From 1 to N

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1811.12316 (PRL) and 1903.00368 (JHEP)

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PLAN

I. Motivation

XXX Integrable Spin chain

Very easy to go From 2 (= Shortest length for periodic spin chain) to N (= any length) while keeping Integrability:

$$H = \sum_{\alpha=1}^{N} \left(\sigma_\alpha^x \sigma_{\alpha+1}^x + \sigma_\alpha^y \sigma_{\alpha+1}^y + \sigma_\alpha^z \sigma_{\alpha+1}^z \right)$$

Is there an analogous statement for classical integrable σ-models ?

To answer this question, one needs to review and use reinterpretation **[B. Vicedo 1701.04856]** of integrable σ-models as realisations of Affine Gaudin Models

II. Classical Finite Gaudin models

Poisson Manifold

Notations:

- Simple Lie algebra $\mathfrak{f}$ - Basis (I^a) - Structure constants $f^{ab}{}_c$
- Opposite of the Killing form $\kappa^{ab} = \kappa(I^a, I^b)$; $\kappa_{ab}\kappa^{bc} = \delta_a^c$
- Quadratic Casimir $C_{\mathbf{12}} = \kappa_{ab} I^a \otimes I^b = I_a \otimes I^a \in \mathfrak{f} \otimes \mathfrak{f}$

Poisson manifold P_G of Gaudin model with N sites
= Cartesian product of N copies of $\mathfrak{f}^*$

- Define for any $J \in \mathfrak{f}$: $J^a = \kappa(I^a, J)$
- The algebra $\mathcal{F}(P_G)$ of functions on P_G is generated by N copies of (J^a) equipped with Kirillov-Kostant Poisson Bracket (P.B.)

Sites

- Each set $\left(J^a_{(\alpha)}\right)$ is formally attached to a site α
- Commutation between different sites

$$\left\{J^a_{(\alpha)}, J^b_{(\beta)}\right\} = \boldsymbol{\delta_{\alpha\beta}}\, f^{ab}{}_c\, J^c_{(\alpha)}$$

This P.B. may be rewritten in tensorial notations as

$$\{J_{(\alpha)\mathbf{1}}, J_{(\beta)\mathbf{2}}\} = \delta_{\alpha\beta}[C_{\mathbf{12}}, J_{(\alpha)\mathbf{1}}]$$

where

$$J_{(\alpha)} = I_a \otimes J^a_{(\alpha)} \quad \in \mathfrak{f} \otimes \mathcal{F}(P_G)$$

Lax matrix

$*$ Fundamental object sustaining Hamiltonian Integrability

$$L(z,t) = \sum_{\alpha=1}^{N} \frac{J_{(\alpha)}(t)}{z - z_\alpha} + \Omega$$

with z the spectral parameter

z_α is the position **in the spectral plane** of the site α

$*$ $\Omega \in \mathfrak{f}$ is a constant (vanishing P.B.)

$*$ Possible to show that

$$\{L_{\mathbf{1}}(z), L_{\mathbf{2}}(z')\} = [r_{\mathbf{12}}(z,z'), L_{\mathbf{1}}(z) - L_{\mathbf{1}}(z')]$$

with

$$r_{\mathbf{12}}(z,z') = C_{\mathbf{12}}/(z - z')$$

classical antisymmetric r-matrix

Quadratic Hamiltonian

Define the spectral parameter dependent quantity

$$\boldsymbol{H(z) = \tfrac{1}{2}\kappa(L(z), L(z))}$$

P.B. of $L(z)$ is a commutator

$$\rightarrow \qquad \{H(z), H(z')\} = 0$$

$$\rightarrow \qquad \{H(z'), L(z)\} = [M(z', z), L(z)]$$

with

$$M(z', z) = \frac{L(z')}{z - z'}$$

Next step: Extract/Define the Hamiltonian from $H(z)$

Quadratic Hamiltonians

$$H(z) = \Delta_\infty + \sum_{\alpha=1}^{N} \left(\frac{\Delta_\alpha}{(z - z_\alpha)^2} + \frac{H_\alpha}{z - z_\alpha} \right)$$

