

Aspects of Gauge-Strings Duality

Carlos Nunez

Santiago de Compostela, June 2019

Outline

- ① I will discuss work in the area of AdS/CFT. The focus will be on the general ideas and outcomes.
- ② The knowledge of field theory results at strong coupling allows us to calculate aspects of a geometry and vice-versa. The viewpoint adopted in this talk will be mostly driven by the QFT.
- ③ The particular examples I will discuss today will be in the context of an $N = 2$ SCFT in four dimensions and $N = (1, 0)$ SCFTs in six dimensions . If time permits, some unpublished results on 2d $(4,0)$ SCFTs will be briefly discussed.
- ④ This talk mostly based on work recently developed with: K. Filippas, Y. Lozano, N. Macpherson, J. van Gorsel, S. Speziali, A. Ramirez, D. Thompson, D. Roychowdhuri and S. Zacarías.

SCFTs in diverse dimensions (8+8 SUSY). An *incomplete* picture.

- $d=6$: Hanany-Zaffaroni, Brunner-Karch —D6-D8-NS5— Gaiotto, Tomasiello, Apruzzi, Fazzi, Rosa, Passias, Cremonesi.
- $d=5$: Aharony-Hanany-Kol —D5-D7-NS5— D'Hoker, Gutperle, Uhlemann, Trivella, Karch.
- $d=4$: Gaiotto —D4-D6-NS5— Gaiotto, Maldacena; Aharony, Berkooz, Berdichevsky; Stefanski, Reid-Edwards.
- $d=3$: Gaiotto-Witten —D3-D5-NS5— D'Hoker, Estes, Gutperle; Assel, Bachas, Gomis.
- $d=2$: (0,4) SCFT —D2-D4-D6-D8-NS5. Lozano, Macpherson, Nunez, Ramirez.
- $d=1$ (not a SCFT): Lin, Lunin, Maldacena —D0-D2-NS5— Lin, Maldacena.

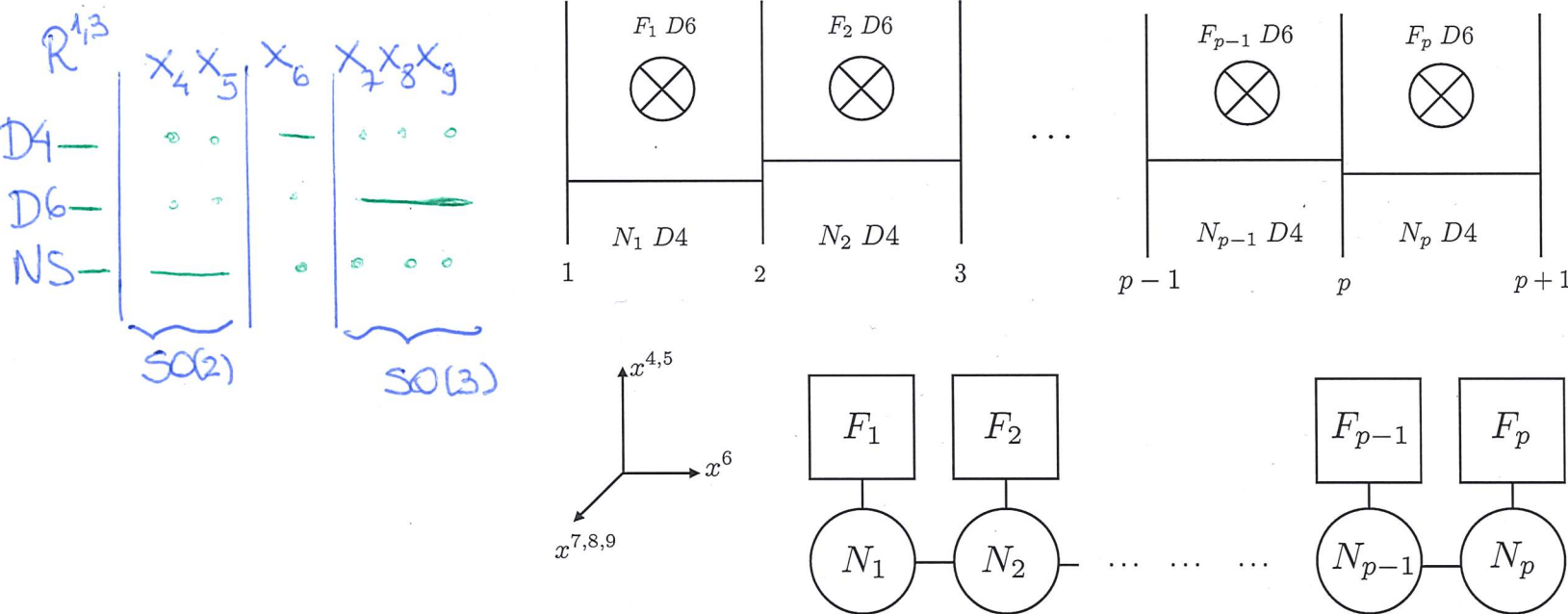
In all these examples, there are eight Poincare SUSYs and (at least) an $SU(2)$ R-symmetry. The dual backgrounds have the form

$$ds^2 \sim f_1 AdS_{d+1} + f_2 d\Omega_2 + f_3 d\Omega_{5-d} + f_4 d\eta^2 + f_5 d\sigma^2; \quad f_i(\sigma, \eta).$$

There are also NS B_2 , Φ and RR fields respecting the isometries above.

