

Quantum information from symmetry, geometry and the effective action

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Based mainly on:

- ▶ work with G. Georgiou, P. Panopoulos & E. Sagkrioti,
[1906.Today \[hep-th\]](#).

General motivation

- ▶ The **quantum behavior** of **interacting field theories** is encoded on how the **couplings** change with **energy** and on how the **operators** acquire **anomalous** dimensions.
- ▶ Studied in the framework of **RG** (since [Wilson 71]). Results into **1st order non-linear differential systems**, i.e. **β -function** eqs.
- ▶ Traditionally computed perturbatively. For example

$$\text{QCD:} \quad \mu \frac{dg}{d\mu} = -\frac{7g^3}{16\pi^2} + \dots$$

A **rare occasion** and very **important** to be able to compute them **exactly**, i.e. re-sum perturbative expansion.

Knowing the β -functions and anomalous dimensions exactly:

- ▶ We may discover new fixed point theories towards the IR.
- ▶ We may investigate the fate of degrees of freedom in the flow from the UV to the IR, i.e. Zamolodchikov's C-theorem.
- ▶ We will learn more on the structure of QFTs.

I will concentrate on 2-dim Conformal Field Theories as a basis:

- ▶ Systematic construction of new (integrable) deformations of (interacting) CFT's having explicit Lagrangian descriptions.
- ▶ Smooth RG flows (UV to IR) between CFTs.
- ▶ β -function and anomalous dimensions of general operators, essentially without loop computations.

Not to cover here at all, but serving as extra motivation:

- ▶ Embedding to type-II supergravity: [KS-Thompson 14, Demulder-KS-Thompson15, Borsato-Tseytlin-Wulff16, Chervonyi-Lunin16].
- ▶ Conceivable usage in an AdS/CFT context.

Outline

- ▶ The theories of interest
- ▶ Effective actions: Self-interacting theories.
- ▶ Exact β -functions and anomalous dimensions.

Perturbative info + symmetry + analyticity \implies exact info

- ▶ A more powerful and easier to apply method.

Effective action + geometry of couplings \implies exact info

- ▶ The anomalous dims of $d_{a_1 \dots a_m}^{(m)} J_+^{a_1} \dots J_+^{a_m}$ and $d_{abc} J_+^a J_+^b J_-^c$.
- ▶ Mutually interacting theories.
- ▶ Zamolodchikov's C-function.
- ▶ Concluding remarks.

The theories of interest

Seeds in old works: The non-Abelian Thirring model

(1+1)-dim fermions ψ^a in the \square of $SU(N)$ [Dashen-Frishman 73 & 75].

- ▶ A non-Abelian generalization of the Thirring model [Thirring 58]

$$\mathcal{L} = \overbrace{\bar{\psi}\gamma^\mu\partial_\mu\psi}^{\text{free action}} - \frac{g_v}{2} \overbrace{J_\mu^a J^{a\mu}}^{\text{interaction}}$$

Fermion bilinear: $J_\mu^a = \bar{\psi}t^a\gamma_\mu\psi$, $a = 1, 2, \dots, N^2 - 1$.

- ▶ **Scale invariant** for $g_v = 0$ and $g_v = \frac{4\pi}{N+1}$.
The **scaling** dimension Δ of the fermions is

$$g_v = 0 : \quad \Delta = \frac{1}{2}$$
$$g_v = \frac{4\pi}{N+1} : \quad \Delta = \frac{1}{2} + \frac{N-1}{N}$$

- ▶ A **non-trivial** conformal point. Fermions get **new** dimension.
Admits a **large N** limit, by keeping **$gN = \text{fixed}$** .

Bosonic version leads to generalization

Generalization via the steps:

- ▶ Replace the **free fermion theory** by a 2-dim **CFT** with action S_k having a **current algebra** for a **group G** (replaces $SU(N)$)
- ▶ Keep the **interaction bilinear** in currents the same

$$S_{k,\lambda} = S_k + \frac{k}{\pi} \int d^2\sigma \lambda_{ab} J_+^a J_-^b, \quad a, b = 1, 2, \dots, \dim G .$$

- ▶ Near the CFT point

$$\partial_{\mp} J_{\pm}^a = \mathcal{O}(\lambda) .$$

- ▶ For $\lambda_{ab} = 0$ the J_{\pm} 's obey the OPE's

$$J^a(z) J^b(w) = i f_{abc} \frac{J^c}{z-w} + k \frac{\delta^{ab}}{(z-w)^2} \quad (\text{Kac-Moody algebra}) .$$

Infinite dimensional extension of the **Lie-algebra**.

