Introduction to the "conformal" and "modular" Bootstrap

- Motivations of studying CFT
  - AdS/CFT correspondence
  - Describe second order phase transition

ex) $$S = \int d^3x \ \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{\lambda}{4!}(\phi^i\phi^i)^2$$

dimensionful coupling constant $\rightsquigarrow$ hard to apply perturbation!

Alternative : $D = 4-\varepsilon$ a.k.a ε-expansion

for $O(N)$ scalar theory,

$$\beta(\lambda) = -\varepsilon\lambda + (N+8)\frac{\lambda^2}{8\pi^2} + \cdots \rightsquigarrow \begin{cases} \lambda^* = 0 & \text{(Gaussian)} \\ \lambda^* = \frac{8\pi^2}{N+8}\varepsilon & \text{(Wilson-Fisher)} \end{cases}$$

@ WF fixed pt.
$$\Delta_\phi = 1 - \frac{\varepsilon}{2} + \frac{N+2}{4(N+8)^2}\varepsilon^2 + \cdots = \frac{D}{2} - 1 + \frac{\eta}{2}$$
$$\Delta_{\phi^2} = 2 - \frac{6}{N+8}\varepsilon + \cdots = D - \frac{1}{\nu}$$

$(\eta, \nu) \Rightarrow$ Critical exponents!

Critical exponents of Ising, O(2) model...

$\boxed{1}$ perturbative computation via ε-expansion

$\boxed{2}$ Monte-Carlo Simulation

$\boxed{3}$ Conformal Bootstrap! (Non-perturbative)

- Conformal Algebra (in Euclidean sig)

$$\text{Poincare } P_\mu, M_{\mu\nu} \longrightarrow \text{Conformal } P_\mu, M_{\mu\nu}, D, K_\mu$$

$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu, \qquad [D, P_\mu] = P_\mu$$
$$[M_{\mu\nu}, K_\rho] = \delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu, \qquad [D, K_\mu] = -K_\mu$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\rho\mu} - \delta_{\mu\sigma}M_{\rho\nu},$$
$$[K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu}$$

Every quantum states in CFT Hilbert space are labled by (Δ, l) !

Δ : conformal dimension : $\mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x)$, $[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0)$

l : spin

$[K_\mu, \mathcal{O}(0)] = 0 \rightsquigarrow$ primary operator $\left(\begin{array}{l} P_\mu : \text{raising} \\ K_\mu : \text{lowering} \end{array} \right)$

$\therefore$ Quantum states in CFT = Primary $\oplus$ Descendants !

- Structure of correlation function (in Euclidean)

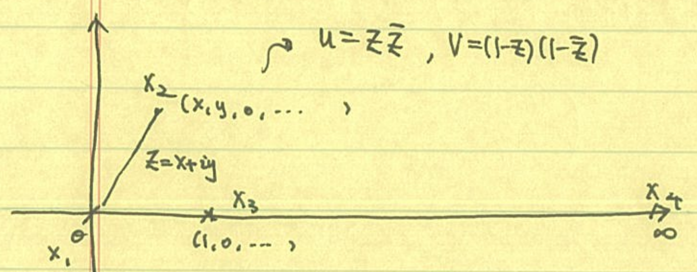
Two pt of scalar op. $\langle \mathcal{O}_a(x_1)\, \mathcal{O}_b(x_2) \rangle = \dfrac{\delta_{ab}}{x_{12}^{2\Delta_\mathcal{O}}}$ $\quad \Delta_\mathcal{O} = \Delta_a = \Delta_b$

Three pt of scalar op. $\langle \mathcal{O}_a(x_1)\, \mathcal{O}_b(x_2)\, \mathcal{O}_c(x_3) \rangle = \dfrac{f_{abc}}{x_{12}^{\Delta_a+\Delta_b-\Delta_c}\, x_{23}^{\Delta_b+\Delta_c-\Delta_a}\, x_{31}^{\Delta_c+\Delta_a-\Delta_b}}$ $\quad f_{abc} \rightsquigarrow$ structure const.

