Integrable deformations of the $\mathbf{AdS_5 \times S^5}$ superstring: resolving some puzzles

Ben Hoare

ETH Zurich

Integrability, duality and beyond **4th June 2019**

Introduction

η deformations and their worldsheet duals

- The η deformation of a σ-model is a **Yang–Baxter deformation** based on a choice of **Drinfel'd–Jimbo R-matrix**. Its worldsheet duals are found by **Poisson–Lie duality**.
  - What is a Yang–Baxter deformation?
  - What is a Drinfel'd–Jimbo R-matrix?
  - What is Poisson–Lie duality?

Introduction

- η deformations are related to a q-deformation of the global symmetry, where the latter is well-understood in the theory of quantum groups.
- Low-dimensional examples first appeared as strong-coupling duals of massive integrable models of interest in statistical mechanics.
- Also within string theory the $AdS_5 \times S^5$ superstring and other integrable string σ-models can be η deformed.

Introduction

- How the choice of R-matrix affects the deformed σ-model?
  - Do different R-matrices give different η deformations?
  - Do different η deformations admit different Poisson–Lie duals?
  - In string theory, for which R-matrices does the deformed background, or its Poisson–Lie duals, define a critical string theory?

Outline

Introduction

The η deformation

Poisson–Lie duality

String theory

The η deformation

- We start from the symmetric space σ-model for the coset G/H

$$\mathcal{S} = -\frac{h}{2}\int d^2\sigma \; \mathrm{Tr}[J_+ P J_-]$$

  - $J_\pm = g^{-1}\partial_\pm g \in \mathrm{Lie}(\mathrm{G}) = \mathfrak{g}$ where $g(\sigma) \in \mathrm{G}$,
  - h is the σ-model coupling,
  - $\mathfrak{g}$ is semi-simple and Tr is the normalised Killing form (or trace),
  - $\mathfrak{g}$ admits a $\mathbb{Z}_2$ grading: $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ where $\mathfrak{g}^{(0)} = \mathrm{Lie}(\mathrm{H})$,
  - P is the projector onto $\mathfrak{g}^{(1)}$.

The η deformation

[Delduc, Magro, Vicedo; Klimčík]

- The η deformed model has the form

$$\mathcal{S} = -\frac{h}{2}\int d^2\sigma \; \mathrm{Tr}[J_+ P \,\frac{1}{1-\eta R_g P}\, J_-]$$

  - $\eta \in \mathbb{R}$ is the deformation parameter,
  - $R_g = \mathrm{Ad}_g^{-1} R\, \mathrm{Ad}_g$ where R is a constant linear map from $\mathfrak{g}$ to itself,
  - the deformation is compatible with the gauge symmetry.

The η deformation

[Delduc, Magro, Vicedo; Klimčík]

- If R is an antisymmetric solution of the classical Yang–Baxter equation, or its modified counterpart, then this is the Yang–Baxter deformation

$$[RX, RY] - R([X, RY] - [RX, Y]) = \sigma[X, Y] \qquad \sigma \in \{-1, 0, +1\}$$

$$\mathrm{Tr}[X\, RY] = -\,\mathrm{Tr}[RX\, Y] \qquad X, Y \in \mathfrak{g}$$

  - $\sigma = 0$ is the classical Yang–Baxter equation,
  - $\sigma = \pm 1$ is its modified counterpart, with $+1$ and -1 known as the non-split and split cases respectively.

- The Yang–Baxter deformation is classically integrable.

The η deformation

- We focus on R-matrices that are solutions of the non-split modified classical Yang–Baxter equation and are of Drinfel'd–Jimbo type.
- We take a Cartan–Weyl basis of $\mathfrak{g}$:
  - h_i are the Cartan generators,
  - e_m and f_m are the positive and negative roots with $\text{Tr}[e_m f_n] = \delta_{mn}$,
- The corresponding Drinfel'd–Jimbo R-matrix is

$$RX = -i \sum_m (\text{Tr}[X f_m] e_m - \text{Tr}[e_m X] f_m) \qquad X \in \mathfrak{g}$$

$$R : h_i \to 0 \qquad R : e_m \to -i e_m \qquad R : f_m \to i f_m$$

The η deformation

- Inequivalent choices of Cartan–Weyl basis lead to different Drinfel'd–Jimbo R-matrices.
- For a complex semi-simple Lie algebra the Cartan–Weyl basis is unique (up to inner automorphisms) and there is one Drinfel'd–Jimbo R-matrix.
- For the compact real form there is also only a single R-matrix.
- For non-compact real forms and for superalgebras this is no longer true!