Effective actions: Self-interacting theories [KS 13]

For a group G and group elements $g, \tilde{g} \in G$

$$S(g, \tilde{g}) = S_k(g) + S_{\text{PCM}, \kappa^2}(\tilde{g}) .$$

- ▶ The **WZW** action $S_k(g)$ is a **CFT**. It has a $G_{L, \text{cur}} \times G_{R, \text{cur}}$ **current** symmetry at **level** $k \in \mathbb{Z}^+$ generated by

$$J_+^a = -i\text{Tr}(t^a \partial_+ g g^{-1}) , \quad J_-^a = -i\text{Tr}(t^a g^{-1} \partial_- g) .$$

- ▶ The **Principal Chiral model** action (PCM) is

$$S_{\text{PCM}, \kappa^2}(\tilde{g}) = -\frac{\kappa^2}{\pi} \int \text{Tr}(\tilde{g}^{-1} \partial_+ \tilde{g} \tilde{g}^{-1} \partial_- \tilde{g}) .$$

It is **integrable** with global $G_L \times G_R$ symmetry.

Introduction of interactions via gauge fields

Gauge the global symmetry group acting as

$$g \rightarrow \Lambda^{-1}g\Lambda, \quad \tilde{g} \rightarrow \Lambda^{-1}\tilde{g}, \quad \Lambda \in G$$

and we consider the action

$$S_{k,\kappa^2}(g, \tilde{g}, A_{\pm}) = S_k(g, A_{\pm}) + S_{\text{PCM}}(\tilde{g}, A_{\pm}).$$

- ▶ The gauged WZW action is

$$S_k(g, A_{\pm}) = S_k(g) + \frac{k}{\pi} \int \text{Tr} \left(A_- \partial_+ g g^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1} - A_- A_+ \right).$$

- ▶ The gauged PCM action is

$$S_{\text{PCM}}(\tilde{g}, A_{\pm}) = -\frac{\kappa^2}{\pi} \int \text{Tr}(\tilde{g}^{-1} \tilde{D}_+ \tilde{g} \tilde{g}^{-1} \tilde{D}_- \tilde{g}),$$

with the covariant derivatives being $D_{\pm} \tilde{g} = \partial_{\pm} \tilde{g} - A_{\pm} \tilde{g}$.

After gauge fixing $\tilde{g} = \mathbb{1}$ and integrating out the gauge fields

$$S_{k,\lambda}(g) = S_k(g) + \frac{k}{\pi} \int J_+^a (\lambda^{-1} \mathbb{I} - D^T)^{-1}_{ab} J_-^b ,$$

the λ -deformed σ -model action, where

$$\lambda = \frac{k}{k + \kappa^2} , \quad 0 < \lambda < 1 ,$$

is the deformation parameter and

$$J_+^a = -i \text{Tr}(t^a \partial_+ g g^{-1}) , \quad J_-^a = -i \text{Tr}(t^a g^{-1} \partial_- g) , \\ D_{ab} = \text{Tr}(t_a g t_b g^{-1}) .$$

- ▶ Since D is orthogonal this action is non-singular.
- ▶ Singularities appear only when $\lambda \rightarrow \pm 1$. Letting also $k \rightarrow \infty$

$$(\lambda + 1)k = \text{finite} , \quad (\lambda + 1)^3 k = \text{finite} ,$$

Non-Abelian & Pseudo-dual limits [KS 13, Georgiou-KS-Siampos 16].

Some properties

- ▶ Generalization to $\lambda\delta_{ab} \rightarrow \lambda_{ab}$ straightforward.
- ▶ For $\lambda \ll 1$

$$S_{k,\lambda}(g) = S_k(g) + \frac{k}{\pi} \int d^2\sigma \lambda_{ab} J_+^a J_-^b + \dots$$

- ▶ The **full action** $S_{k,\lambda}(g)$ has a **duality-type** symmetry [Itsios-KS-Siampos 14]

$$\boxed{k \rightarrow -k, \quad \lambda \rightarrow \lambda^{-1}, \quad g \rightarrow g^{-1}}.$$

It should be reflected as a **symmetry** in physical quantities.