Four pt of identical scalar op.

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\, \mathcal{O}(x_3)\, \mathcal{O}(x_4) \rangle = \frac{1}{x_{12}^{2\Delta_\mathcal{O}}\, x_{34}^{2\Delta_\mathcal{O}}}\, g(u, v)$$

$u = \dfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $\quad v = \dfrac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$ $\quad$ cross ratio

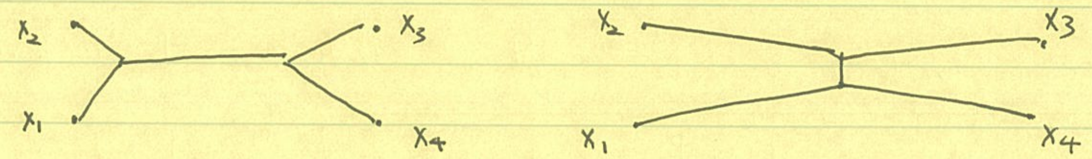


$\rightsquigarrow$ regardless of spacetime dimension, dynamics of 4-pt function always constrained in 2-dim subplane!

- Crossing symmetry constraint (Euclidean) ★

$x_1 \leftrightarrow x_3$

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\, \mathcal{O}(x_3)\, \mathcal{O}(x_4) \rangle = \langle \mathcal{O}(x_3)\, \mathcal{O}(x_2)\, \mathcal{O}(x_1)\, \mathcal{O}(x_4) \rangle$$



s - channel = t - channel

Crossing symmetry gives,

$$\frac{g(u,v)}{x_{12}^{2\Delta_0} x_{34}^{2\Delta_0}} = \frac{\sum f_{OO\tilde{O}}^2 G_{\Delta,l}^{\tilde{O}}(u,v)}{x_{12}^{2\Delta_0} x_{34}^{2\Delta_0}}$$

$x_1 \leftrightarrow x_3$ $\|$

$$\frac{g(v,u)}{x_{23}^{2\Delta_0} x_{14}^{2\Delta_0}} = \frac{\sum f_{OO\tilde{O}}^2 G_{\Delta,l}^{\tilde{O}}(v,u)}{x_{23}^{2\Delta_0} x_{34}^{2\Delta_0}}$$

$$\boxed{v^{\Delta_0} \sum_{\tilde{O}} f_{OO\tilde{O}}^2 G_{\Delta,l}(u,v) = u^{\Delta_0} \sum_{\tilde{O}} f_{OO\tilde{O}}^2 G_{\Delta,l}(v,u)} \quad \cdots (*)$$

$G_{\Delta,l}(u,v)$ is referred to as conformal block.

- For even D, conformal block is known to have closed form.

e.g. $G_{\Delta,l}^{D=2}(u,v) = K_{\Delta+l}(z) K_{\Delta-l}(\bar{z}) + K_{\Delta-l}(z) K_{\Delta+l}(\bar{z})$

$G_{\Delta,l}^{D=4}(u,v) = \frac{z\bar{z}}{z-\bar{z}} \left(K_{\Delta+l}(z) K_{\Delta-l-2}(\bar{z}) - K_{\Delta-l-2}(z) K_{\Delta+l}(\bar{z}) \right)$

where $K_\beta(x) \equiv x^{\frac{\beta}{2}} {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}; \beta; x\right)$

- For odd D, <u>No closed form</u>. However, the recursive formula exists.

$$G_{\Delta l}(u,v) = \boxed{\text{positive factor}} \times (\text{polynomal of } u, v)$$

One can rewrite (*) as

Bootstrap Equation

$$\boxed{\underbrace{v^{\Delta_0}}_{\text{identity contribution}} + v^{\Delta_0} \sum_{\substack{\tilde{O} \\ I \neq \tilde{O}}} f_{OO\tilde{O}}^2 G_{\Delta,l}^{\tilde{O}}(u,v) = \overbrace{u^{\Delta_0}}^{\text{identity contribution}} + u^{\Delta_0} \sum_{\substack{\tilde{O} \\ I \neq \tilde{O}}} f_{OO\tilde{O}}^2 G_{\Delta,l}^{\tilde{O}}(v,u)}$$

* few remarks

- For Hermitian local operator $O(t, \vec{x})$, $O_E(t_E, \vec{x})^\dagger = O_E(-t_E, \vec{x})$ (operator in Euclidean sig.)

