Entanglement transfer using local operations

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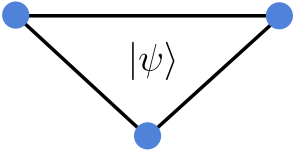
Introduction: quantum entanglement

Entanglement = global property of multipartite systems



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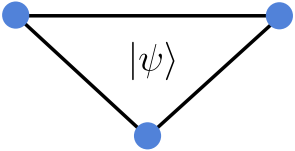

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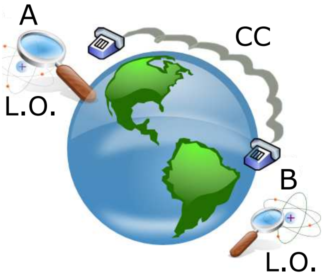
➥ A system of locally prepared states cannot be entangled!

Local manipulation of entanglement

Distant experimenters sharing the parties of a multipartite state can:

- perform local operations (e.g. measurements)
- share and exploit information through classical communication



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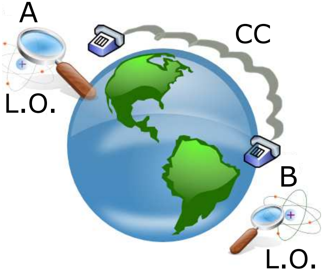

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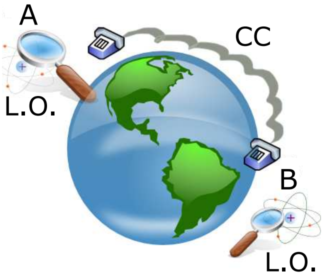

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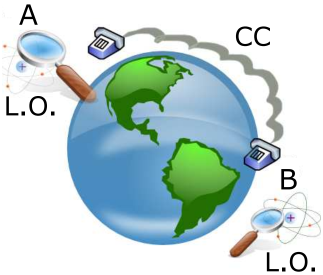

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Example: Transform entangled state $|\psi\rangle$ into a new entangled state $|\phi\rangle$

$$|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$$

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Bipartite states

Majorization condition [M. Nielsen, PRL **83**, 436 (1999)]

For bipartite states $|\psi\rangle,\ |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$|\psi\rangle \xrightarrow{LOCC} |\phi\rangle \text{ iff } \lambda_\psi \prec \lambda_\phi,$$

where λ_ψ (λ_ϕ) are the eigenvalues of $\mathrm{Tr}_1|\psi\rangle\langle\psi|$ ($\mathrm{Tr}_1|\phi\rangle\langle\phi|$).

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  Maximally entangled state

  - Universal resource in the bipartite LOCC framework.

Universal resource for multipartite states?

For states of 3 qubits:

- There is no maximally entangled state.
- BUT there exists a maximally entangled set.

Maximally entangled set (MES) [J. De Vicente *et al.*, PRL **111**,110502 (2013)]

In any quantum system, the MES contains the minimal number of states such that any state of the system can be reached form a state of the MES (but no state in the MES can be reached form a state outside the MES).

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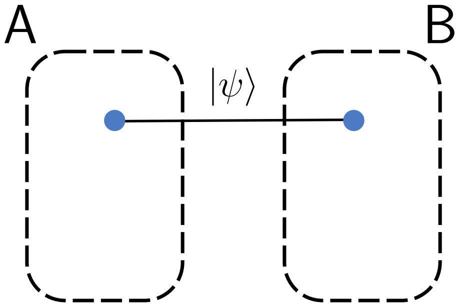
However,

MES in homogeneous systems [D. Sauerwein *et al.*, PRX **8**,031020 (2018)]

In any N-qudit system (with $N \geq 4$), the MES contains almost all states of the Hilbert space.

Increasing the resource of the parties



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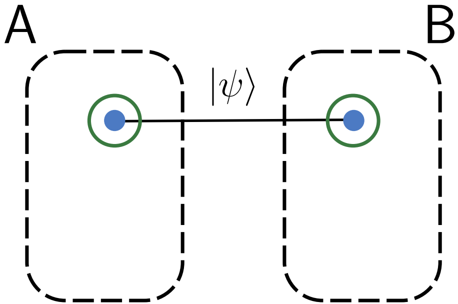

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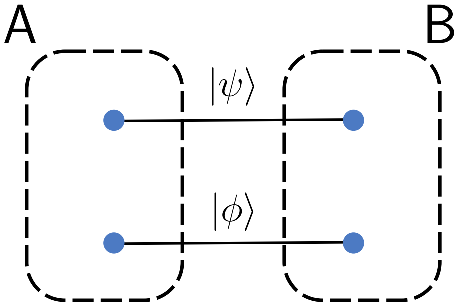

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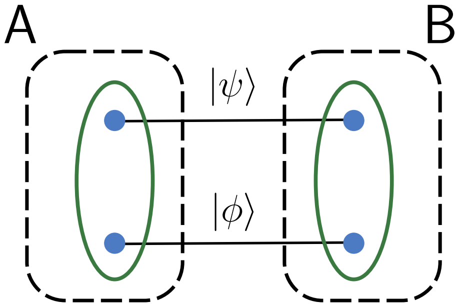