Non-compact real forms

- The simplest example is AdS_3

$$\mathrm{AdS}_3 \cong \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(1,2)} \cong \frac{\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)}{\mathrm{SU}(1,1)}$$

for which there are two inequivalent R-matrices

$$R = \begin{pmatrix} R_{\mathrm{SU}(1,1)} & \\ & R_{\mathrm{SU}(1,1)} \end{pmatrix} \qquad R = \begin{pmatrix} R_{\mathrm{SU}(1,1)} & \\ & -R_{\mathrm{SU}(1,1)} \end{pmatrix}$$

- The two models are related by a real η dependent field redefinition, up to a total derivative.

Non-compact real forms

- Therefore, let us focus on AdS_5

$$\mathrm{AdS}_5 \cong \frac{\mathrm{SO}(2,4)}{\mathrm{SO}(1,4)} \cong \frac{\mathrm{SU}(2,2)}{\mathrm{Sp}(1,1)}$$

for which there are three inequivalent R-matrices. [Delduc, Magro, Vicedo]

- These correspond to three inequivalent Cartan–Weyl bases of $\mathfrak{su}(2,2)$.
- They are inequivalent in $\mathfrak{su}(2,2)$, but equivalent in $\mathfrak{sl}(4;\mathbb{C})$.

Non-compact real forms

- Working with 4×4 matrices in $\mathfrak{sl}(4;\mathbb{C})$, these R-matrix can be constructed from one R-matrix

$$R : \begin{pmatrix} h & & e \\ & \ddots & \\ f & & h \end{pmatrix} \to \begin{pmatrix} 0 & & -ie \\ & \ddots & \\ if & & 0 \end{pmatrix}$$

and imposing the real form using three different signature matrices

$$\Sigma_0 = (+1, +1, -1, -1)$$
$$\Sigma_1 = (+1, -1, +1, -1)$$
$$\Sigma_2 = (+1, -1, -1, +1)$$

Non-compact real forms

[BH, van Tongeren]

- The three deformed backgrounds are related by analytic continuation.

$$ds_0^2 = \frac{1}{1-\eta^2\rho^2}\Big(-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2}\Big) + \frac{1}{1+\eta^2\rho^4x^2}\Big(\rho^2(1-x^2)d\psi_1^2 + \frac{\rho^2dx^2}{1-x^2}\Big) + \rho^2x^2d\psi_2^2$$

$$B_0 = \frac{\eta\rho}{1-\eta^2\rho^2}\,dt\wedge d\rho + \frac{\eta\rho^4x}{1+\eta^2\rho^4x^2}\,d\psi_1\wedge dx$$

- ds_1^2 and B_1 given by

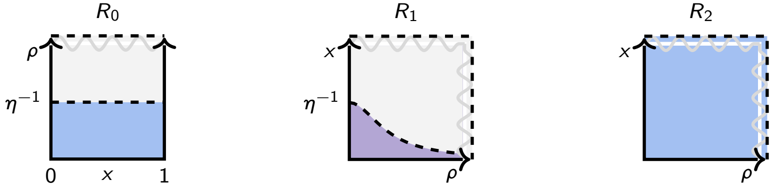
$$t\to\psi_1 \quad \psi_1\to t \quad \psi_2\to\psi_2 \quad \rho\to i\sqrt{1+\rho^2} \quad x\to ix$$

- ds_2^2 and B_2 given by

$$t\to\psi_2 \quad \psi_1\to\psi_1 \quad \psi_2\to t \quad \rho\to i\sqrt{1+\rho^2} \quad x\to\sqrt{1+x^2}$$

Non-compact real forms

[BH, van Tongeren]



- The dashed lines represent singularities. We focus on the blue and purple regions, which are connected to AdS_5 in the undeformed limit.
- The backgrounds from R_0 and R_2 are related by a real η dependent diffeomorphism, up to a closed B-field.
- Naively, the background from R_1 is unrelated to the others given the structure of the H-flux.