Let $|\psi\rangle = O_E(-t_1) O_E(-t_2) \cdots O_E(-t_{E_n})|0\rangle$

then, $\langle 0| O_E(t_{E_n}) \cdots O_E(t_2) O_E(t_1) O_E(-t_1) O_E(-t_2) \cdots O_E(-t_n)|0\rangle \geq 0$

$\rightsquigarrow$ Reflection Positivity

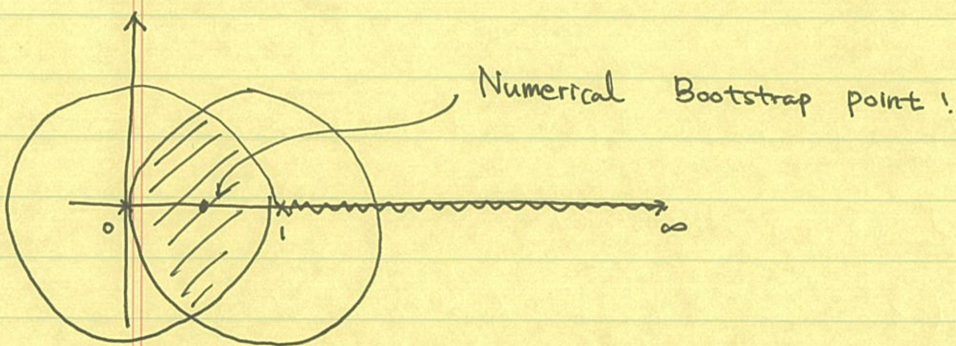
Osterwalder - Schrader reconstruction theorem

; Given a collection of Euclidean correlators satisfying reflection positivity, one can construct an unitary Lorentzian quantum field theory by analytica continuation.

· Unitarity bound

$\Delta \geq D/2 - 1$ for spin 0, $\Delta \geq D-2+l$ for spin l.

· Convergence



$u = \frac{1}{4}$, $v = \frac{1}{4}$

$\left(z = \frac{1}{2}, \bar{z} = \frac{1}{2} \right)$

· For unitary theory: OPE coefficient is always real!

∘ Solving constraint.

Let us introduce linear functional $\alpha \equiv \alpha^{m,n} \frac{\partial}{\partial z^m} \frac{\partial}{\partial \bar{z}^n}$

and notation $\mathcal{F}^{vac}(u,v) \equiv v^{\Delta_O} - u^{\Delta_O}$

$$\mathcal{F}^{\tilde{O}}_{\Delta,l}(u,v) \equiv v^{\Delta_O} \left(G^{\tilde{O}}_{\Delta,l}(u,v) \right) - u^{\Delta_O} \left(G^{\tilde{O}}_{\Delta,l}(v,u) \right)$$

Apply the linear functional to the bootstrap equation gives

$$\alpha(\mathcal{F}^{vac}) + \sum_{\substack{\tilde{O} \\ I \neq \tilde{O}}} f^2_{OO\tilde{O}} \alpha(\mathcal{F}^{\tilde{O}}_{\Delta,l}(u,v)) = 0 \quad \cdots (\#)$$

Now, try to find $\alpha_{m,n}$ such that

$\alpha(\mathcal{F}^{vac}) \geq 0$ & $\alpha(\mathcal{F}^{\tilde{O}}_{\Delta,l}(u,v)) > 0$ for all operators $\tilde{O}$

If such linear functional exist $\longrightarrow$ (#) is Violated!!