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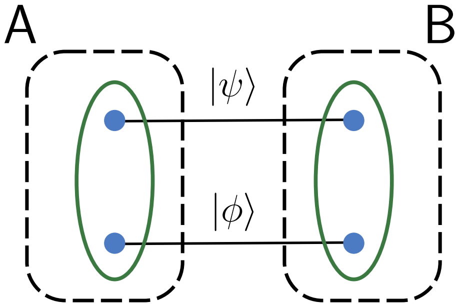

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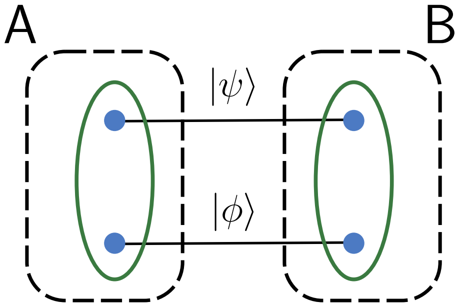

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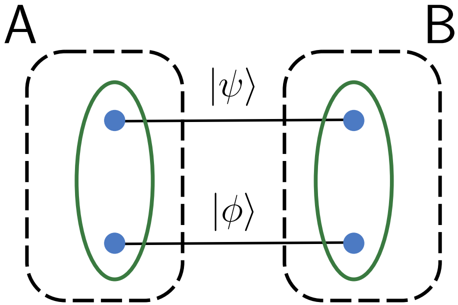

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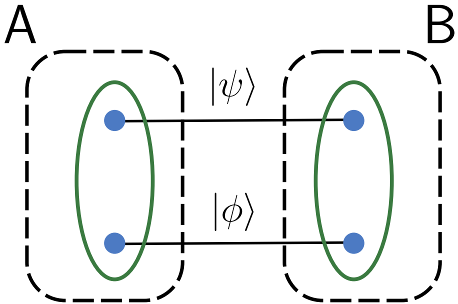

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Full characterization still open.

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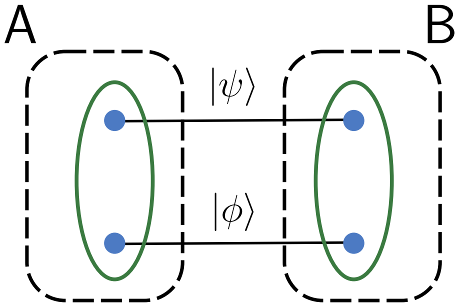

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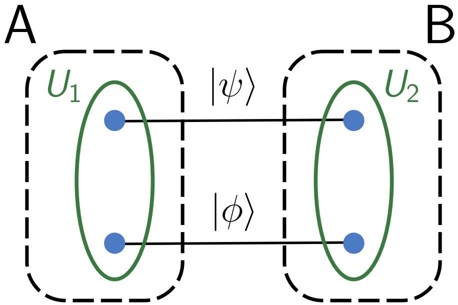

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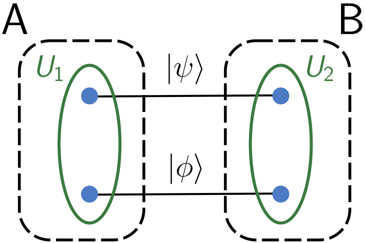
2-Qubit state transformation



$$|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \overset{LU}{\simeq} \sum_{i=1}^{2} \sqrt{\lambda_i}\, |i\rangle|i\rangle$$

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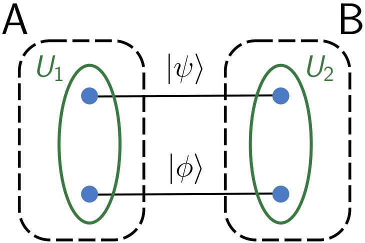

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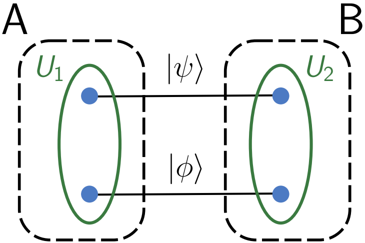

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$d = 2$

- $|\psi\rangle \otimes |\phi\rangle \leftrightarrow |\psi\rangle \otimes |\phi\rangle, \quad U_1, U_2 = Id$
- $|\psi\rangle \otimes |\phi\rangle \leftrightarrow |\phi\rangle \otimes |\psi\rangle, \quad U_1, U_2 = swap$

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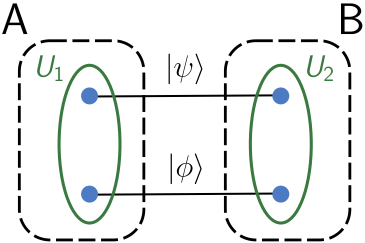