- ▶ **Integrable** for special forms of λ_{ab} ,
eg. [KS 13, Hollowood-Miramontes-Schmidt 14]

$$\lambda_{ab} = \lambda\delta_{ab}.$$

Not integrable at any **finite** order in λ .

More integrable cases [KS-Siampos 14, KS-Siampos-Thompson 15].

Exact β -function and anomalous dims [Georgiou-Siampos-KS 15 & 16]

CFT and symmetry approach

- ▶ We want to compute the **2-point functions**

$$\begin{aligned}\langle J^a(x_1)J^b(x_2)\rangle_\lambda &= \langle J^a(x_1)J^b(x_2)e^{-\frac{\lambda}{\pi}\int J^a\bar{J}^a}\rangle, \\ \langle J^a(x_1)\bar{J}^b(x_2)\rangle_\lambda &= \langle J^a(x_1)\bar{J}^b(x_2)e^{-\frac{\lambda}{\pi}\int J^a\bar{J}^a}\rangle,\end{aligned}$$

perturbatively in λ by expanding the exponential.

- ▶ The **basic correlators** are

$$\langle J^a(x_1)J^b(x_2)\rangle = \frac{\delta_{ab}}{x_{12}^2}, \quad \langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle = \frac{1}{\sqrt{k}} \frac{f_{abc}}{x_{12}x_{13}x_{23}}$$

and similarly for the \bar{J}^a 's. **Mixed $J\bar{J}$** correlators **vanish**.

- ▶ For higher correlators use **Ward identities**

The perturbative β -function and anomalous dimensions

With effort using “conventional” perturbative techniques:

- ▶ The β -function is

$$\beta = \frac{1}{2}\mu \frac{d\lambda}{d\mu} = -\frac{c_G}{2k}(\lambda^2 - 2\lambda^3) + \mathcal{O}(\lambda^4).$$

c_G : quadratic Casimir in the adjoint, i.e. $f_{acd}f_{bcd} = c_G\delta_{ab}$.

- ▶ The anomalous dimension of the currents is

$$\gamma^{(J)} = \frac{c_G}{k}(\lambda^2 - 2\lambda^3) + \mathcal{O}(\lambda^4).$$

Task: Extend these exactly in λ and leading in $1/k$.

Analyticity: λ -dependence of physical quantities

- ▶ Recall, that the action is **singular** at $\lambda = \pm 1$.
- ▶ The β -function & anomalous dims may have **poles** at $\lambda = \pm 1$.
- ▶ The β -function & anomalous dims should be **invariant** under

$$k \rightarrow -k, \quad \lambda \rightarrow \frac{1}{\lambda}, \quad (\text{for } k \gg 1).$$

- ▶ **Perturbative information** to $\mathcal{O}(\lambda^2)$ and the above **symmetry** are enough to **determine** the β -function and the anomalous dimensions **exactly in λ** and to leading order in k .

The exact β -function and anomalous dimensions

The exact β -function and anomalous dimensions are of the form

$$\beta_\lambda = -\frac{c_G}{2k} \frac{f(\lambda)}{(1+\lambda)^2}, \quad \gamma^{(J)} = \frac{c_G}{k} \frac{g(\lambda)}{(1-\lambda)(1+\lambda)^3},$$

where $f(\lambda)$ and $g(\lambda)$ are analytic in λ .

- ▶ They have a well defined **non-Abelian** and **pseudodual** limits as $\lambda \rightarrow \pm 1$ and $k \rightarrow \infty$, in a **correlated** way.
- ▶ Due to the symmetry $(k, \lambda) \mapsto (-k, \lambda^{-1})$ we have that

$$\lambda^4 f(1/\lambda) = f(\lambda), \quad \lambda^4 g(1/\lambda) = g(\lambda).$$

- ▶ $f(\lambda)$ and $g(\lambda)$ are polynomials of degree four. It is fixed by the **above symmetry** and the **two-loop** perturbative result.