Non-compact real forms

[Arutyunov, Borsato, Frolov]
[BH, van Tongeren]

- Consider the η deformation of $\mathrm{AdS}_5 \times \mathrm{S}^1_\phi$.
- Fix light-cone gauge around the BMN-like geodesic $t = \phi = \tau$.
- Four massive "transverse" degrees of freedom Z^i

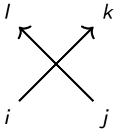
$$\mathcal{L} = \partial_+ Z_i \partial_- Z^i - (1+\eta^2) Z_i Z^i + \text{interactions}$$

- The interactions break Lorentz invariance.
- Compute the tree-level $2 \to 2$ S-matrix in the blue and purple regions.

Non-compact real forms

[Arutyunov, Borsato, Frolov]
[BH, van Tongeren]

$S_{ij}^{kl}(p_1, p_2) \quad =$



- The two S-matrices are different, but both have all the expected properties including satisfying the classical Yang-Baxter equation.
- They are related by a unitary one-particle momentum-dependent change of basis.

$$|Z^i(p)\rangle \to U^i{}_j(p)|Z^j(p)\rangle \qquad UU^\dagger = 1 \quad p \in \mathbb{R}$$

- Not equivalent to a standard field redefinition!

Non-compact real forms

- For AdS_3 we have two R-matrices. The backgrounds are related by a **real η dependent diffeomorphism**, up to a closed B-field.
- For AdS_5 we have three R-matrices.
  - Two of the backgrounds are related by a **real η dependent diffeomorphism**, up to a closed B-field.
  - The final one appears to be independent.
  - However, the light-cone gauge S-matrices are related by a **momentum-dependent change of basis**.
- Different backgrounds, but physics is related.

Superspheres

[Alfimov, BH, Feigin, Litvinov (in progress)]

- One of the simplest examples of supermanifolds is the supersphere

$$S^{N|2m} = \frac{\mathrm{OSp}(N+1|2m)}{\mathrm{OSp}(N|2m)}$$

- The undeformed and deformed actions are the same as before, with the trace, Tr, replaced by the supertrace, STr.

- The deformed model solves the one-loop RG flow equations with $(\nu = \eta h^{-1})$

$$\dot{\nu} = 0 \qquad \dot{\eta} = -\nu(N - 2m - 1)(1 + \eta^2)$$

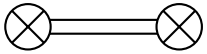
Superspheres

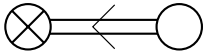
[Alfimov, BH, Feigin, Litvinov (in progress)]

- Superalgebras often have multiple Dynkin diagrams with inequivalent Cartan–Weyl bases. This may lead to different deformed models.

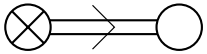
- The two simplest superspheres whose superisometry algebra has more than one Dynkin diagram are $S^{1|2}$ and $S^{2|2}$

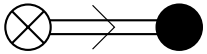
$$\mathrm{S}^{1|2} = \frac{\mathrm{OSp}(2|2)}{\mathrm{OSp}(1|2)}$$





$$\mathrm{S}^{2|2} = \frac{\mathrm{OSp}(3|2)}{\mathrm{OSp}(2|2)}$$





- For $\mathrm{S}^{1|2}$ the two models are related by a real η dependent field redefinition, up to a total derivative.

Superspheres

[Alfimov, BH, Feigin, Litvinov (in progress)]

- Let us focus on $S^{2|2}$
  - bosonic truncation is S^2 with coordinates r and ϕ,
  - two Grassmann-odd coordinates, ψ^1 and ψ^2,
  - $\chi \cdot \chi' = \epsilon_{ab}\chi^a\chi'^b$ and $\chi \wedge \chi' = \delta_{ab}\chi^a\chi'^b$.
- Embedded in $\mathbb{R}^{3|2}$ it is defined by the constraint

$$X_1^2 + X_2^2 + X_3^2 - \Psi \cdot \Psi = 1$$

Superspheres

[Alfimov, BH, Feigin, Litvinov (in progress)]