Let us start with the four dimensional case. Consider 4d $\mathcal{N} = 2$ SCFTs.

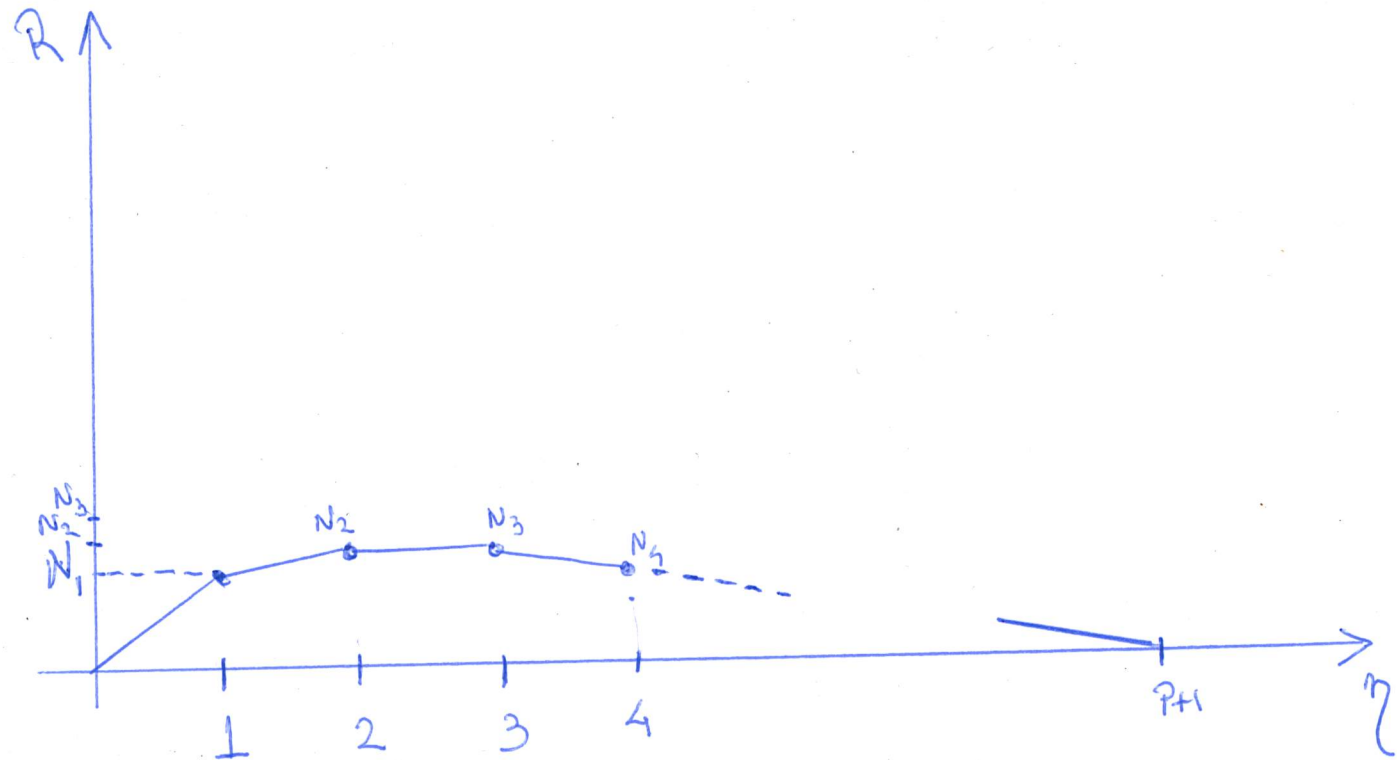
We shall focus on the CFTs for which we can give a quiver-like description and a Hanany-Witten set-up.



The beta function of these theories is $\beta \sim (2N_c - N_f)$. This implies

$$2N_1 = N_2 + F_1, \quad 2N_2 = N_1 + N_3 + F_2, \dots, 2N_P = F_P + N_{P-1}.$$

One can define a 'Rank-function' $R(\eta)$, that due to the condition above turns out to be a convex polygonal.



The holographic description captures more Physics. For four dimensional $\mathcal{N} = 2$ CFTs, Lin, Lunin and Maldacena wrote in 2005 the Type IIA backgrounds ($\alpha' = g_s = 1$)

$$ds_{10}^2 = 4f_1 ds_{AdS_5}^2 + f_2(d\sigma^2 + d\eta^2) + f_3 d\Omega_2(\chi, \xi) + f_4 d\beta^2.$$

$$B_2 = f_5 d\Omega_2(\chi, \xi), \quad C_1 = f_6 d\beta, \quad A_3 = f_7 d\beta \wedge d\Omega_2, \quad e^{2\phi} = f_8.$$

The functions $f_i(\sigma, \eta)$ can be all written in terms of a function $V(\sigma, \eta)$ and its derivatives, $f_i \sim f_i(V, \partial_\sigma V, \partial_\eta V)$. } using $\delta\psi_\mu = \delta\lambda = 0$.