$\Delta_\infty = \frac{1}{2}\kappa(\Omega, \Omega)$ is a constant and $\Delta_\alpha = \frac{1}{2}\kappa(J_{(\alpha)}, J_{(\alpha)})$ is an invariant of the Kirillov-Kostant Poisson Bracket

$\rightarrow$ Residue of $H(z)$ in z_α

$$H_\alpha = \sum_{\beta \neq \alpha} \frac{\kappa(J_{(\alpha)}, J_{(\beta)})}{z_\alpha - z_\beta} + \kappa(J_{(\alpha)}, \Omega)$$

$$H_\alpha = \sum_{\beta \neq \alpha} \frac{\kappa_{ab} J^a_{(\alpha)} J^b_{(\beta)}}{z_\alpha - z_\beta} + \kappa(J_{(\alpha)}, \Omega)$$

The H_α's are in involution:

$$\{H_\alpha, H_\beta\} = 0$$

Hamiltonian and Lax Pair

Hamiltonian = Linear combination of the H_α's:

$$H = \sum_{\alpha=1}^{N} c_\alpha H_\alpha$$

All this implies

$$\frac{dL(z)}{dt} \equiv \{H, L(z)\} = [M(z), L(z)]$$

$\Longrightarrow$ Lax Pair

$$L(z) = \sum_{\alpha=1}^{N} \frac{J_{(\alpha)}}{z - z_\alpha} + \Omega$$

$$M(z) = \sum_{\alpha=1}^{N} c_\alpha \frac{J_{(\alpha)}}{z - z_\alpha}$$

Example of realisation for $\mathfrak{f} = \mathfrak{sl}(2, \mathbb{R})$

$$L(z) = \sum_{\alpha=1}^{N} \frac{1}{z - z_\alpha} \begin{pmatrix} J^H_{(\alpha)} & 2J^F_{(\alpha)} \\ 2J^E_{(\alpha)} & -J^H_{(\alpha)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Realisation: Do N times the following

- Take canonical coordinates (x, p)
- Define

$$J^H = xp \qquad J^E = -\tfrac{1}{2}p^2 \qquad J^F = \tfrac{1}{2}x^2$$

$\Longrightarrow$ Poisson Map from $\mathbb{R}^{2N}$ to $P_G \simeq \mathbb{R}^{3N}$

- Quadratic Hamiltonians

$$H_\alpha = x_\alpha^2 + \sum_{\beta \neq \alpha} \frac{(x_\alpha p_\beta - x_\beta p_\alpha)^2}{z_\alpha - z_\beta}$$

Coupling and Decoupling

$*$ Coupling between the N particles in expression of H_α comes from the second term in

$$H_\alpha = x_\alpha^2 + \sum_{\beta \neq \alpha} \frac{(x_\alpha p_\beta - x_\beta p_\alpha)^2}{\boldsymbol{z_\alpha - z_\beta}}$$

$*$ Coupling depends on how far the sites of the Gaudin model are from each other (in the spectral plane)

$*$ Decoupling of the last particle:

Maintain to fixed value all z_α for $\alpha = 1, \cdots, N-1$
while sending $z_N \to \infty$

$*$ How to "implement" the N particles is very simple !

$*$ **Same kind of mechanism** for Affine Gaudin Models and thus for Integrable σ-models

Remarks: (Unreduced) Neumann Model

- The quadratic Hamiltonians

$$H_\alpha = x_\alpha^2 + \sum_{\beta\neq\alpha} \frac{(x_\alpha p_\beta - x_\beta p_\alpha)^2}{z_\alpha - z_\beta}$$

  correspond (for $z_\alpha \equiv \omega_\alpha^2$) to the Uhlenbeck's [Uhlenbeck '82] conserved quantities of the Neumann model
- 'Unreduced' Neumann model is a realisation of $\mathfrak{sl}(2,\mathbb{R})$ Gaudin model [Kuznetsov' 1992] (see also [S. Lacroix's PhD thesis, Integrable models with twist function and affine Gaudin models, 1809.06811])
- Remark: Neumann model ($N = 3$) and spinning strings in $AdS_5 \times S^5$ [G. Arutyunov, S. Frolov, J. Russo and A. Tseytlin '03, G. Arutyunov, J. Russo and A. Tseytlin '04]