- Two choices of parametrisation for the undeformed model

$$X_1 + iX_2 = \sqrt{1+\psi\cdot\psi}\sqrt{1-r^2}\, e^{i\phi} \qquad X_3 = \sqrt{1+\psi\cdot\psi}\, r \qquad \Psi = \psi$$

$$\mathcal{L}^{(i)} = (1+\psi\cdot\psi)\Big(\frac{\partial_+ r\partial_- r}{1-r^2} + (1-r^2)\partial_+\phi\partial_-\phi\Big) - (1+\tfrac{1}{2}\psi\cdot\psi)\partial_+\psi\cdot\partial_-\psi$$

$$X_1 + iX_2 = \sqrt{1-r^2}\, e^{i\phi} \qquad X_3 = \sqrt{1+\psi\cdot\psi}\, r \qquad \Psi = r\psi$$

$$\mathcal{L}^{(ii)} = \frac{\partial_+ r\partial_- r}{1-r^2} + (1-r^2)\partial_+\phi\partial_-\phi - r^2(1+\tfrac{1}{2}\psi\cdot\psi)\partial_+\psi\cdot\partial_-\psi$$

Superspheres

[Alfimov, BH, Feigin, Litvinov (in progress)]

- The two deformed Lagrangians are

$$\mathcal{L}_\eta^{(i)} = \frac{1+\eta^2 r^2 + (1-\eta^2 r^2)\psi\cdot\psi}{(1+\eta^2 r^2)^2}\Big(\frac{\partial_+ r \partial_- r}{1-r^2} + (1-r^2)\partial_+\phi\partial_-\phi + \eta r(1+\psi\cdot\psi)(\partial_+ r\partial_-\phi - \partial_+\phi\partial_- r)\Big) - \frac{1+\eta^2+\frac{1}{2}(1-\eta^2)\psi\cdot\psi}{(1+\eta^2)^2}\Big(\partial_+\psi\cdot\partial_-\psi - \eta(1+\tfrac{1}{2}\psi\cdot\psi)\partial_+\psi\wedge\partial_-\psi\Big)$$

$$\mathcal{L}_\eta^{(ii)} = \frac{1}{1+\eta^2 r^2}\Big(\frac{\partial_+ r \partial_- r}{1-r^2} + (1-r^2)\partial_+\phi\partial_-\phi + \eta r(\partial_+ r\partial_-\phi - \partial_+\phi\partial_- r)\Big) - \frac{r^2(1+\eta^2 r^4 + \frac{1}{2}(1-\eta^2 r^4)\psi\cdot\psi)}{(1+\eta^2 r^4)^2}\Big(\partial_+\psi\cdot\partial_-\psi - \eta r^2(1+\tfrac{1}{2}\psi\cdot\psi)\partial_+\psi\wedge\partial_-\psi\Big)$$

- Naively, the models are unrelated given the structure of the H-flux.

Superspheres

[Alfimov, BH, Feigin, Litvinov (in progress)]

- Curiously both can be found from the η deformation of S^4

$$\mathcal{L} = \frac{1}{1+\eta^2 r_1^2}\left(\frac{\partial_+ r_1 \partial_- r_1}{1-r_1^2} + (1-r_1^2)\partial_+\phi_1\partial_-\phi_1 + \eta r_1(\partial_+ r_1\partial_-\phi_1 - \partial_+\phi_1\partial_- r_1)\right)$$
$$+ \frac{r_1^2}{1+\eta^2 r_1^4 r_2^2}\left(\frac{\partial_+ r_2 \partial_- r_2}{1-r_2^2} + (1-r_2^2)\partial_+\phi_2\partial_-\phi_2 + \eta r_1^2 r_2(\partial_+ r_2\partial_-\phi_2 - \partial_+\phi_2\partial_- r_2)\right)$$

  – Choose one of the two-spheres, (r_1, ϕ_1) or (r_2, ϕ_2),

  – change to stereographic coordinates

$$z = \sqrt{2}\sqrt{\frac{1-r_i}{1+r_i}}\, e^{i\phi_i}$$

  – and then formally replace

$$z\bar{z} \to -\tfrac{1}{2}\psi\cdot\psi \qquad \partial_+ z\partial_-\bar{z} + \partial_+\bar{z}\partial_- z \to -\partial_+\psi\cdot\partial_-\psi$$
$$\partial_+ z\partial_-\bar{z} - \partial_+\bar{z}\partial_- z \to i\partial_+\psi\wedge\partial_-\psi$$

Superspheres

[Alfimov, BH, Feigin, Litvinov (in progress)]

- Also works as a way of constructing the three η deformations of $S^{4|2}$ from the η deformation of S^6.