- ▶ The **final result** for the beta-function is

$$\beta^\lambda = -\frac{c_G}{2k} \frac{\lambda^2}{(1+\lambda)^2} \leq 0.$$

- ▶ For the anomalous dimension

$$\gamma^{(J)} = \frac{c_G}{k} \frac{\lambda^2}{(1-\lambda)(1+\lambda)^3} \geq 0.$$

- ▶ **Agreement** with perturbation theory to $\mathcal{O}(\lambda^3)$ and $\mathcal{O}(\lambda^4)$.
- ▶ Similarly for **current 3-point** functions and correlators of **primary fields** [Georgiou-KS-Siampos 16].
- ▶ Extended for the β -function for **general** λ_{ab} [Itsions-KS-Siampos 14, KS-Siampos 14, KS-Siampos-Sagkrioti 18] using the gravitational approach of [Ecker-Honerkamp 71, Friedan 80]

$$\frac{1}{2}\mu \frac{dG_{\mu\nu}}{d\mu} = R_{\mu\nu} + \dots$$

Questions and limitations

- ▶ What about the anomalous dimensions of general **composite operators** of the type

$$\mathcal{O}^{(m,n)} = S_{a_1 \dots a_m; b_1 \dots b_n} J_+^{a_1} \dots J_+^{a_m} J_-^{b_1} \dots J_-^{b_n} ,$$

where the S -tensor belongs to **some irrep** of G .

- ▶ How are the **operators dressed/modified** in the process of the deformation?
- ▶ Clearly the previous approach is **limited**.
- ▶ How can we use the **effective action** to perform such computations?

Anomalous dims from the effective action

[Georgiou-Panopoulos-Sagkrioti-KS, Today]

Recall the action we started with

$$S_{k,\kappa^2}(g, \tilde{g}, A_{\pm}) = S_k(g, A_{\pm}) + S_{\text{PCM}}(\tilde{g}, A_{\pm}) ,$$

to which we add the term

$$\frac{k\mathbf{s}}{\pi} \int S_{a_1 \dots a_m; b_1 \dots b_n} (\tilde{g}^{-1} D_+ \tilde{g})^{a_1} \dots (\tilde{g}^{-1} D_+ \tilde{g})^{a_m} (\tilde{g}^{-1} D_- \tilde{g})^{b_1} \dots (\tilde{g}^{-1} D_- \tilde{g})^{b_n} ,$$

where \mathbf{s} is a **coupling**.

- ▶ This action is still **gauge invariant**, so we gauge fix as $\tilde{g} = \mathbb{1}$.
- ▶ At the end set $\mathbf{s} = 0$; the λ -deformed action is recovered.
- ▶ The **anomalous dimension** of $\mathcal{O}^{(m,n)}$ will be extracted.
- ▶ The eqs. of motion are **non-linear**. Need the $\mathcal{O}(s)$ -terms.

The dressed operators

Integrating out the gauge fields we obtain the action

$$S = S_{k,\lambda}(g) - \frac{ks}{\pi} \int d^2\sigma \mathcal{O}_\lambda^{(m,n)} + \mathcal{O}(s^2) ,$$

where

$$\mathcal{O}_\lambda^{(m,n)} = S_{a_1 \dots a_m; b_1 \dots b_n} A_+^{a_1} \dots A_+^{a_m} A_-^{b_1} \dots A_-^{b_n} .$$

with the classical values for the gauged fields

$$A_+ = i(\lambda^{-1}\mathbb{1} - D)^{-1} J_+ , \quad A_- = -i(\lambda^{-1}\mathbb{1} - D^T)^{-1} J_- .$$

- Obviously, as $\lambda \rightarrow 0$ we have that $A_\pm \sim J_\pm$. Therefore

$$\mathcal{O}_\lambda^{(m,n)} \sim S_{a_1 \dots a_m; b_1 \dots b_n} J_+^{a_1} \dots J_+^{a_m} J_+^{b_1} \dots J_-^{b_n} , \quad \text{as } \lambda \rightarrow 0 .$$

- In the λ -deformed theory the operators are modified .

The RG flow eqs: Essentials of the method

The eqs. of motion can be cast **solely** in terms of the **gauge fields**

$$\begin{aligned}\partial_+ A_- - \lambda \partial_- A_+ &= [A_+, A_-] - m s \lambda D_+ \mathcal{O}_\lambda^{(m',n)} , \\ \partial_- A_+ - \lambda \partial_+ A_- &= -[A_+, A_-] - n s \lambda D_- \mathcal{O}_\lambda^{(m,n')} .\end{aligned}$$

The prime implies a free index and D_\pm are covariant derivatives.