Generalisation and Summary

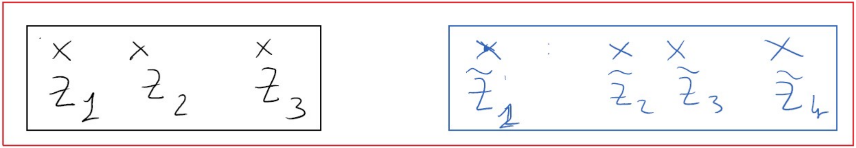
- Generalisation of Gaudin models to higher-order poles known [Feigin Frenkel Toledano Laredo '10]
- How to go from 1 to N ? Add sites
- How to decouple one site ? Send it to ∞
- Important: $\{J_{(\alpha)}, J_{(\beta)}\} \propto \delta_{\alpha\beta}$

$$L(z) = \sum_{\alpha=1}^{N} \frac{J_{(\alpha)}}{z - z_\alpha} + \Omega$$

$$H = \frac{1}{2} \sum_{\alpha=1}^{N} c_\alpha \operatorname*{res}_{z_\alpha} \kappa(L(z), L(z))$$

Summary: Assembling LEGO bricks

$$\color{red}{L(z)} = \sum_{\alpha=1}^{3} \frac{J_{(\alpha)}}{z - z_\alpha} + \color{blue}{\sum_{\widetilde{\alpha}=1}^{4} \frac{J_{(\widetilde{\alpha})}}{z - \widetilde{z}_\alpha}} + \Omega$$



Possible to couple Gaudin models associated with
same Lie algebra $\mathfrak{f}$

III. Classical Affine Gaudin Models

∗ Defined at Hamiltonian level
∗ Models which are integrable **by construction**
∗ One of the main difficulties = Perform the Legendre transform
∗ Many levels:

- Formal: Affine Kac-Moody algebra associated with $\mathfrak{f}$
- **Intermediate:** $\mathfrak{f}$-valued connections
- **Realisation level:** Integrable σ-models as realisations of Affine Gaudin Models

At the end, we present an **action** and **Lagrangian expression** of the Lax pair satisfying the zero curvature equation

$$[\partial_x + \mathcal{L}(z,x,t)\,,\,\partial_t + \mathcal{M}(z,x,t)] = 0$$

Spatial component $\mathcal{L}(z,x)$ of Lax pair $\equiv$ Lax matrix

Analogy

Finite case	Affine case
$J_{(\alpha)} = I_a \otimes J^a_{(\alpha)}$	$\ell_{(\alpha)}\partial_x + J_{(\alpha)}(x)$ $J_{(\alpha)}(x) = I_a \otimes \sum_{n\in\mathbb{Z}} \left(J^a_{(\alpha)}\right)_n e^{-inx}$
$L(z) = \sum_{\alpha=1}^{N} \frac{J_{(\alpha)}}{z-z_\alpha}$	$\boldsymbol{\varphi(z)\partial_x + \Gamma(z,x)}$

$$\varphi(z) = \sum_{\alpha=1}^{N} \frac{\ell_{(\alpha)}}{z - z_\alpha} \quad \text{and} \quad \Gamma(z,x) = \sum_{\alpha=1}^{N} \frac{J_{(\alpha)}(x)}{z - z_\alpha}$$

- φ is called the **Twist function**
- Γ is called the **Gaudin Lax matrix**

Lax matrix

- So far, we have $\varphi(z)\partial_x + \Gamma(z,x)$
- For field theory we need to construct $\partial_x + \mathcal{L}(z,x)$

$\Longrightarrow$ The Lax matrix of the integrable field theory is

$$\mathcal{L}(z,x) = \varphi(z)^{-1}\Gamma(z,x)$$

- Zeroes of $\varphi(z)$ correspond to poles of $\mathcal{L}(z)$
- P.B. $\{\Gamma,\Gamma\}$ and $\{\mathcal{L},\mathcal{L}\}$ are **determined by** φ and they **ensure Hamiltonian integrability**

φ controls the algebraic structure behind integrability

Datum defining an **Affine Gaudin Model**

With each site α one associates:

- Its **position** z_α in spectal plane
- Its **multiplicity** $m_\alpha \in \mathbb{N}^*$
- m_α numbers ℓ^α_p, called the **levels**, $p \in \{0, \cdots, m_\alpha - 1\}$
- m_α **currents** $J^\alpha_p(x)$

$$\varphi(z) = \sum_\alpha \sum_{p=0}^{m_\alpha - 1} \frac{\ell^\alpha_p}{(z - z_\alpha)^{p+1}} - \ell^\infty$$

$$\Gamma(z, x) = \sum_\alpha \sum_{p=0}^{m_\alpha - 1} \frac{J^\alpha_p(x)}{(z - z_\alpha)^{p+1}}$$

$\ell^\alpha_{m_\alpha - 1} \neq 0 \Longrightarrow$ **Multiplicity = Order of the pole** z_α of $\varphi(z)$

- Choice of Linear Combination of Quadratic Hamiltonians

Datum defining an **Integrable σ-model**

With each site α one associates:

- Its **position** z_α in spectal plane
- Its **multiplicity** $m_\alpha \in \mathbb{N}^*$
- m_α numbers ℓ_p^α, called the **levels**, $p \in \{0, \cdots, m_\alpha - 1\}$
- m_α **currents** $J_p^\alpha(x)$

$$\varphi(z) = \sum_\alpha \sum_{p=0}^{m_\alpha - 1} \frac{\ell_p^\alpha}{(z - z_\alpha)^{p+1}} - \ell^\infty$$

$$\Gamma(z, x) = \sum_\alpha \sum_{p=0}^{m_\alpha - 1} \frac{J_p^\alpha(x)}{(z - z_\alpha)^{p+1}}$$

$\ell_{m_\alpha - 1}^\alpha \neq 0 \Longrightarrow$ **Multiplicity = Order of the pole** z_α of $\varphi(z)$

- Choice of Linear Combination of Quadratic Hamiltonians
- **Choice of Realisation for the currents $J_p^\alpha(x)$**

Coupling two Affine Gaudin Models

- Start with
  - $\text{AGM}_{\mathbf{1}}$ for Lie algebra $\mathfrak{f}$ with sites $z_\alpha^{\mathbf{1}}$ and levels $\ell_p^{\alpha\,\mathbf{1}}$
  - $\text{AGM}_{\mathbf{2}}$ for same Lie algebra $\mathfrak{f}$ with sites $z_\alpha^{\mathbf{2}}$ and levels $\ell_p^{\alpha\,\mathbf{2}}$
- What is easy to understand:

  Build the AGM with sites $\left(z_\alpha^{\mathbf{1}}, z_\alpha^{\mathbf{2}} + \frac{1}{\gamma}\right)$ and levels $\left(\ell_p^{\alpha\,\mathbf{1}}, \ell_p^{\alpha\,\mathbf{2}}\right)$
- Result:

  Decoupling in the limit $\gamma \to 0$
- What requires more work: Choice of linear combination for the Hamiltonian which is coherent with this scenario
- At the Hamiltonian level, more or less the end of the story !

Takiff currents and their realisations

$*$ Poisson brackets

$$\left\{ J^{\alpha}_{p\,\mathbf{1}}(x), J^{\beta}_{q\,\mathbf{2}}(y) \right\} = \delta_{\alpha\beta} \Big(\big[C_{\mathbf{12}}, J^{\alpha}_{p+q\,\mathbf{1}}(x) \big]\, \delta_{xy} - \ell^{\alpha}_{p+q}\, C_{\mathbf{12}}\, \delta'_{xy} \Big)$$

$$\text{if } p+q < m_\alpha$$

$$\left\{ J^{\alpha}_{p\,\mathbf{1}}(x), J^{\alpha}_{q\,\mathbf{2}}(y) \right\} = 0 \text{ if } p+q \geq m_\alpha$$

$*$ For σ-models, Realisations of these currents done in terms of (g, X) with fields g and X valued respectively in Lie group F and Lie algebra $\mathfrak{f}$