Finding $\alpha \xrightarrow{\text{mapped}}$ semi-definite programming

∘ Semi-definite programming

Start from $\alpha[\mathcal{F}_{\Delta,\ell}(u,v)] \Rightarrow$ (positive) $\otimes$ [polynomial of u & v & Δ]

Since we put $u=v=1/4$, this is polynomial of Δ

We will set $\Delta = \Delta_0 + x$ ($x \in \mathbb{R}^+$), and write as $f(x)$. → polynomial of x

Hilbert theorem says,

$$f(x) = \underbrace{\left\{ \sum_{n=0}^{d_1} b_n x^n \right\}^2}_{\text{even order}} + \underbrace{x \left\{ \sum_{n=0}^{d_2} c_n x^n \right\}^2}_{\text{odd order}}$$

Const → matrix of b_n

$[x]_d^T \equiv (1, x, x^2, \cdots x^d)$

$$= [x]_{d_1}^T \underbrace{(B^T B)}_{\substack{|||\\ \mathcal{B}\text{; real symmetric}}} [x]_{d_1} + x\, [x]_{d_2}^T \underbrace{(C^T C)}_{\substack{|||\\ \mathcal{C}\text{; real sym}}} [x]_{d_2}$$

$\therefore f(x) \geq 0 \Leftrightarrow \exists$ semi-definite matrix $\mathcal{B}$ & $\mathcal{C}$!

Summary

We know the expression of conformal block

$\Rightarrow$ Can find matrix $\mathcal{B}$ & $\mathcal{C}$

$\Rightarrow$ Check if they are semi-definite matrix!

SDPB, David Simmons-Duffin

* few remarks

a) invariance under $s \leftrightarrow u$ or $t \leftrightarrow u$?

· $t \leftrightarrow u$ constraint can be expressed in terms of $s \leftrightarrow u$ & $s \leftrightarrow t$.

· $s \leftrightarrow u$ correspond to $g(u,v) = g(u/v, \frac{1}{v})$

$\Rightarrow$ identity of conformal block.

b) $\tilde{\mathcal{O}}$ only impose even spin states.

$$Z_\mu = \frac{2x_\mu}{x^2}$$

$$\langle \mathcal{O}(x)\, \mathcal{O}(-x)\, \hat{\mathcal{O}}_\mu(0) \rangle = \frac{\lambda_{\mathcal{O}\mathcal{O}\tilde{\mathcal{O}}}}{|x|^{2\Delta_{\mathcal{O}} - \hat{\Delta}_{\mathcal{O}} + \ell}\, |x|^{2\Delta_{\mathcal{O}} - 2\ell}} Z_{\mu_1} \cdots Z_{\mu_\ell}$$

→ odd spin cannot satisfy crossing $\mathcal{O}(x) \leftrightarrow \mathcal{O}(-x)$.

· Finally, we impose some further assumptions on the spectrum.

Namely, we impose a gap for scalar operator, $\Delta \geq \Delta^*$ ($\Delta^* \geq D/2 - 1$)

and impose an unitary bound for spin operator. $\Delta \geq D-2+l$ for $l \neq 0$

Now, we have two tunable parameters (Δ_ϕ & Δ^*)

Example 1) 3D Bootstrap (1203.6064), Figure 3.

(Note that $\Delta_\sigma = \Delta_\phi$, $\Delta^* = \Delta_\varepsilon$)

Example 2) Mixed Correlator Bootstrap (1406.4858), Figure 2.

$\rightsquigarrow$ assuming only two relevant operators.

Example 3) O(N) Bootstrap (1307.6856) Figure 2.

$\rightsquigarrow$ Critical exponents for O(N) vector models.

Example 4) Bound on the central charge (1203.6064), Figure 11

$\rightsquigarrow$ (C/C_{Free}) @ ising model is given by 0.946534(11).

⊙ Modular Bootstrap

· Conformal algebra in two dimension ⇒ Virasoro Algebra.