Background field method (in this context by [Appadu-Hollowood 15])

- ▶ Choose a particular group element and construct a classical solution $A_\pm^{(0)}$ exact in λ and leading in s . Consider **fluctuations** around this δA_\pm .
- ▶ Cast them in the form

$$\hat{D} \begin{pmatrix} \delta A_- \\ \delta A_+ \end{pmatrix} = 0 , \quad \hat{D} \text{ is } \mathbf{1st\ order} .$$

To compute the contribution to the classical action

- ▶ Pass to the **Euclidean** regime and to **momentum** space.
- ▶ Split $\hat{D} = \hat{C} + \hat{F}$, where in \hat{C} all **momentum** dependence.
- ▶ **Integrating** out the fluctuations, gives the **effective** Lagrangian

$$-\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \int^{\mu} \frac{d^2 p}{(2\pi)^2} \ln(\det \hat{D})^{-1/2} .$$

- ▶ **Logarithmically divergent** with respect to the **UV mass scale** μ . We perform the **large momentum expansion** and keep terms proportional to $1/|p|^2$.
- ▶ Since \hat{C} grows with $|p|$ we use the fact that

$$\ln \det \hat{D} = \ln \det \hat{C} + \text{Tr}(\hat{C}^{-1} \hat{F}) - \frac{1}{2} \underbrace{\text{Tr}(\hat{C}^{-1} \hat{F})^2}_{\text{relevant-term}} + \dots .$$

The β -functions are obtained by demanding that

$$\frac{1}{2}\mu \frac{d\mathcal{L}_{\text{eff}}}{d\mu} = 0 .$$

- ▶ That gives

$$\beta^\lambda = -\frac{c_G}{2k} \frac{\lambda^2}{(1+\lambda)^2} .$$

- ▶ In general there is **operators mixing** under the RG flow.
- ▶ We will restrict to classes of operators where mixing doesn't occur. This will provide β^5 .

The geometry of coupling space

For **small values** of λ and s

$$S = S_k(g) + \frac{k}{\pi} \int d^2\sigma \left(\lambda J_+^a J_-^a + \tilde{\lambda} \mathcal{O}^{(m,n)} \right) + \dots ,$$

where $\tilde{\lambda} \sim s\lambda^{m+n}$.

- ▶ Consider a **general perturbation** with operators

$$\lambda^i \mathcal{O}_i , \quad \text{beta functions } \beta^i .$$

- ▶ There is **metric** in the space of couplings G_{ij} defined via

$$G_{ij}(\lambda) = x^{d_i+d_j} \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle .$$

- ▶ Assuming renormalizability and using the **Callan-Symanzik** eq. the **anomalous dimension matrix** is [Kutasov 89]

$$\gamma_i^j = \partial_i \beta^j + G^{(0)jm} \left(G_{in}^{(0)} \partial_m \beta^n + \beta^n \partial_n G_{im}^{(0)} \right) .$$

Consider the two couplings: $\mathcal{O}_\lambda = J_+^a J_-^a$ and $\mathcal{O}_{\tilde{\lambda}} = \mathcal{O}^{(m,n)}$.

- ▶ $\mathcal{O}_{\tilde{\lambda}} = \mathcal{O}^{(m,n)}$ **breaks** Lorentz and/or scale invariance.
- ▶ **Operator mixing** and an anomalous dimension matrix γ_i^j .
- ▶ When $\tilde{\lambda} \rightarrow 0$, **no mixing** and only $\gamma_\lambda^\lambda, \gamma_{\tilde{\lambda}}^{\tilde{\lambda}} \neq 0$.
- ▶ Near $\tilde{\lambda} = 0$ the metric becomes **diagonal**

$$G_{ij}^{(0)}(\lambda, \tilde{\lambda}) = \delta_{ij} g_{ii}^{(0)}(\lambda) + \mathcal{O}(\tilde{\lambda}) .$$

This is consequence of the decoupling at $\tilde{\lambda} = 0$.