$*$ Exception = Non-abelian T-dual

List of known Building blocks

Model	Twist function $\varphi(z)$	Realisation of the Gaudin Lax matrix $\Gamma(z,x)$
PCM + WZ	$\frac{K-K^{-1}k^2}{(z-K^{-1}k)^2} - \frac{2k}{z-K^{-1}k} - K$	$\frac{(K-K^{-1}k^2)\,j(x)}{(z-K^{-1}k)^2} + \frac{X(x)-k\,j(x)-k\,W(x)}{z-K^{-1}k}$
PCM	$\frac{K}{z^2} - K$	$\frac{K\,j(x)}{z^2} + \frac{X(x)}{z}$
hYB	$\frac{K}{z^2} - K$	$\frac{K\,j(x)-R_g X(x)}{z^2} + \frac{X(x)}{z}$
NATD	$\frac{K}{z^2} - K$	$\frac{\mathfrak{m}(x)}{z^2} + \frac{K\,\partial_x \mathrm{v}(x)+[\mathfrak{m}(x),\mathrm{v}(x)]}{z}$
iYB	$\frac{K/(2c\eta)}{z-c\eta} + \frac{-K/(2c\eta)}{z+c\eta} - \frac{K}{1-c^2\eta^2}$	$\frac{1}{2c}\frac{cX(x)-R_g X(x)+\frac{K}{\eta}j(x)}{z-c\eta} + \frac{1}{2c}\frac{cX(x)+R_g X(x)-\frac{K}{\eta}j(x)}{z+c\eta}$
λ	$\frac{K/(2\alpha)}{z-\alpha} + \frac{-K/(2\alpha)}{z+\alpha} - \frac{K}{1-\alpha^2}$	$\frac{X(x)-k\,j(x)-k\,W(x)}{z-\alpha} - \frac{g(x)(X(x)+k\,j(x)-k\,W(x))g(x)^{-1}}{z+\alpha}$

$j = g^{-1}\partial_x g$ and $I_{WZ}[g] = \iint dt\,dx\,\kappa\big(W, g^{-1}\partial_t g\big)$

Difference between PCM and (PCM + WZ)

Model	Twist function $\varphi(z)$	Gaudin Lax matrix $\Gamma(z,x)$
PCM + WZ	$\frac{K-K^{-1}k^2}{(z-K^{-1}k)^2} - \frac{2k}{z-K^{-1}k} - K$	$\frac{(K-K^{-1}k^2)\,j(x)}{(z-K^{-1}k)^2} + \frac{X(x)-k\,j(x)-k\,W(x)}{z-K^{-1}k}$
PCM	$\frac{K}{z^2} - K$	$\frac{K\,j(x)}{z^2} + \frac{X(x)}{z}$

$*$ Both PCM and (PCM + WZ) correspond to a **single site of multiplicity 2** (**double pole** of φ, which admits also **two zeros** in both cases)

$$\varphi(z) = \frac{\ell_1}{(z-z_0)^2} + \frac{\ell_0}{z-z_0} - \ell^\infty$$

$$\Gamma(z,x) = \frac{J_1(x)}{(z-z_0)^2} + \frac{J_0(x)}{z-z_0}$$

$*$ But **for PCM $\ell_0 = 0$** (which is clear from P.B. of currents)

Poles and zeroes of the twist function

$*$ Two ‘starting’ sets of parameters

- Poles z_α of $\varphi(z)$ and levels ℓ_p^α
- Poles z_α and zeroes ζ_i of $\varphi(z)$

$$\varphi(z) = -\ell^\infty \frac{\prod_{i=1}^{M}(z-\zeta_i)}{\prod_{\alpha}(z-z_\alpha)^{m_\alpha}}$$

with

$$M = \sum_\alpha m_\alpha$$

Suppose all ζ_i are real and simple

$*$ Advantage and drawback of each set

Poles and zeroes of the twist function

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Suppose all ζ_i are real and simple

* Advantage and drawback of each set

Basis for Hamiltonian

- Most convenient basis of **commuting quadratic charges:**

$$Q_i = -\frac{1}{2\varphi'(\zeta_i)} \int dx\, \kappa\big(\Gamma(\zeta_i, x), \Gamma(\zeta_i, x)\big)$$