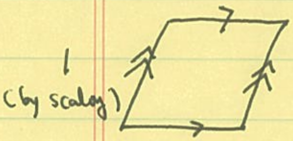
$$\begin{cases} [L_m, L_n] = (m-n) L_{m+n} - \frac{c(c^2-1)}{12} \delta_{m+n,0} & \rightsquigarrow \text{Left mover} \\ [\bar{L}_m, \bar{L}_n] = (m-n) \bar{L}_{m+n} - \frac{c(c^2-1)}{12} \delta_{m+n,0} & \rightsquigarrow \text{right mover} \\ [L_m, \bar{L}_n] = 0 & \end{cases}$$

* Conformal weight h & $\bar{h}$ labels quantum state.

Primary state ; $\overset{(-)}{L}_m |h, \bar{h}\rangle = 0$, $\left\{ \begin{matrix} L_{-m} : \text{create} \\ L_m : \text{annihilate} \end{matrix} \right\}$

Highest weight state (c, h) ⊕ desendants

Now, let us put 2D CFT on Torus



$\tau \rightarrow$ complex structure

- Observables of this theory have dependance on τ.
- It should be inv under $T: \tau \to \tau+1$ & $S: \tau \to -\frac{1}{\tau}$.

T & S generate modular group $SL(2,\mathbb{Z})$

$$SL(2,\mathbb{Z}) = \left\{ \tau \longmapsto \frac{a\tau+b}{c\tau+d} \;,\; ad-bc=1,\; a,b,c,d \in \mathbb{Z} \right\}$$

We define torus partition function

$$Z(\tau,\bar{\tau}) \equiv \mathrm{Tr}_{\mathcal{H}} \left[e^{\underbrace{2\pi i \tau}_{\equiv -\beta}(L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau}(\bar{L}_0 - \frac{c}{24})} \right]$$

Computation Example:

Let us suppose $\mathcal{H}_{vac}$ = Vac $\oplus$ Descendants.

$$\mathrm{Tr}_{\mathcal{H}_{vac}}\left[e^{-\beta(L_0-\frac{c}{24})}\right] = \langle 0| e^{-\beta(L_0-\frac{c}{24})}|0\rangle$$

$$+ \langle 0| L_1 e^{-\beta(L_0-\frac{c}{24})} L_{-1}|0\rangle \rightarrow 0, \text{ because } \langle 0|L_0|0\rangle = 0,$$

Level 1

2 degeneracy $\rightarrow$ $+ \langle 0|L_1^2 e^{-\beta(L_0-\frac{c}{24})} L_{-1}^2|0\rangle + \langle 0|L_2 e^{-\beta(L_0-\frac{c}{24})} L_{-2}|0\rangle$

Level 2

$+ \cdots$

$q \equiv e^{-\beta}$

$$= q^{-\frac{c}{24}} + 2q^{2-\frac{c}{24}} + \cdots$$

$$= q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1}{1-q^n} \equiv \chi_{vac}(\tau) \rightarrow \text{Virasoro Vacuum character.}$$

If $\mathcal{H}$ = primary of h $\oplus$ Descendants,

$$\mathrm{Tr}_{\mathcal{H}_h}\left[e^{-\beta(L_0-\frac{c}{24})}\right] = q^{h-\frac{c}{24}} + q^{h+1-\frac{c}{24}} + 2q^{h+2-\frac{c}{24}} + \cdots$$

$$= q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \equiv \chi_h(\tau) \rightarrow \text{Virasoro primary character.}$$

Character Decomposition

$$Z(\tau,\bar\tau) = \underbrace{\chi_0(\tau)\bar\chi_0(\bar\tau)}_{\substack{\text{Unique vacuum} \\ \text{: CFT axiom}}} + \sum_{h,\bar h} \underbrace{d_{h,\bar h}}_{\hookrightarrow \text{ degeneracy : positive integer}} \chi_h(\tau)\bar\chi_{\bar h}(\bar\tau)$$

$$+ \underbrace{\sum_j d_{j,0}\chi_j(\tau)\bar\chi_0(\bar\tau) + \sum_{\bar j} d_{0,\bar j}\chi_0(\tau)\bar\chi_{\bar j}(\bar\tau)}_{\hookrightarrow \text{ for conserved currents}}$$

