Entanglement transfer using local operations

A. Neven, D. Gunn and B. Kraus

Institut für Theoretische Physik, Universität Innsbruck, 6020 Innsbruck, Austria

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Introduction: quantum entanglement

Entanglement = global property of multipartite systems

\[ |\psi\rangle \text{ entangled} \iff |\psi\rangle \neq |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle \]
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A system of locally prepared states cannot be entangled!
Local manipulation of entanglement

Distant experimenters sharing the parties of a multipartite state can:

- perform local operations (e.g. measurements)
- share and exploit information through classical communication

source: image by Alvaro Feito for wikipedia.org
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BUT entangled states can be manipulated locally.

Example: Transform entangled state $|\psi\rangle$ into a new entangled state $|\phi\rangle$

$|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$

source: image by Alvaro Feito for wikipedia.org
Bipartite states

**Majorization condition** [M. Nielsen, PRL 83, 436 (1999)]

For bipartite states $|\psi\rangle, |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle \text{ iff } \lambda_\psi \prec \lambda_\phi,$$

where $\lambda_\psi$ ( $\lambda_\phi$ ) are the eigenvalues of $\text{Tr}_1|\psi\rangle\langle\psi|$ ( $\text{Tr}_1|\phi\rangle\langle\phi|$ ).
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$$|\psi\rangle \overset{\text{LOCC}}{\longrightarrow} |\phi\rangle \iff \lambda_\psi < \lambda_\phi,$$

where $\lambda_\psi (\lambda_\phi)$ are the eigenvalues of $\text{Tr}_1 |\psi\rangle\langle\psi| \ (\text{Tr}_1 |\phi\rangle\langle\phi|)$.

Simple characterization of LOCC transformations.
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The state $|\Phi^+\rangle = 1/\sqrt{d} \sum_{i=1}^{d} |i\rangle_1 |i\rangle_2$ can reach any other state via LOCC.
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**Maximally entangled state**
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- Simple characterization of LOCC transformations.
- The state $|\Phi^+\rangle = 1/\sqrt{d} \sum_{i=1}^{d} |i\rangle_1 |i\rangle_2$ can reach any other state via LOCC.

**Maximally entangled state**

- Universal resource in the bipartite LOCC framework.
Universal resource for multipartite states?

For states of 3 qubits:

- There is no maximally entangled state.
- BUT there exists a maximally entangled set.

**Maximally entangled set (MES)** [J. De Vicente et al., PRL 111,110502 (2013)]

In any quantum system, the MES contains the minimal number of states such that any state of the system can be reached from a state of the MES (but no state in the MES can be reached from a state outside the MES).
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However,

MES in homogeneous systems [D. Sauerwein et al., PRX 8,031020 (2018)]

In any $N$-qudit system (with $N \geq 4$), the MES contains almost all states of the Hilbert space.
Increasing the resource of the parties

\[ |\psi\rangle \]
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\[ |\psi\rangle \]

\[ \psi \]
Increasing the resource of the parties

A

|ψ⟩

|φ⟩

B
Increasing the resource of the parties

\[ |\psi\rangle \]

\[ |\phi\rangle \]
Increasing the resource of the parties

Given $|\psi_1\rangle$, $|\psi_2\rangle$:

$$|\psi_1\rangle \xrightarrow{\text{LOCC}} |\psi_2\rangle,$$
Increasing the resource of the parties

![Diagram showing entanglement transfer using local operations](image)

- Given $|\psi_1\rangle$, $|\psi_2\rangle$: $|\psi_1\rangle \xrightarrow{\text{LOCC}} |\psi_2\rangle$,

- $\exists\, |\phi_1\rangle$, $|\phi_2\rangle$: $|\psi_1\rangle \otimes |\phi_1\rangle \xrightarrow{\text{LOCC}} |\psi_2\rangle \otimes |\phi_2\rangle$
Increasing the resource of the parties

Given $|\psi_1\rangle$, $|\psi_2\rangle$: $|\psi_1\rangle \xrightarrow{\text{LOCC}} |\psi_2\rangle$,

$\exists \phi_1, \phi_2$: $|\psi_1\rangle \otimes |\phi_1\rangle \xrightarrow{\text{LOCC}} |\psi_2\rangle \otimes |\phi_2\rangle$

$|\phi_1\rangle = |\phi_2\rangle \Rightarrow$ catalytic LOCC transformation
Increasing the resource of the parties

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Full characterization still open.
Increasing the resource of the parties

- Given $|\psi_1\rangle, |\psi_2\rangle : |\psi_1\rangle \xrightarrow{LOCC} |\psi_2\rangle$,
- $\exists \, |\phi_1\rangle, |\phi_2\rangle : |\psi_1\rangle \otimes |\phi_1\rangle \xrightarrow{LOCC} |\psi_2\rangle \otimes |\phi_2\rangle$
Increasing the resource of the parties

Given $|\psi_1\rangle$, $|\psi_2\rangle$:

$|\psi_1\rangle \xrightarrow{\text{LOCC}} |\psi_2\rangle$,

$\exists \phi_1, \phi_2$:

$|\psi_1\rangle \otimes |\phi_1\rangle \xrightarrow{\text{LU}} |\psi_2\rangle \otimes |\phi_2\rangle$
2-Qubit state transformation

\[ |\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \Leftrightarrow \sum_{i=1}^{2} \sqrt{\lambda_i} |i\rangle |i\rangle \]

\[ |\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \Leftrightarrow \sum_{i=1}^{d} \sqrt{\mu_i} |i\rangle |i\rangle \]
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Entanglement parameter of \(|\psi\rangle\): \(a_\psi \equiv \frac{\lambda_2}{\lambda_1} \in (0, 1]\)
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\(d = 2\)

- \(|\psi\rangle \otimes |\phi\rangle \leftrightarrow |\psi\rangle \otimes |\phi\rangle, \quad U_1, U_2 = Id\)
- \(|\psi\rangle \otimes |\phi\rangle \leftrightarrow |\phi\rangle \otimes |\psi\rangle, \quad U_1, U_2 = swap\)
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\[ |\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow{LU} \sum_{i=1}^{2} \sqrt{\lambda_i} |i\rangle |i\rangle \]

\[ |\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \xrightarrow{LU} \sum_{i=1}^{d} \sqrt{\mu_i} |i\rangle |i\rangle \]

\( d \) even: possible to split \(|\phi\rangle\) in \(|\phi_1\rangle \otimes |\phi_2\rangle \in (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^{d/2} \otimes \mathbb{C}^{d/2}) \)

- \(|\psi\rangle \otimes (|\phi_1\rangle \otimes |\phi_2\rangle) \leftrightarrow |\phi_1\rangle \otimes (|\psi\rangle \otimes |\phi_2\rangle)\), \( U_1, U_2 = \text{partial swap} \)
2-Qubit state transformation

\[ |ψ⟩ \in \mathbb{C}^2 \otimes \mathbb{C}^2 \overset{LU}{\cong} \sum_{i=1}^{2} \sqrt{\lambda_i} |i⟩ |i⟩ \]

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If \( E(ψ) \neq E(φ_1) \), some entanglement is transfered!
2-Qubit state transformation

\[ |\psi\rangle \mapsto |\phi\rangle \]

\[ |\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow{LU} \sum_{i=1}^{2} \sqrt{\lambda_i} |i\rangle |i\rangle \]

\[ |\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \xrightarrow{LU} \sum_{i=1}^{d} \sqrt{\mu_i} |i\rangle |i\rangle \]

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\( \bullet \) \( |\psi\rangle \otimes (|\phi_1\rangle \otimes |\phi_2\rangle) \leftrightarrow |\phi_1\rangle \otimes (|\psi\rangle \otimes |\phi_2\rangle), \quad U_1, U_2 = \text{partial swap} \)

\( \iff \) If \( E(\psi) \neq E(\phi_1) \), some entanglement is transferred!

Using Von Neumann entropy: \( S(\psi \otimes \phi) = S(\psi) + S(\phi) \)

Amount of entanglement transferred: \( \Delta S(\psi) = -\Delta S(\phi) \)
Entanglement transfer for odd dimensions

- No partial swap for entanglement transfer...
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Example: $|\phi\rangle \in C^3 \otimes C^3$ and transformation $|\psi_i\rangle \otimes |\phi_i\rangle \leftrightarrow |\psi_f\rangle \otimes |\phi_f\rangle$
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Only possible if $\tilde{\lambda}_{\phi_i} = \left( \sqrt{\frac{1}{1+a+a^2}}, \sqrt{\frac{a}{1+a+a^2}}, \sqrt{\frac{a^2}{1+a+a^2}} \right)$
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Initial vs final qubit entanglement

![Graph](image-url)
Entanglement transfer for odd dimensions

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Entanglement transfer

$$S(\psi_i) - S(\psi_f)$$
Entanglement transfer for odd dimensions

For higher odd dimensions:

- several possible $\tilde{\lambda}_{\phi_i}$
- several possible entanglement transfer
Entanglement transfer for odd dimensions

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Initial vs final qubit entanglement, $d = 7$
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Entanglement transfer, $d = 7$
Entanglement transfer for odd dimensions

For higher odd dimensions:

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**General result**

Given a 2-qubit state $|\psi_i\rangle$ with Schmidt coefficients $\{1, a\}/(1 + a)$ characterized by the entanglement parameter $a \in (0, 1]$, and a 2-qudit state $|\phi_i\rangle$ with odd dimension $d \geq 3$, then there is entanglement transfer protocol using LU operations if and only if the final 2-qubit state $|\psi_f\rangle$ is characterized by the entanglement parameter

$$a^{d_1/d_2},$$

with $d_1, d_2$ any odd integers satisfying $1 \leq d_2 < d_1 \leq d$. 
Conclusions and outlook

In the LOCC framework:

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Applications:

- Entanglement distribution in qubit states networks
- Authentication protocols based on the specificity of the allowed transformations
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Thank you for your attention!