The function $V(\sigma, \eta)$ satisfies a Laplace-like equation with certain given boundary conditions that avoid nasty singularities,

$$\sigma \partial_\sigma [\sigma \partial_\sigma V] + \sigma^2 \partial_\eta^2 V = 0,$$

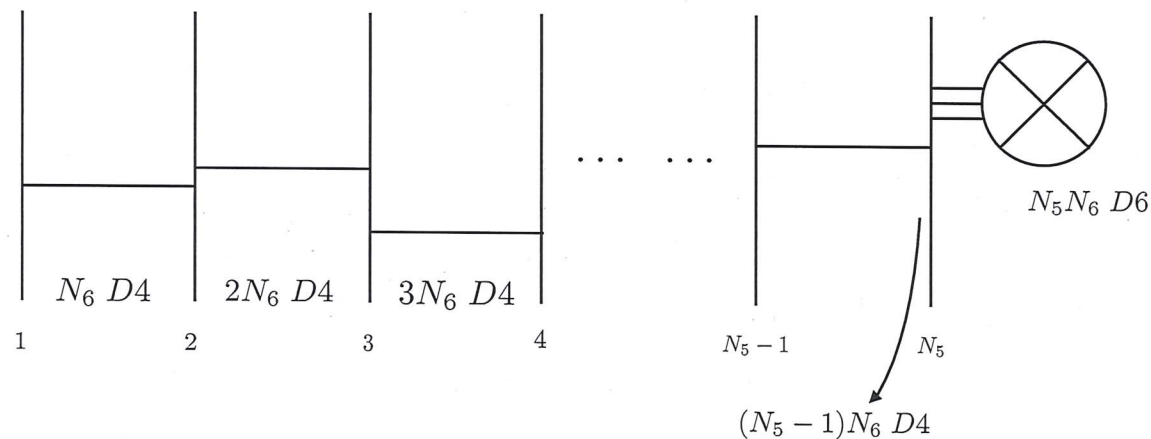
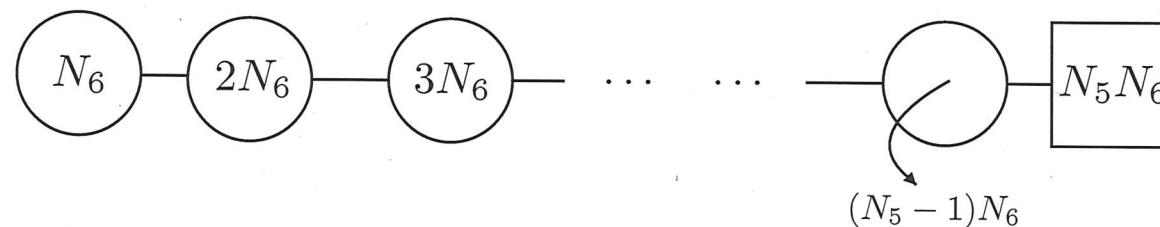
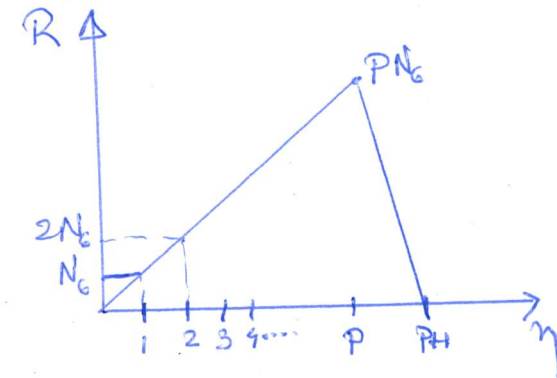
$$V(\sigma \rightarrow \infty, \eta) \rightarrow 0,$$

$$R(\eta) = \sigma \partial_\sigma V(\sigma, \eta)|_{\sigma=0}, \quad R(0) = R(P+1) = 0.$$

Various aspects of the Physics of the CFT are encoded in the function $R(\eta)$.

A simple example of an acceptable $R(\eta)$ is, $(N_5 = P + 1)$

$$R(\eta) = N_6 \begin{cases} \eta & 0 \leq \eta \leq P \\ P(P + 1 - \eta) & P \leq \eta \leq P + 1. \end{cases}$$



Given $R(\eta)$, one can write the solution for $V(\sigma, \eta)$ as a Fourier series

$$V(\sigma, \eta) = - \sum_{n=1}^{\infty} \frac{c_n}{w_n} K_0(w_n \sigma) \sin(w_n \eta), \quad w_n = \frac{n\pi}{P+1}.$$
$$c_n = \frac{n\pi}{(P+1)^2} \int_{-(P+1)}^{P+1} R(\eta) \sin(w_n \eta) d\eta.$$

These backgrounds are trustable when the numbers P, N_6 are large.

Using this, one can calculate the Page charges and make a correspondence with the Hanany-Witten brane set-up. The number of D4, D6 and NS branes comes out as function of the function $R(\eta)$. Similarly the linking numbers of these branes can be computed. More interestingly, the central charge and Entanglement Entropy have expressions in terms of integrals of $R(\eta)$.

Let me assure you that they coincide exactly with what can be calculated using either the Hanany-Witten set up, or non-perturbative results of $\mathcal{N} = 2$ SCFTs.