For simplicity, let us drop the contribution of c.c. Then, modular inv says,

$$\left.\begin{array}{c} Z(\tau,\bar\tau) \\ \shortparallel \\ Z(-\frac{1}{\tau},-\frac{1}{\bar\tau}) \end{array}\right\} \rightarrow 0 = \underbrace{\chi_0(\tau)\bar\chi_0(\bar\tau) - \chi_0(-\tfrac{1}{\tau})\bar\chi_0(-\tfrac{1}{\bar\tau})}_{\equiv F_{vac}} + \sum_{h,\bar h} d_{h,\bar h}\Big[\underbrace{\chi_h(\tau)\bar\chi_{\bar h}(\bar\tau) - \chi_h(-\tfrac{1}{\tau})\bar\chi_{\bar h}(-\tfrac{1}{\bar\tau})}_{F_{h,\bar h}}\Big] \quad \cdots\cdots (**)$$

Then, this has exactly same form with bootstrap equation!

Moreover, we can refine the partition function (multiplying $\tau^{\frac{1}{2}}\eta(q)$) to make polynomial F_{vac} & $F_{h,\bar h}$. $\rightarrow$ Semi-definite programming!

Denote $\tau = ie^{z}$, modular point correspond to $z=0$.

Introduce $\alpha \equiv \alpha^{m,n}\frac{\partial}{\partial z^m}\frac{\partial}{\partial \bar z^n}$ and apply to $(**)$.

$$\alpha(F_0) + \sum_{h,\bar h} d_{h,\bar h}\,\alpha(F_{h,\bar h}) = 0 .$$

Find linear functional such that $\alpha(F_0) \geq 0$ & $\alpha(F_{h,\bar h}) > 0$

We assume twist gap : give a gap on the twist $\tau \equiv \Delta - l$.

i.e. $\Delta - l \geq \Delta_t$ $\Rightarrow$ We have two parameters : $\underline{c \ \& \ \Delta_t}$ //

Main Result : Figure 1,2 of 1708.08815

I) $C \leq 8$; level-1 Wess-Zumino-Witten (WZW) models

at $\left\{ \begin{array}{l} C = 1,\ 2,\ \frac{14}{5},\ 4,\ \frac{26}{5},\ 6,\ 7,\ \frac{38}{5},\ 8 \\ G = A_1,\ A_2,\ G_2,\ D_4,\ F_4,\ E_6,\ E_7,\ E_{7\frac{1}{2}},\ E_8 \end{array} \right\}$

Extremal Functional Method (EFM) confirms partition ftns of above theories!

- Modular Differential Equation

For n-th order rational CFT (RCFT), extended characters can be computed by below n-th order MDE.

modular weight of test ftn.

$$D_\tau^n f(\tau) + \sum_{k=0}^{n-1} \underbrace{\phi_{2(n-k)}(\tau)}_{\downarrow} D_\tau^k f(\tau) = 0 \quad , \quad D_\tau \equiv \partial_\tau - \frac{1}{6} i\pi r E_2(\tau)$$

determined by requiring uniform modular weight.

ex) $n=2$. $\underbrace{D_\tau^2 f(\tau)}_{\text{Weigh 4}} + \cancel{\mu_1 \underbrace{E_2}_{2} \underbrace{D_\tau f(\tau)}_{\text{weigt 2}}} + \mu_2 \underbrace{E_4}_{4} \underbrace{f(\tau)}_{0} = 0$

$\because E_2$ is not a modular form.

Try $f(\tau) = q^\alpha (a_0 + a_1 q + a_2 q^2 + \cdots)$ and check order by order.

$\rightarrow$ Only when $C = 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, \frac{38}{5}, 8$, all a_i can be positive integer!

Namely, modular bootstrap catches all possible $n=2$ ($\ell=0$) RCFTs!

II) Peak @ $C=24$ $\Rightarrow$ Monster CFT

$$Z(\tau, \bar{\tau}) = J(\tau) J(\bar{\tau})$$

$\hookrightarrow$ Klein - j inv.