- Conjectured to work as a way of constructing $\binom{\lfloor \frac{N}{2} \rfloor + m}{m}$ η deformations of $S^{N|2m}$ from the η deformation S^{N+2m}.

  — For even N this matches the number of Dynkin diagrams of $\mathfrak{osp}(N+1|2m)$.

  — However, there are $\binom{\frac{N+1}{2} + m}{m}$ Dynkin diagrams of $\mathfrak{osp}(N+1|2m)$ for odd N.

  — Therefore, either some deformations cannot be found via this trick, or they are equivalent to those that can be found. For $S^{1|2}$ it is the latter.

- Reminiscent of the relation between the three η deformations of AdS_5 and the η deformation of S^5 by analytic continuation!

Superspheres

- Different Dynkin diagrams have inequivalent Cartan–Weyl bases and may give different deformed models.
- Is there a sense in which they are equivalent?
  - Are the q-deformed algebras isomorphic?
  - Are the deformed $\mathrm{OSp}(N|2m)$ S-matrices related?
  - What are the dual integrable models at strong coupling?

[Alfimov, BH, Feigin, Litvinov (in progress)]

Poisson–Lie duality

- The undeformed symmetric space σ-model on the coset G/H can be dualised in any subgroup of G via the usual gauging procedure.
- It is well-known that the η deformation has Poisson–Lie symmetry.
- Which dualities survive the deformation?

Poisson–Lie duality

[Klimčík, Ševera; Sfetsos; Squellari]

- To Poisson–Lie dualise we lift the action of the η deformed symmetric space σ-model to a first-order action on the doubled space $G^{\mathbb{C}}/H$.
- We start from the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{b}$$

where $\mathfrak{b}$ is the Borel subalgebra picked out by the Cartan–Weyl basis. For simplicity, we take $\mathfrak{g}$ to be compact and simple. Then

$$\mathfrak{g} = \text{span}_{\mathbb{R}}\{ih_i, i(e_m + f_m), -(e_m - f_m)\} \qquad \mathfrak{b} = \text{span}_{\mathbb{R}}\{h_i, e_m, ie_m\}$$

Poisson–Lie duality

[Klimčík, Ševera; Sfetsos; Squellari]

- The two real subalgebras $\mathfrak{g}$ and $\mathfrak{b}$ are maximally isotropic with respect to the non-degenerate ad-invariant bilinear form on $\mathfrak{g}^{\mathbb{C}}$

$$\langle X, Y \rangle = \operatorname{Im} \operatorname{Tr}[XY] \qquad \langle \mathfrak{g}, \mathfrak{g} \rangle = \langle \mathfrak{b}, \mathfrak{b} \rangle = 0$$

Note that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and hence is also isotropic.

- The first-order action of Klimčík and Ševera on $\mathrm{G}^{\mathbb{C}}/\mathrm{H}$ is

$$\mathcal{S} = \frac{1}{2} \int d\tau d\sigma \big(\langle l^{-1}\partial_\sigma l, l^{-1}\partial_\tau l \rangle - K(l^{-1}\partial_\sigma l) \big) + \mathrm{WZ}(l) \qquad l \in \mathrm{G}^{\mathbb{C}}$$

  – K is a bilinear form defined in terms of the R-matrix and respects the H gauge invariance. $\mathrm{WZ}(l)$ is the standard WZ term.

Poisson–Lie duality

[Klimčík, Ševera]

- Given any maximally isotropic subalgebra $\tilde{\mathfrak{k}} \subset \mathfrak{g}^{\mathbb{C}}$, that is $\langle \tilde{\mathfrak{k}}, \tilde{\mathfrak{k}} \rangle = 0$ and $\dim \tilde{\mathfrak{k}} = \dim \mathfrak{g}$ we can parametrise

$$l = \tilde{k} k \qquad \tilde{k} \in \tilde{\mathrm{K}} \quad k \in \tilde{\mathrm{K}} \backslash \mathrm{G}^{\mathbb{C}} / \mathrm{H}$$

- The first-order action only depends on $\tilde{k}$ through $p = \tilde{k}^{-1} \partial_\sigma \tilde{k}$. Integrating out p gives a second-order Lorentz-invariant model on

$$\boxed{\tilde{\mathrm{K}} \backslash \mathrm{G}^{\mathbb{C}} / \mathrm{H}}$$

- Taking $\tilde{\mathrm{K}} = \mathrm{B}$ we have $\mathrm{B} \backslash \mathrm{G}^{\mathbb{C}} \cong \mathrm{G}$ and we recover the η deformation.
- However, in general $\tilde{\mathrm{K}} \backslash \mathrm{G}^{\mathbb{C}}$ need not be a group.