- **Sufficient condition for Lorentz invariance:**

$$H = \sum_{i=1}^{M} \epsilon_i\, Q_i \text{ with } \epsilon_i^2 = 1$$

- For examples worked out: To be able to perform Legendre transform, **same number of ϵ_i equal to 1 than to -1**

$$\Longrightarrow \mathcal{L}_\pm(z, x) = \mathcal{M} \pm \mathcal{L} = \pm 2 \sum_{i \in I_\pm} \frac{1}{\varphi'(\zeta_i^\pm)} \frac{\Gamma(\zeta_i^\pm, x)}{z - \zeta_i^\pm}$$

Recall for PCM

$$\varphi(z) = K \frac{1 - z^2}{z^2} \qquad \text{and} \qquad \mathcal{L}_\pm(z) = \frac{g^{-1}\partial_\pm g}{1 \mp z}$$

Basis for Hamiltonian

- Most convenient basis of **commuting quadratic charges:**

$$Q_i = -\frac{1}{2\varphi'(\zeta_i)} \int dx\ \kappa\big(\Gamma(\zeta_i, x), \Gamma(\zeta_i, x)\big)$$

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- Importance of the zeroes of φ for constructing higher order local charges [S. Lacroix, M.M., B. Vicedo '17]

IV. From 1 PCM_k to N coupled PCM_{k_r}

- One (PCM + WZ): One double pole z_1 and two simple zeroes $\zeta_1^{\pm}$
- N (PCM + WZ): N double poles z_r and 2N simple zeroes $\zeta_r^{\pm}$

Difficulty: Legendre Transform

$$\varphi_\pm(z) \equiv \frac{\prod_{r=1}^N (z-\zeta_r^\pm)}{\prod_{r=1}^N (z-z_r)} \quad \text{and} \quad \varphi(z) \equiv -\ell^\infty \varphi_+(z)\varphi_-(z)$$

$$S[g^{(1)},\dots,g^{(N)}] = \iint dt\, dx \sum_{r,s=1}^N \rho_{rs}\, \kappa\big(j_+^{(r)}, j_-^{(s)}\big) + \sum_{r=1}^N k_r\, I_{WZ}[g^{(r)}]$$

$$\rho_{rr} = \tfrac{1}{4}\ell^\infty \Big(\varphi'_{+,r}(z_r)\varphi_{-,r}(z_r) - \varphi_{+,r}(z_r)\varphi'_{-,r}(z_r)\Big)$$

$$\rho_{rs} = \tfrac{1}{2}\ell^\infty \frac{\varphi_{+,r}(z_r)\varphi_{-,s}(z_s)}{z_r - z_s} \qquad \text{for} \quad r \neq s$$

$$k_r = \tfrac{1}{2}\ell^\infty \Big(\varphi'_{+,r}(z_r)\varphi_{-,r}(z_r) + \varphi_{+,r}(z_r)\varphi'_{-,r}(z_r)\Big)$$

$$\varphi_{\pm,r}(z) = (z - z_r)\varphi_\pm(z)$$

All $g^{(r)}$ take values in the same Lie group F and

$$j_\pm^{(r)} = g^{(r)-1}\partial_\pm g^{(r)} \in \mathfrak{f}$$

* **Number of parameters:**
  - $2N$ zeros $\zeta_i^{\pm}$ and N poles z_r of φ together with ℓ^{∞}
  - Redundacy (Dilatation and Translation)

$$\Longrightarrow \mathbf{3N - 1 \text{ free parameters}}$$

  - While $N^2 + N$ coefficients in the action (ρ_{rs} and k_r)

* **Lagrangian expression of Lax pair:**

$$\mathcal{L}_{\pm}(z, x, t) = \sum_{r=1}^{N} \frac{\varphi_{\pm,r}(z_r)}{\varphi_{\pm,r}(z)} j_{\pm}^{(r)}(x, t)$$

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0$$

Note that

$$\mathcal{L}_{\pm}(z_r, x, t) = j_{\pm}^{(r)}(x, t)$$

Global symmetries

$$S\big[g^{(1)},\dots,g^{(N)}\big] = \iint dt\,dx \sum_{r,s=1}^{N} \rho_{rs}\,\kappa\big(j_+^{(r)},j_-^{(s)}\big) + \sum_{r=1}^{N} k_r\; I_{\rm WZ}\big[g^{(r)}\big]$$