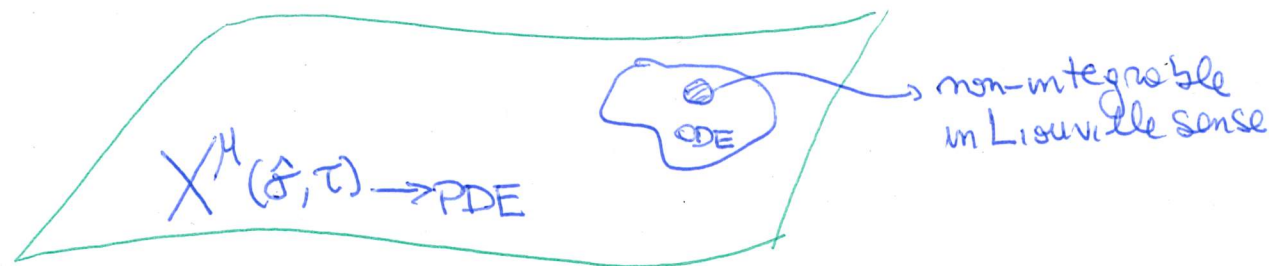
Let me focus on a particular aspect of these systems: Integrability.

One can show that the string sigma model in a given background is classically integrable, if the equations of motion can be written in terms of a Lax pair. In general, it is very difficult to find such Lax pair.

It is much easier to 'disprove Integrability'. By proposing a semiclassical string soliton $X^\mu(\tilde{\sigma}, \tau)$ and studying the coupled non-linear partial differential equations of motion. This is still quite complicated to do in practice!

A more modest approach is to consider a simple string soliton, whose equations of motion admit a one-dimensional truncation and reduce to ordinary differential equations.

If this truncation is Liouville non-integrable, the whole sigma model is also non-Integrable.



Consider the NS sector of the $\mathcal{N} = 2$ Gaiotto-Maldacena solutions

$$ds_{10}^2 = 4f_1 ds_{AdS_5}^2 + f_2(d\sigma^2 + d\eta^2) + f_3 d\Omega_2(\chi, \xi) + f_4 d\beta^2.$$
$$B_2 = f_5 d\Omega_2(\chi, \xi).$$

Propose a string solution of the form,

$$t = t(\tau), \quad \sigma = \sigma(\tau), \quad \eta = \eta(\tau), \quad \chi = \chi(\tau); \quad \xi = k\tilde{\sigma}, \quad \beta = \lambda\tilde{\sigma}.$$

Carefully studying the equations of motion and Virasoro constraint, one finds a set of non-linear and couple ordinary differential equations for $\ddot{\sigma}, \ddot{\eta}, \ddot{\chi}$, in terms of first derivatives and the potential function $V(\sigma, \tau)$.

One solution is $\sigma = 0, \eta = E\tau, \chi = 0, \dot{t} = E/f_1$.

One then follows developments by mathematicians.

Consider the previous simple solution and a small-order ϵ -variation of it

$$\eta(\tau) = \eta_s = E\tau, \quad \chi = 0 + \epsilon z(\tau).$$

This leads us (at second order in ϵ) to a linear second order differential equation

$$\ddot{z}(\tau) + \mathcal{B}\dot{z}(\tau) + \mathcal{A}z(\tau) = 0,$$
$$\mathcal{A} = \left(k^2 - k\dot{\eta} \frac{\partial_{\eta} f_5}{f_3}\right)|_{\eta=\eta_s}, \quad \mathcal{B} = (\dot{\eta} \partial_{\eta} \log f_3)|_{\eta=\eta_s}.$$

There are criteria due to Kovacic to decide the Liouvillian integrability (or not) of this differential equation.

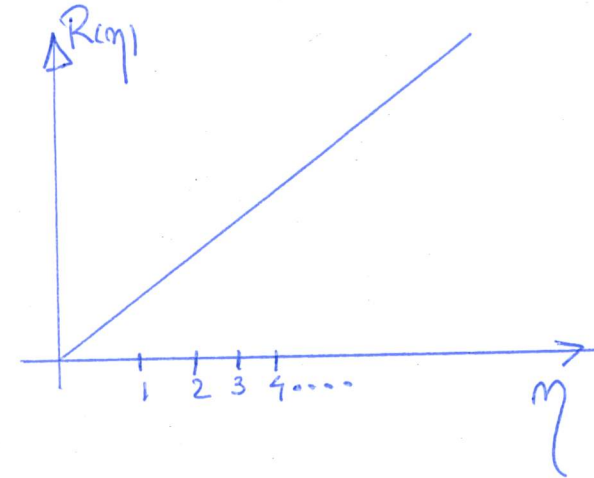
When applied to the potentials written in terms of a Fourier-Bessel series, we find that—unless the wrapping $k = 0$ —all of them are non-integrable.

Except for one $V(\sigma, \eta)$!

The only potential, for which one finds an integrable soliton is very simple,

$$V_{ST} = N_c \left(\eta \log \sigma - \eta \frac{\sigma^2}{2} + \frac{\eta^3}{3} \right).$$

$$R_{ST} = \sigma \partial_\sigma V_{ST}|_{\sigma=0} = N_c \eta.$$



Notice that we do not satisfy the boundary conditions $V(\sigma \rightarrow \infty, \eta) = 0$, $R(P+1) = 0$. We expect to find a badly singular behaviour.