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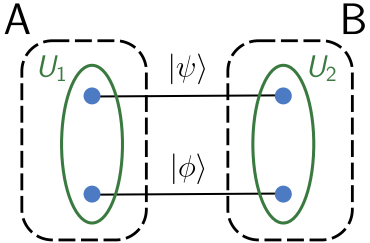

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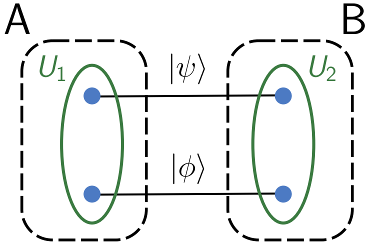

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Using Von Neumann entropy: $S(\psi \otimes \phi) = S(\psi) + S(\phi)$

Amount of entanglement transfered: $\Delta S(\psi) = -\Delta S(\phi)$

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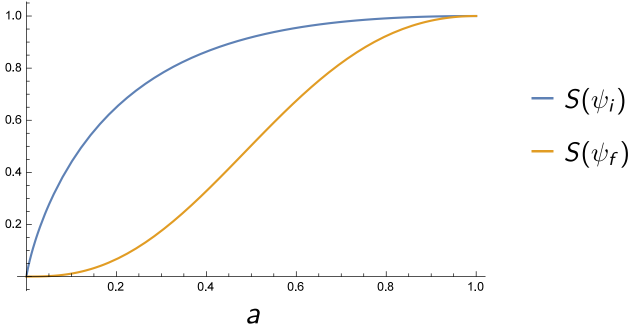

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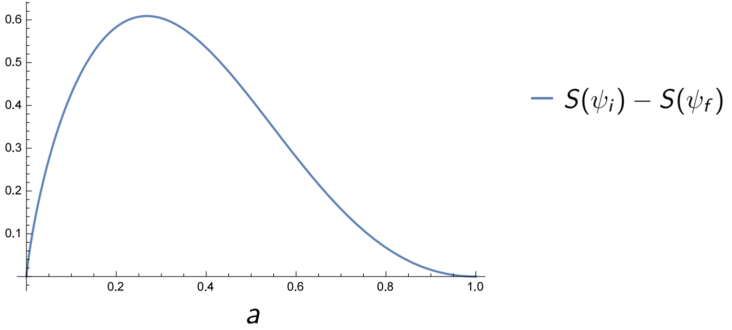
Entanglement transfer



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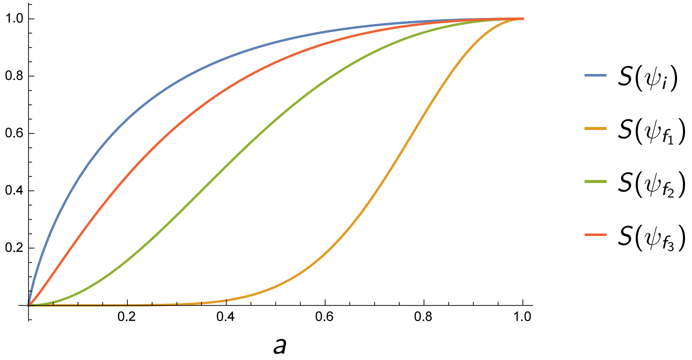
Initial vs final qubit entanglement, $d = 7$



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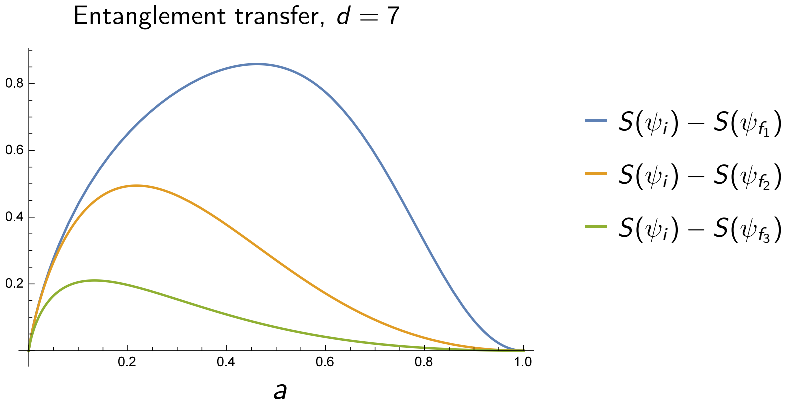

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General result

Given a 2-qubit state $|\psi_i\rangle$ with Schmidt coefficients $\{1, a\}/(1+a)$ characterized by the entanglement parameter $a \in (0, 1]$, and a 2-qudit state $|\phi_i\rangle$ with odd dimension $d \geq 3$, then there is entanglement transfer protocol using LU operations if and only if the final 2-qubit state $|\psi_f\rangle$ is characterized by the entanglement parameter

$$a^{d_1/d_2},$$

with d_1, d_2 any odd integers satisfying $1 \leq d_2 < d_1 \leq d$.

Conclusions and outlook

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