha_1}A_{\beta_1}\nabla^{\alpha_2}A_{\beta_2}\nabla^{\alpha_3}A_{\beta_3}\,. \tag{26}$$

After simplifications it can be written as

$$\mathcal{L}_4^{\mathrm{N}} = -\left[2XG_{N,X}(X) + G_N(X)\right]\delta^{\mu_1\mu_2}_{\nu_1\nu_2}(\nabla_{\mu_1}A^{\nu_1})(\nabla^{\nu_2}A_{\mu_2}) + G_N(X)R_{\mu\nu}A^{\mu}A^{\nu}\,. \tag{27}$$

Thus, we may conclude that our theory is equivalent to the BGP theory in the case of the $\mathcal{L}_4^{\mathrm{P}}$ Proca sector.

Beyond Horndeski theory

The Horndeski $\mathcal{L}_4^{\mathrm{H}}$ and beyond Horndeski $\mathcal{L}_4^{\mathrm{BH}}$ Lagrangians are written [3]

$$\mathcal{L}_4^H = G_4(\phi, X)R - G_{4,X}(\phi, X)\Big((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\Big)\,, \tag{28}$$

$$\mathcal{L}_4^{BH} = f_4^{\mathrm{N}}(\phi,\, X)\Big[X\Big((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\Big)+2\phi_\mu\phi_\nu\left(\phi^{\mu\alpha}\phi^\nu_\alpha - \Box\phi\,\phi^{\mu\nu}\right)\Big]\,, \tag{29}$$

where $X \equiv -\nabla_\mu\phi\,\nabla^\mu\phi/2$, $\phi_\mu \equiv \nabla_\mu\phi$, $\phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi$, R is the Ricci scalar, and the G_i's are arbitrary functions of ϕ and X.

In the scalar limit $A_\mu \to \nabla_\mu \phi$, our test Lagrangian takes the form

$$\mathcal{L}_{\text{test}} \to -2G_{4,X}(X)\Big((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\Big) + f_4^{\text{N}}(X)\Big[X\Big((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\Big) + 2\phi_\mu\phi_\nu\left(\phi^{\mu\alpha}\phi^\nu_\alpha - \Box\phi\,\phi^{\mu\nu}\right)\Big], \tag{30}$$

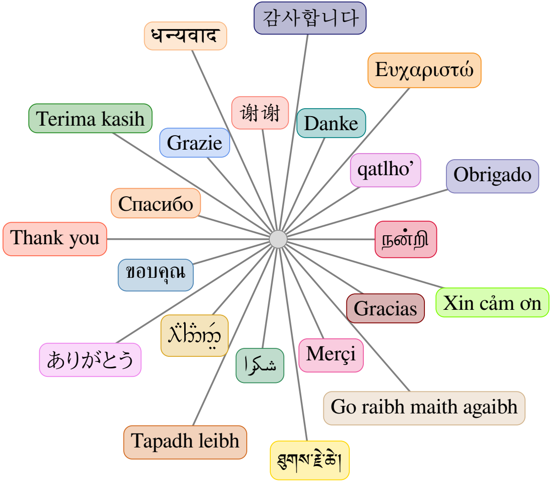
such that it reduces to the Horndeski and beyond Horndeski theories, except for the term proportional to the Ricci scalar.

Final Lagrangian

Our final $\mathcal{L}_4^{\rm P}$ Lagrangian is [1]

$$\mathcal{L}_4^{\rm P} = G_4(X)R - G_{4,X}(X)\delta^{\mu_1\mu_2}_{\nu_1\nu_2}(\nabla_{\mu_1}A^{\nu_1})(\nabla^{\nu_2}A_{\mu_2}) + f_4^{\rm N}(X)\delta^{\beta_1\beta_2\beta_3\gamma_4}_{\alpha_1\alpha_2\alpha_3\gamma_4}A^{\alpha_1}A_{\beta_1}\nabla^{\alpha_2}A_{\beta_2}\nabla^{\alpha_3}A_{\beta_3}\,. \tag{31}$$

- We generate parity-violating terms.
- The method can be applied to any of the other pieces of the generalized Proca theory, such as $\mathcal{L}_5^{\mathrm{P}}$ or $\mathcal{L}_6^{\mathrm{P}}$.
- It can also be applied to extensions of the generalized Proca theory, such as the scalar-vector-tensor theory developed in Ref. [4] or the generalized SU(2) Proca theory of Ref. [5].
- Total derivatives in FST could no longer be total derivatives in CST. Thus, a few terms were ignored that we have unveiled, finding out that they produce the BGP terms.
- The construction of the beyond generalized SU(2) Proca theory will be discussed in a forthcoming paper [6].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Ευχαριστώ
Terima kasih
谢谢
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Thank you
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