The background that is derived from this potential reads,

$$ds^2 = AdS_5 + \frac{d\sigma^2 + d\eta^2}{1 - \sigma^2} + \frac{\eta^2(1 - \sigma^2)}{4\eta^2 + (1 - \sigma^2)^2} d\Omega_2 + \sigma^2 d\beta^2,$$

$$e^{-2\phi} = (1 - \sigma^2)[4\eta^2 + (1 - \sigma^2)^2], \quad B_2 = \frac{2\eta^3}{4\eta^2 + (1 - \sigma^2)^2} d\Omega_2,$$

$$A_1 = 2(1 - \sigma^2)^2 d\beta, \quad F_4 = B_2 \wedge F_2.$$

This background was written by Sfetsos and Thompson in 2011.

They obtained it by applying non-Abelian T-duality on $AdS_5 \times S^5$.

If one computes the Ricci scalar, one finds that there is an ugly singularity at $\sigma = 1$. Close to that point there is a continuous of NS five branes.

Also, since the quiver never ends, various observables in the CFT are not well-defined.



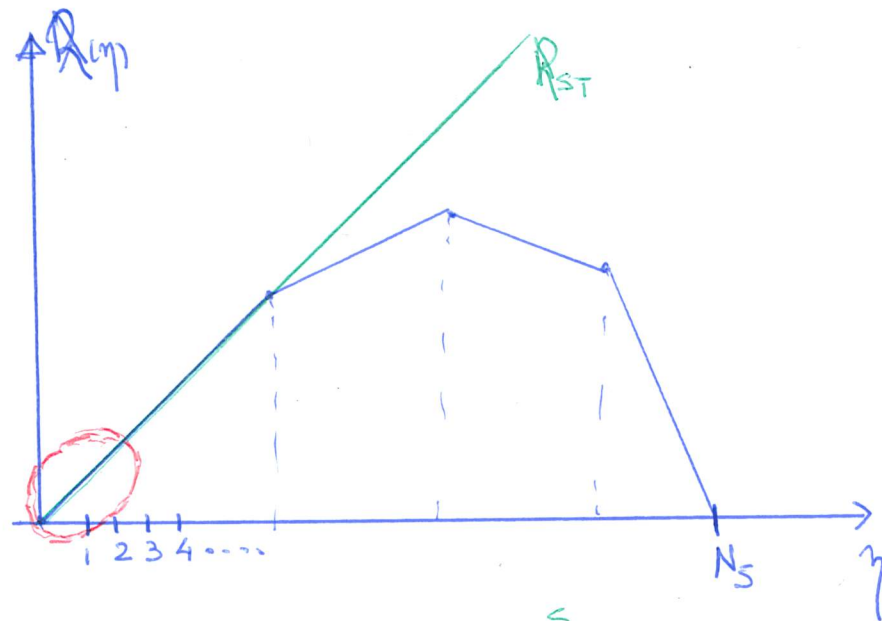
Interestingly, for this background a Lax pair was written by Borsato and Wulff (2017). The system is classically integrable.

One may discuss how to cure the singularity and turn this background into a good dual to 4d SCFTs (Lozano-Nunez, 2016).

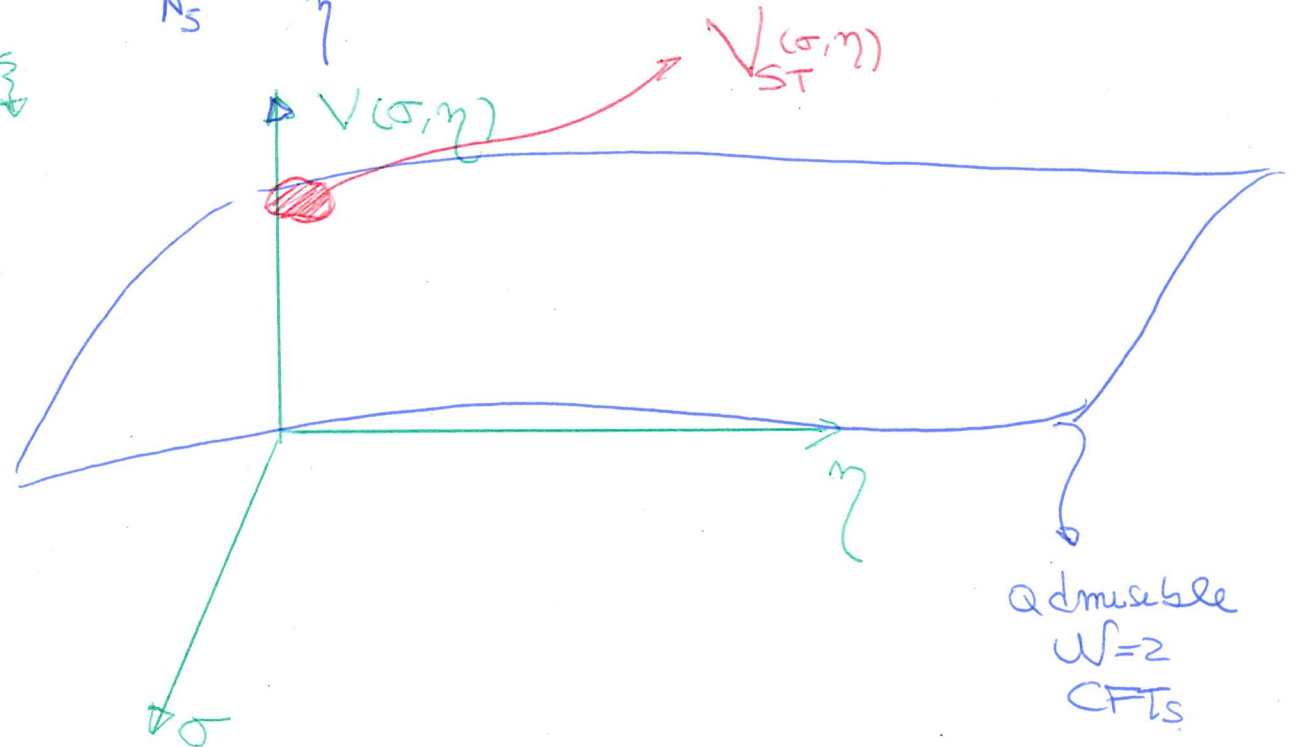
I will not elaborate on this today. Instead, let me be pictorial about what is done,