Poisson–Lie duality

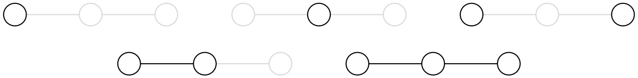
[BH, Seibold; Osten Lüst]

- For which subalgebras $\mathfrak{g}_0 \subset \mathfrak{g}$ does the orthogonal complement $\mathfrak{b}_1 \subset \mathfrak{b}$ also form a subalgebra?
- In this case $\tilde{\mathfrak{k}} = \mathfrak{g}_0 \oplus \mathfrak{b}_1$ is a maximally isotropic subalgebra of $\mathfrak{g}^{\mathbb{C}}$.
- If $\mathfrak{g}_0$ corresponds to a sub-Dynkin diagram (up to additional Cartan generators) then $\mathfrak{b}_1$ is a subalgebra.
- In these cases it transpires that $[\mathfrak{g}_0, \mathfrak{b}_1] \subset \mathfrak{b}_1$ as well.
- This implies that the Poisson–Lie symmetry consistently truncates to the subalgebra.

Poisson–Lie duality

[BH, Seibold]

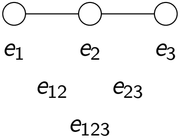
- As a first example we take the η deformation of $S^5 \cong \frac{SO(6)}{SO(5)} \cong \frac{SU(4)}{Sp(2)}$.
- We can dualise in SU(2), SU(2) $\times$ SU(2), SU(3) or SU(4)



- Adding additional Cartan generators amounts to additional abelian T-dualities in these directions.
- However, it is not clear if we can dualise in Sp(2) $\subset$ SU(4), or a different SU(2) subgroup, for example.

Poisson–Lie duality

- This is reminiscent of the structure of q-deformed Hopf algebras



- The coproduct of a non-simple root depends on its constituent roots.

For example, the coproduct of e_{12} depends on e_1 and e_2.

- Sub-Hopf algebras can also be constructed from sub-Dynkin diagrams.

Poisson–Lie duality

[Klimčík, Ševera; Sfetsos]

- Now consider the η deformation of $\mathrm{S}^2 \cong \dfrac{\mathrm{SO}(3)}{\mathrm{SO}(2)} \cong \dfrac{\mathrm{SU}(2)}{\mathrm{U}(1)}$.
- The doubled algebra is

$$\mathfrak{sl}(2;\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{b}(2) = \mathrm{span}_{\mathbb{C}}\{h, e, f\}$$

$$\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\{ih, i(e+f), -(e-f)\} \qquad \mathfrak{b}(2) = \mathrm{span}_{\mathbb{R}}\{h, e, ie\}$$

- Choosing $\tilde{\mathrm{K}} = \mathrm{B}(2)$ gives the η deformation of S^2

$$ds^2 = \frac{1}{1+\eta^2 r^2}\left(\frac{dr^2}{1-r^2} + (1-r^2)d\phi^2\right)$$

$$B = -\frac{\eta r}{1+\eta^2 r^2}\, d\phi \wedge dr$$

Poisson–Lie duality

[Klimčík, Ševera; Sfetsos]

- Now consider the η deformation of $\mathrm{S}^2 \cong \dfrac{\mathrm{SO}(3)}{\mathrm{SO}(2)} \cong \dfrac{\mathrm{SU}(2)}{\mathrm{U}(1)}$.
- The doubled algebra is

$$\mathfrak{sl}(2;\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{b}(2) = \mathrm{span}_{\mathbb{C}}\{h, e, f\}$$

$$\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\{ih, i(e+f), -(e-f)\} \qquad \mathfrak{b}(2) = \mathrm{span}_{\mathbb{R}}\{h, e, ie\}$$

- Choosing $\tilde{\mathrm{K}} = \mathrm{SU}(2)$ gives the Poisson–Lie dual of the η deformation

$$ds^2 = \frac{1}{p^2 - q^2 - 1}\left(\frac{dp^2}{\eta^2} + dq^2\right)$$

- This is the analytic continuation of the λ deformation of S^2.