In the limit $\tilde{\lambda} \rightarrow 0$, the anomalous dimensions are:

$$\gamma_{\mathcal{O}_\lambda} = \gamma_\lambda = 2\partial_\lambda \beta^\lambda + \beta^\lambda \partial_\lambda \ln g_{\lambda\lambda}^{(0)}$$

and that

$$\gamma_{\mathcal{O}_{\tilde{\lambda}}} = \gamma_{\tilde{\lambda}} = 2\partial_{\tilde{\lambda}} \beta^{\tilde{\lambda}} + \beta^{\tilde{\lambda}} \partial_{\tilde{\lambda}} \ln g_{\tilde{\lambda}\tilde{\lambda}}^{(0)} .$$

- ▶ Hence, when $\tilde{\lambda} = 0$, we have the λ -deformed theory and as a **bonus** the **anomalous** dimension of $\mathcal{O}_{\tilde{\lambda}}$.
- ▶ In our case the metric has

$$g_{\lambda\lambda}^{(0)} = \frac{\dim G}{(1 - \lambda^2)^2} , \quad g_{\tilde{\lambda}\tilde{\lambda}}^{(0)} \sim \frac{1}{(1 - \lambda^2)^{m+n}} .$$

Computable with **free field** contractions (**strict $k \rightarrow \infty$ limit**).

- ▶ We first extract γ for $\mathcal{O}_\lambda = J_+^a J_-^a$ [Georgiou, KS, Siampos 15]

$$\gamma_{J_+ J_-} = -\frac{2c_G}{k} \frac{\lambda(1 - \lambda(1 - \lambda))}{(1 - \lambda)(1 + \lambda)^3} = -2\frac{c_G}{k} \lambda + \dots .$$

Example I: Arbitrary number of same chirality operators

We are interested in the anomalous dimension of

$$\mathcal{O}^{(m,0)} = d_{a_1 \dots a_m}^{(m)} J_+^{a_1} \dots J_+^{a_m} ,$$

where $d_{a_1 \dots a_m}^{(m)}$ is the completely symmetric rank- m tensor of $SU(N)$.

- ▶ At CFT point has $\dim m$ [Bais-Bouwknegt-Surridge-Schoutens 88].
- ▶ $d_{a_1 a_2 \dots a_m}^{(m)} t^{a_1} t^{a_2} \dots t^{a_m}$ is a Casimir operator. We derive

$$d_{ab(a_1 \dots a_{m-2} f_c)bd} f_{dae} = -\frac{c_G}{m-1} d_{cea_1 \dots a_{m-2}}^{(m)} , \quad m = 2, 3, \dots$$

and we may show there is no operator mixing under RG flow.

- ▶ Apply the formalism and define $\tilde{\lambda} = s\lambda^m$ (effective coupling)

$$-\mathcal{L}_{\text{eff}} = -\frac{k}{2\pi} \left(\frac{1+\lambda}{1-\lambda} J_+ J_- + \frac{2\tilde{\lambda}}{(1-\lambda)^m} \mathcal{O}^{(m,0)} \right) \\ - \frac{c_G}{2\pi} \ln \mu^2 \frac{\lambda^2}{(1-\lambda^2)^2} \left(J_+ J_- + \frac{m\tilde{\lambda} \mathcal{O}^{(m,0)}}{(1-\lambda)^{m-1}(1+\lambda)} \right) + \mathcal{O}(\tilde{\lambda}^2) .$$

- ▶ Then, demanding $\partial_{\ln \mu^2} \mathcal{L}_{\text{eff}} = 0$ we obtain

$$\beta^{\tilde{\lambda}} = \frac{c_G}{2k} \frac{m \tilde{\lambda} \lambda^3}{(1-\lambda)(1+\lambda)^3} + \mathcal{O}(\tilde{\lambda}^2).$$

- ▶ Using that we obtain

$$\gamma_{\mathcal{O}_\lambda^{(m,0)}} = 0, \quad m = 2, 3, \dots$$

This is quite **general** and **robust**.

- ▶ For $m = 1$: The anomalous dimension of a **single current**

$$\gamma_{\mathcal{O}_\lambda^{(1,0)}} = \gamma_{J_+} = \frac{c_G}{k} \frac{\lambda^2}{(1-\lambda)(1+\lambda)^3},$$

as before.

Conserved currents

- ▶ Due to the classical equations of motion

$$\partial_{\mp} A_{\pm} = \mp \frac{1}{1 + \lambda} [A_{+}, A_{-}] ,$$

as well as of $f_{ab(a_1 a_2 a_3 \dots a_m) b} d_{a_2 a_3 \dots a_m}^{(m)} = 0$, **chiral conservation laws**

$$\partial_{-} \mathcal{O}_{\lambda}^{(m,0)} = 0 , \quad m = 2, 3, \dots .$$

The first one $\mathcal{O}_{\lambda}^{(2,0)} \sim T_{++}$.