$$V_{ST} = N \left[\eta \log \sigma + \eta \frac{\sigma^2}{2} + \frac{\eta^3}{3} \right] ; \quad \lambda_{ST}(\eta) = \eta \cdot N \leadsto \textcircled{N} - \textcircled{2N} - \textcircled{3N} - \textcircled{4N} - \dots$$

(NOT a good CFT in 4d!)



\Downarrow



Once we cure the singularity of the background (in any proposed way), we make the 4d SCFT sensible, but we spoil integrability!

Notice two peculiar things about the Sfetsos-Thompson background

- The prefactor in front of the AdS_5 is a constant, independent of the coordinates (σ, η) .
- The flavour structure of the SCFT dual to the background is very simple, there are actually no-flavours!

This suggest that ALL $\mathcal{N} = 2$ SCFTs contain a 'core' or a sector that is integrable.

In other words, the generic Gaiotto-Maldacena background contains a Sfetsos-Thompson background inside of it.

Let me draw some general lessons from this.

We started by considering $\mathcal{N} = 2$ linear quiver SCFTs. We worked with their dual string description.

All the dynamical information is encoded in a function $V(\sigma, \eta)$ solving a Laplace equation with boundary condition in terms of $R(\eta)$.

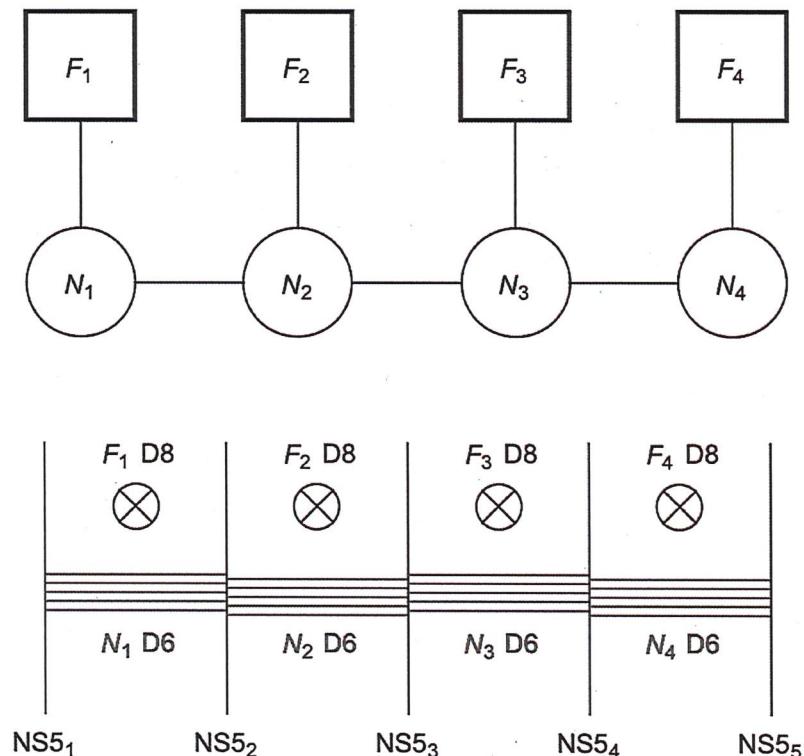
Inside all of these different potentials $V(\sigma, \eta)$ and functions $R(\eta)$, there is a particular one. It is the potential leading to the Sfetsos-Thompson background.

The 4d field theoretical interpretation of such isolated background is not solid. Aside from this, the Sfetsos-Thompson background has the special property of having an integrable sigma model.

The Sfetsos-Thompson solution should be understood as 'needing a completion' that the Gaiotto-Maldacena backgrounds provide. I can elaborate on this if there is interest and time at the end of this talk.

Let us now discuss the situation in six dimensional $N = (1, 0)$ SCFTs

These SCFTs appear when considering an array of D6 (colour) branes, D8 (flavour) branes and NS-five branes. In 1997, it was suggested by Seiberg that these theories reach a conformal point in the UV. We can draw a generic quiver and its associated Hanany-Witten set-up



The conditions for these field theories to be non-anomalous is (again!) that for each node $N_f = 2N_c$.