[Hollowood, Miramontes, Schmidtt; Sfetsos]
[Vicedo; BH, Tseytlin; Sfetsos, Siampos, Thompson; Klimčík]

Poisson–Lie duality

- Now consider the η deformation of $\mathsf{S}^2 \cong \dfrac{\mathsf{SO}(3)}{\mathsf{SO}(2)} \cong \dfrac{\mathsf{SU}(2)}{\mathsf{U}(1)}$.
- The doubled algebra is

$$\mathfrak{sl}(2;\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{b}(2) = \text{span}_{\mathbb{C}}\{h, e, f\}$$

$$\mathfrak{su}(2) = \text{span}_{\mathbb{R}}\{ih, i(e+f), -(e-f)\} \qquad \mathfrak{b}(2) = \text{span}_{\mathbb{R}}\{h, e, ie\}$$

- Another maximally isotropic subalgebra is $\mathfrak{iso}(2) = \text{span}_{\mathbb{R}}\{ih, e, ie\}$.

Choosing $\tilde{\mathsf{K}} = \mathsf{ISO}(2)$ gives the T-dual of the η deformation

$$ds^2 = \frac{1}{1-r^2}\left(dr^2 - 2r\,dr\,d\tilde{\phi} + \eta^{-2}(1+\eta^2 r^2)d\tilde{\phi}^2\right)$$

[BH, Seibold]

Poisson–Lie duality

[BH, Tseytlin; BH, Seibold]

- Setting $p = \cosh\tilde{\phi}$ and $q = r\sinh\tilde{\phi}$ in the Poisson–Lie dual metric

$$ds^2 = \frac{1}{1-r^2}\Big(dr^2 + 2r\coth\tilde{\phi}\,dr\,d\tilde{\phi} + \eta^{-2}(1+\eta^2 r^2\coth^2\tilde{\phi})d\tilde{\phi}^2\Big)$$

$$\Big\downarrow \quad \tilde{\phi} \to \tilde{\phi} - \gamma \qquad \gamma \to \infty$$

$$ds^2 = \frac{1}{1-r^2}\Big(dr^2 - 2r\,dr\,d\tilde{\phi} + \eta^{-2}(1+\eta^2 r^2)d\tilde{\phi}^2\Big)$$

- The T-dual is the scaling limit of the Poisson–Lie dual.

- This is analogous to similar limits in non-abelian duality. For example, the T-dual in the Hopf fibre of S^3 is a limit of the SU(2) dual.

[Macpherson, Núñez, Thompson, Zacarias; Lozano, Núñez]

Poisson–Lie duality

[BH, Seibold]

- Algebraically this corresponds to the contraction limit $\mathfrak{su}(2) \to \mathfrak{iso}(2)$

$$\{ih, i(e+f), -(e-f)\}$$

$$\Big\downarrow \text{ deform the embedding of } \mathfrak{su}(2)$$

$$e^{-\gamma h}\{ih, ie^{2\gamma}(e+f), -e^{2\gamma}(e-f)\}e^{\gamma h}$$

$$\Big\downarrow \ \gamma \to -\infty$$

$$\{ih, ie, -e\}$$

- In the $\gamma \to -\infty$ limit the Poisson–Lie dual becomes the T-dual.

What happens in string theory?

- We conclude with various general comments on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring and its η deformations.

- The worldsheet action in the Green-Schwarz formalism is described by the semi-symmetric space σ-model on the supercoset

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1)\times\mathrm{Sp}(2)}$$

[Metsaev, Tseytlin]

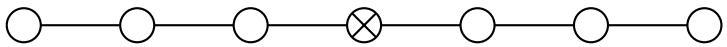
- This model also admits a Yang–Baxter deformation.

[Delduc, Magro, Vicedo]
[Kawaguchi, Matsumoto, Yoshida]

- $\mathfrak{psu}(2,2|4)$ has 35 Dynkin diagrams (at least in one way of counting)!

What happens in string theory?

- The distinguished Dynkin diagram is



- The dual Borel algebra is not unimodular and hence the η deformed background is not a supergravity solution. [Arutyunov, Borsato, Frolov; Borsato, Wulff]

- Instead the background solves a set of generalised equations. [Arutyunov, Frolov, BH, Roiban, Tseytlin; Tseytlin, Wulff]

- Weyl invariance and critical string theory remain unclear, but there has been interesting work in this direction. [Fernández-Melgarejo, Sakamoto, Sakatani, Yoshida; Mück]

- The light-cone gauge tree-level S-matrix is not immediately consistent with integrability. [Arutyunov, Borsato, Frolov]

What happens in string theory?