Recall that $j_\pm^{(r)} = g^{(r)-1}\partial_\pm g^{(r)}$

- **On the left:** $F \times \cdots \times F$

$$\big(g^{(1)},\dots,g^{(N)}\big) \longmapsto \big(h_1 g^{(1)},\dots,h_N g^{(N)}\big), \quad h_r \in F$$

- **On the right:** F_{diag}

$$\big(g^{(1)},\dots,g^{(N)}\big) \longmapsto \big(g^{(1)}h,\dots,g^{(N)}h\big), \quad h \in F$$

Examples for $N = 2$

* First example:

$$z_1 = -z_2 = \gamma^{-1}, \quad \zeta_1^{\pm} = \pm 1 + \gamma^{-1}, \quad \zeta_2^{\pm} = \pm 1 - \gamma^{-1}$$

$$\rho_{11} = \rho_{22} = \frac{\ell^\infty}{4}(2 - \gamma^2), \qquad k_1 = -k_2 = -\frac{\ell^\infty}{8}\gamma^3,$$
$$\rho_{12} = -\frac{\ell^\infty}{16}\gamma(\gamma - 2)^2, \qquad \rho_{21} = \frac{\ell^\infty}{16}\gamma(\gamma + 2)^2.$$

* Second example: For

$$z_1 = -z_2 = \gamma^{-1}, \quad \zeta_1^{\pm} = \pm\sqrt{1 + \gamma^{-2} + \sqrt{1 + 4\gamma^{-2}}},$$
$$\zeta_2^{\pm} = \mp\sqrt{1 + \gamma^{-2} - \sqrt{1 + 4\gamma^{-2}}}.$$

$k_1 = k_2 = 0$

V. Conclusion

Comments

- Construction of Integrable σ-models **from a given set of parameters ...**
- ... encoded in a generic rational function with N double poles and 2N simple zeroes
- Very nice to see **how these parameters of (Hamiltonian) integrability show up in the expressions of both the action and the Lagrangian Lax pair !**
  $\rightarrow$ Importance of the twist function

Hamiltonian integrability no more 'hidden'
in Lagrangian formulation

- For N=1: Two free parameters = Same number as the number of coefficients (ρ_{11} and k_1) in the action

Key point = observables associated with different sites mutually Poisson commute

Model	Twist function $\varphi(z)$	Realisation of the Gaudin Lax matrix $\Gamma(z,x)$
iYB	$\frac{K/(2c\eta)}{z-c\eta} + \frac{-K/(2c\eta)}{z+c\eta} - \frac{K}{1-c^2\eta^2}$	$\frac{1}{2c}\frac{cX(x)-R_g X(x)+\frac{K}{\eta}j(x)}{z-c\eta} + \frac{1}{2c}\frac{cX(x)+R_g X(x)-\frac{K}{\eta}j(x)}{z+c\eta}$
λ	$\frac{K/(2\alpha)}{z-\alpha} + \frac{-K/(2\alpha)}{z+\alpha} - \frac{K}{1-\alpha^2}$	$\frac{X(x)-k\,j(x)-k\,W(x)}{z-\alpha} - \frac{g(x)(X(x)+k\,j(x)-k\,W(x))g(x)^{-1}}{z+\alpha}$

[T.J. Hollowood J.L. Miramontes D.M. Schmidtt '14, B. Vicedo '15]

Other models

- We have also determined the action coupling **One hYB to $N-1$ (PCM+WZ)** and the expression of the Lax pair at Lagrangian level
  Twist function is the same as for One PCM + $N-1$ (PCM+WZ)
- One should recover and extend example of **two Coupled integrable λ-models** constructed in [G. Georgiou and K. Sfetsos 1809.03522 1812.04033]
- **Playground is infinite...**
- But remember that **Same** Lie algebra

Generalisation to Symmetric space σ-models ?