The SCFTs have the (bosonic part) global symmetries $SO(2,6) \times SU(2)$. They also preserve eight Poincare supercharges. Based on this, Tomasiello and various collaborators (most notably, Cremonesi), wrote the most generic background with the needed isometries to be a holographic dual to these SCFTs.

$$ds^2 = f_1(\eta) ds_{AdS_7}^2 + f_2(\eta) d\eta^2 + f_3(\eta) d\Omega_2^2(\chi, \xi),$$

$$B_2 = f_4(\eta) \text{Vol}_{\Omega_2}, \quad F_2 = f_5(\eta) \text{Vol}_{\Omega_2}, \quad e^\phi = f_6(\eta), \quad F_0 = F_0(\eta).$$

Where $d\Omega_2^2(\chi, \xi) = d\chi^2 + \sin^2 \chi d\xi^2$ and $\text{Vol}_{\Omega_2} = \sin \chi d\chi \wedge d\xi$.

They analysed the BPS equations for the functions $f_1(\eta), \dots, f_8(\eta)$ and found that all these functions can be written in terms of just one function $\alpha(\eta)$. } $\delta\psi_\mu = \delta\lambda = 0$

Imposing SUSY preservation, Tomasiello and coauthors found that the function $\alpha(\eta)$ has to satisfy the differential equation

$$\alpha''' = -162\pi^3 F_0.$$

The function $\alpha(\eta)$ must be piece-wise continuous. F_0 can be piece-wise constant and discontinuous.

The internal space $\mathcal{M}_3 = (\eta, \Omega_2)$ is topologically S^3 . The warp factor $f_3(\eta)$ must vanish at the beginning and at the end of the η -interval such that the two-sphere shrinks at those points.

As in the four dimensional case, we define a 'Rank-function'. For the sample quiver we had above, the Rank function is,

$$R(\eta) = \begin{cases} N_1\eta & 0 \leq \eta \leq 1, \\ N_1 + (N_2 - N_1)(\eta - 1) & 1 \leq \eta \leq 2, \\ N_2 + (N_3 - N_2)(\eta - 2) & 2 \leq \eta \leq 3, \\ N_3 + (N_4 - N_3)(\eta - 3) & 3 \leq \eta \leq 4, \\ N_4 - N_4(\eta - 4) & 4 \leq \eta \leq 5. \end{cases}$$

The connection between the quiver and the geometry comes by postulating

$$R(\eta) = -\frac{1}{81\pi^2}\alpha''(\eta).$$

Then, imposing that the function $\alpha(\eta)$ and $\alpha'(\eta)$ are continuous and that $\alpha(\eta = 0) = \alpha(P + 1 = 5) = 0$, one finds the full function $\alpha(\eta)$.

In analogy with the four dimensional case discussed above, we can find nice formulas to calculate the number of NS, D8 and D6 branes. Similarly, Linking numbers, central charge of the SCFT and Entanglement Entropy are computed in terms of (integrals of) $R(\eta)$.

If there is interest, I can discuss the expressions after the talk.

We may wonder about using solitonic strings to display non-integrable behaviours in the string backgrounds and the associated CFTs.

Proposing a string soliton that moves in global AdS_7 and the space $M_3 = [\eta, \chi, \xi]$

$$ds^2 = f_1(\eta) [-dt^2 \cosh \rho + d\rho^2 + \sinh^2 \rho (d\varphi^2 + \cos^2 \varphi d\theta^2 + \sin^2 \varphi d\Omega_3)] \\ + f_2(\eta) d\eta^2 + f_3(\eta) (d\chi^2 + \sin^2 \chi d\xi^2), \quad B_2 = f_4(\eta) \sin \chi d\chi \wedge d\xi.$$

The proposed embedding is,

$$t = t(\tau), \rho = \rho(\tau), \varphi = \varphi(\tau), \theta = \mu\sigma, \eta = \eta(\tau), \chi = \chi(\tau), \xi = \kappa\sigma.$$

The integers κ and μ indicate how many times the string wraps around the ξ and θ -directions respectively.

A careful study of the equations of motion of this soliton reveals that it will be Liouville non-integrable, unless $f_1(\eta)$ is constant.

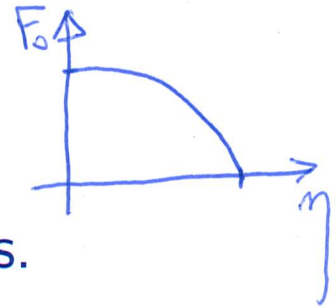
$$f_1(z) = 8\sqrt{2}\pi \sqrt{-\frac{\alpha}{\alpha''}} = \frac{8\sqrt{2}\pi}{\omega} \rightarrow \alpha(\eta) = A \sin(\omega\eta).$$

For this $\alpha(\eta)$, the full background reads

$$ds^2 = \frac{\sqrt{2}\pi}{\omega} \left(8AdS_7 + \omega^2 d\eta^2 + \left(\frac{\sin^2 \omega\eta}{1 + \sin^2 \omega\eta} \right) d\Omega_2 \right),$$

$$e^{-2\phi} = e^{-2\phi_0} (1 + \sin^2 \omega\eta), \quad B_2 = \pi \left(-\eta + \frac{\sin \omega\eta \cos \omega\eta}{\omega(1 + \sin^2 \omega\eta)} \right) d\Omega_2,$$

$$F_0 = \frac{A\omega^3 \cos \omega\eta}{162\pi^3}, \quad F_2 = -\frac{A\omega^2}{81\pi^2} \left(\frac{\sin^3 \omega\eta}{1 + \sin^2 \omega\eta} \right) d\Omega_2.$$