- The bosonic Dynkin diagram is a sub-diagram

- All background fields, the metric, B-field, dilaton and RR fluxes, respect the bosonic Poisson–Lie symmetry.

- This is why it is this background that is found via two space-time approaches to determining the RR fluxes:
  - invariance of Page forms, [Araujo, Ó Colgáin, Yavartanoo]
  - bosonic Poisson–Lie symmetry. [Demulder, Hassler, Thompson]

What happens in string theory?

- We can dualise in the bosonic subalgebra

- In this case $\tilde{\mathfrak{k}}$ is unimodular and we expect a supergravity* solution.

[BH, Seibold (for $AdS_2 \times S^2 \times T^6$)]

- The background can be found by generalising the bosonic non-abelian duality rules. The metric and B-field are those of the λ deformation up to analytic continuation.

[Sfetsos, Thompson; Demulder, Sfetsos, Thompson]
[Hollowood, Miramontes, Schmidtt; Sfetsos]

- Scaling in all Cartan directions gives the complete T-dual of this η deformation, which is also a supergravity* solution. [BH, Tseytlin (for $AdS_2 \times S^2 \times T^6$)]

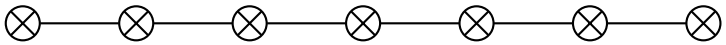
What happens in string theory?

- We can also dualise in the full superalgebra

- Again $\tilde{\mathfrak{k}}$ is unimodular and we expect a supergravity* solution.
- Is this related to the λ deformation? [Hollowood, Miramontes, Schmidtt]

What happens in string theory?

- The fermionic Dynkin diagram is



- The dual Borel algebra is unimodular and hence the η deformed background is a supergravity solution. This is the only Dynkin diagram for which this is true.

[BH, Seibold; Lunin, Roiban, Tseytlin; Borsato, Wulff]

- Extracting the RR fluxes is computationally expensive.

- The bosonic Dynkin diagram is not a sub-diagram. The RR fluxes do not respect the bosonic Poisson–Lie symmetry. Therefore, it is not clear if it is possible to dualise in the bosonic subalgebra.

- Is the dual in the full superalgebra related to the λ deformation?

- What about the light-cone gauge tree-level S-matrix?

Thank you!

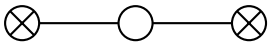
What happens in string theory?

- To gain some insights we can look at $\mathrm{AdS}_2 \times \mathrm{S}^2 \times \mathrm{T}^6$ and the supercoset

$$\frac{\mathrm{PSU}(1,1|2)}{\mathrm{SU}(1,1) \times \mathrm{SU}(2)}$$

- $\mathfrak{psu}(1,1|2)$ has three Dynkin diagrams

distinguished: 

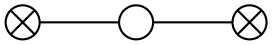
the second one: 

fermionic: 

What happens in string theory?

- There is a known double scaling limit of the λ deformation that, up to analytic continuation, gives the complete T-dual of the η deformation for the distinguished Dynkin diagram. [Borsato, Tseytlin, Wulff]

- However, analysing this limit we find that actually it should give the complete T-dual of the η deformation for the Dynkin diagram



- This is consistent as this η deformation turns out to be the same as that for the distinguished Dynkin diagram, up to a closed B-field. [BH, Seibold]

What happens in string theory?

- Only doing the first scaling should give the dual in $\mathfrak{su}(1|1) \oplus \mathfrak{su}(1|1)$

- The second scaling can then be done in two ways. In one case the bosonic node generates the $\mathfrak{su}(1,1)$ of AdS_2 and in the other, the $\mathfrak{su}(2)$ of S^2. Indeed, this Dynkin diagram has an asymmetry between AdS_2 and S^2.

- It is not known if it is possible to recover the bosonic dual of the η deformation for the distinguished Dynkin diagram, or the complete T-dual of the η deformation for the fermionic Dynkin diagram.

- This suggests that the λ deformation is related to dual of the η deformation in the full superalgebra for the second Dynkin diagram. However, it could also be related to this dual for the remaining two Dynkin diagrams.