- Their nature as Affine Gaudin Models is different
- Called Cyclotomic Affine Gaudin Models
- Comes from $\mathbb{Z}_2$ grading
- More precisely Gauge invariance associated with a constraint
- Comes from special role of infinity in spectral plane
- If F/G symmetric space, one expects to be able to construct a

$$F^{\times N}/G_{diag} \text{ integrable } \sigma\text{-model}$$

- Corresponds to gauging a subgroup of F_{diag} (Symmetry on the right)

Quantum level

- Renormalisation group flow: we already know it will be interesting from [G. Georgiou and K. Sfetsos 1809.03522 1812.04033] [G. Georgiou P. Panopoulos E. Sagkrioti and K. Sfetsos 1906.00984]
- If integrability preserved at the quantum level: S-matrix ?
- Use of Affine Gaudin Model Approach at quantum level [B. Vicedo ’17, S. Lacroix, B. Vicedo, and C. A. S. Young ’18 ’18, S. Lacroix PhD thesis]

From **N** to **1**...

Apóstol Santiago,
fuiste el primero en beber
el cáliz del Señor,
de los peregrinos;
haznos fuertes en la fe
alegres en la esperanza,
en nuestro caminar
de peregrinos
siguiendo el camino
de la vida cristiana
y aliéntanos para que,
finalmente,
alcancemos la gloria
de Dios Padre.
Amén
Caminos de Europa
Oslo
Tonsberg
Lödöse
Alborg
Viborg
Arhus
Helsingborg
Varde
Odense
Lund
Soro
Galway
Dublín
Droitwich
Great Yarmouth
Leeuwarden
Lüberck
Rostock
Bideford
Bristol
Woodstock
London
Groingen
Hamburg
Amsterdam
Bremen
Padstow
Truro
Dartmouth
Winchester
Haarlem
Utrecht
Minden
Hannover
Plymouth
Southampton
Canterbury
Dover
Brugge
Antwerpen
Müster
Goslar
Berlin
Boulogne
Dunkerque
Brucelles
Köln
Paderborn
Magdeburg
Leipzig
Görlitz
Landerneau
Tréguier
Fecamp
Dieppe
Amiens
Namur
Bonn
Gotha
Lubin
Mont Saint Michel
Dinan
Caen
Rouen
Reims
Koblenz
Marburg
Weimar
Dresden
Frankfurt
Hof
Luxembourg
Bamberg
Vannes
Le Mans
París
Metz
Speyer
Praha
Nantes
Chartres
Stuttgart
Nürnberg
Pilsen
Bourgneuf
Orléans
Strabbourg
Eichtatt
Tours
Tübingen
Ulm
Passau
Partenay
Vézelay
München
Wien
Budapest
La Rochelle
Poitier
Nevers
Dijón
Konstanz
Linz
Saintes
Cluny
Chalons/s/saone
Liechtenstein
Salzburg
Soulac
Limoges
Bern
Chor
Innsburck
A Coruña
Ribadeo
Bordeaux
Périgueux
Lyon
Genève
Villach
Graz
Lugo
Oviedo
Rocamadour
Padrón
Cebreiro
Santander
Le Puy
Ljubljana
Pontevedra
Ponferrada
Bilbao
Ostabat
Milano
Valença
Ourense
León
Donostia
Moissac
Conques
Valence
Torino
Verona
Trieste
Zagreb
Barcelos
Sahagún
Vitoria
Roncesvalles
Piacenza
Braga
Toulouse
Briançon
Venezia
Porto
Bragança
Carrión
Pamplona
Olorón
Montpellier
Burgos
Saint Gaudens
Génova
Bologna
Coimbra
Lamego
Zamora
Logroño
Jaca
Arles
Ventimuglia
Viseu
Viella
Narbonne
Pistoia
Salamanca
Soria
Calahorra
Huesca
Fréjus
Firenze
Guarda
Bejar
Ripoll
Perpignan
Marseille
Lucca
Tarazona
Siena
Loreto
Lleida
Assisi
Alpalhão
Cáceres
Girona
Toledo
Tarragona
Lisboa
Viterbo
Évora
Poblet
Barcelona
Mérida
Almendralejo
Roma
Beja
Fuente de Cantos
Benevento
Faro
Bari
Nápoli
Sevilla
Brindisi