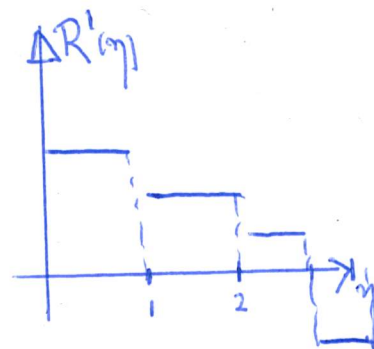
The form of F_0 indicates a continuous distribution of D8 branes.

There are two nice ways to think about this particular solution

First, we can consider the derivative of the Rank function

presented above and remind the identification $R(\eta) = -\frac{1}{81\pi^2} \alpha''(\eta)$,

$$R'(\eta) = -\frac{1}{81\pi^2} \alpha'''(\eta) = \begin{cases} N_1 & 0 \leq \eta \leq 1, \\ (N_2 - N_1) & 1 \leq \eta \leq 2, \\ (N_3 - N_2) & 2 \leq \eta \leq 3, \\ (N_4 - N_3) & 3 \leq \eta \leq 4, \\ -N_4 & 4 \leq \eta \leq 5. \end{cases}$$



We can extend the function α''' even-periodically and expand it in Fourier series

$$\alpha'''(\eta) = \sum_{k=0}^{\infty} c_k \cos\left(\frac{k\pi}{P+1}\eta\right) \rightarrow \alpha(\eta) = \sum_{k=0}^{\infty} b_k \sin\left(\frac{k\pi}{P+1}\eta\right).$$

In a sense, the function $\alpha \sim \sin(\omega\eta)$ is the 'core' of the most generic solution, that is a superposition. For each of those 'cores' the soliton is integrable.

Notice that each of the 'modes' the factor in front of AdS_7 becomes constant and also that the 'flavour structure'—read from $F_0 \sim \alpha'''$ is very simple. We have a bunch of $U(1)$'s.

The superposition of these 'modes' leads to an η -dependent function in front of AdS_7 and to a better defined flavour structure.

The second way of thinking about the special background is more interesting.

The bosonic string sigma model on this background can be shown to be classically integrable. By finding a Lax pair, that encodes the sigma model equations of motion.

This Lax pair was obtained by observing that the internal space $M_3 = [\eta, \chi, \xi]$ for this particular solution is what is called a λ deformation of a WZW model on S^3 .

In fact, in 2014 Sfetsos invented a deformation of an integrable WZW model, preserving integrability. He parametrised it in terms of λ . One can check that for $\lambda = 3 - 2\sqrt{2}$, the λ -deformation of the WZW model on S^3 gives the manifold parametrised by $[\eta, \chi, \xi]$. Why this particular value of λ is privileged, is mysterious to me.

These are beautiful formal aspects of this work that I will not discuss in detail today. Instead, let me present a summary and some conclusions

Summary

The formalism to construct string duals to SCFTs with eight Poincare supercharges in dimensions four and six is very similar.

One defines a "Rank-function" (capturing the rank of gauge groups and hence flavour groups). This Rank function serves as boundary condition for a 'potential function' $V(\sigma, \eta)$ or $\alpha(\eta)$, that solves a linear differential equation. The gravity background is fully expressed in terms of the potential function and its derivatives.

We have successfully matched various observables of the CFT (including some of non-perturbative nature) with calculations on the Type II backgrounds. We studied the non-integrability of string solitons. The particular soliton chosen is very efficient at detecting non-integrable and chaotic behaviour. Imposing that the soliton is Liouville-integrable selects special backgrounds. We have discussed that the full bosonic sigma model is integrable in that case.

In these special backgrounds, the *AdS* factor 'disconnects' from the rest of the space. The flavour structure of the SCFTs dual to the special backgrounds is specially simple, either there are no flavours or there is a product of $U(1)$'s.

This structure repeats in the case of 2d SCFTs with $(4,0)$ SUSY. I have not discussed this, but if there is interest and time, I can discuss the ideas. Same about the case of matrix models $(0+1)$ QFTs. Both are work in progress. It would be interesting to see if and how the ideas above are realised in these cases

Some conclusions

The features discussed here for 4d $\mathcal{N} = 2$ SCFTs and 6d $\mathcal{N} = (1, 0)$ SCFTs: dual description in terms of a single function V (or α) and its derivatives. Writing observable quantities in terms of V (or α). Linearity of the PDE to determine V , etc.

Repeat in the cases of SUSY CFTs in 1d, 2d, 3d, 5d. There should be 'core' solution, with very special features regarding its 'flavour structure'. This core solution should be integrable.

These systems can be thought in terms of D_p - D_{p+2} - NS_5 branes. There is always an AdS and an S^2 , realising the $SO(2, p)$ and $SU(2)$ global symmetries. The relations with Integrability are new aspects of special CFTs with these characteristics.

It may be useful unify the taxonomy of these backgrounds and their deformations. It should be interesting to study how integrability behaves under RG flows or other deformations.