- ▶ Since $\gamma_{\mathcal{O}^{(m,0)}} = 0$ this law holds **up to $\mathcal{O}(1/k)$** thanks to

$$\langle \mathcal{O}^{(m,0)}(x_1) \mathcal{O}^{(n,0)}(x_2) \rangle \sim \frac{\delta_{mn}}{x_{12}^{2m}} + \mathcal{O}(1/k^2) ,$$

exactly in λ and to no operator mixing.

Example II: Mixed chirality operators

We are interested in the anomalous dimension of

$$\mathcal{O}^{(2,1)} = d_{abc} J_+^a J_+^b J_-^c ,$$

where d_{abc} is the **symmetric** tensor of $SU(N)$ of rank 3.

At CFT point this has **dim (2, 1)** (since $Q^a = d_{abc} J_+^b J_+^c$ has **dim 2** [Bais-Bouwknegt-Surridge-Schoutens 88]).

Following a similar procedure:

- ▶ We find that

$$\gamma_{\mathcal{O}_\lambda^{(2,1)}} = -\frac{2c_G}{k} \frac{\lambda(1-\lambda(1-\lambda))}{(1-\lambda)(1+\lambda)^3} .$$

- ▶ This is the same as that for

$$\gamma_{\mathcal{O}_\lambda^{(2,1)}} = \gamma_{J_+ J_-} = \gamma_{\mathcal{O}_\lambda^{(1,1)}} ,$$

quite **unexpected**.

Perturbative checks

Important to check the above exact against perturbative results.

This is a **laborious task** ...

- ▶ We have computed

$$\begin{aligned}\gamma_{\mathcal{O}(m,0)} &= \mathcal{O}(\lambda^3) , \quad m = 3, 4, \dots , \\ \gamma_{\mathcal{O}(2,1)} &= -\frac{2c_G}{k}\lambda(1 - 3\lambda) + \mathcal{O}(\lambda^3) .\end{aligned}$$

- ▶ In particular, for $\mathcal{O}(2,0)$

$$\gamma_{\mathcal{O}(2,0)} = \mathcal{O}(\lambda^4) .$$

In full **agreement** with the exact results.

The above result is **robust** and has a **simple explanation**.

- ▶ The anomalous dimension of any operator has the form

$$\gamma_{\mathcal{O}} = \frac{c_G}{k} \frac{\lambda^n f(\lambda)}{(1-\lambda)(1+\lambda)^3}, \quad n = 0, 1, \dots .$$

n is determined by the **leading order perturbative** result.

- ▶ The overall function $f(\lambda)$ is **analytic** in λ .
- ▶ **Invariance** under the **symmetry** $k \rightarrow -k$ and $\lambda \rightarrow \lambda^{-1}$ implies

$$\lambda^{2(2-n)} f(1/\lambda) = f(\lambda) .$$

- ▶ For $n = 0, 1, 2$, $f(\lambda)$ is a 4th, 2nd and 0th order **polynomial**.
- ▶ For $n \geq 3$ **not possible**, unless $f(\lambda) = 0$ leading to $\gamma_{\mathcal{O}} = 0$.
- ▶ Hence, vanishing at $\mathcal{O}(\lambda^2)$ **implies** $\gamma_{\mathcal{O}} = 0$ altogether.

Large N -limit

Consider groups admitting a large rank limit, i.e. $SU(N)$.

In that case $c_G \gg 1$.

- ▶ To **make sense** of the β -function, let $\lambda \rightarrow 0$ and keep $\zeta = c_G \lambda$ finite.

- ▶ Then

$$\frac{1}{2} \mu \frac{d\zeta}{d\mu} = -\frac{\zeta^2}{2k}.$$

- ▶ The anomalous dimensions of $\mathcal{O}(\lambda^2)$ **vanish** in that limit. Those behaving as $\mathcal{O}(\lambda)$ **remain finite**.

Unequal level mutually interacting models [Georgiou-KS 17]

Consider two WZW models with levels k_1 and k_2 .

- ▶ The simplest action encoding interaction between them is

$$S_{k,\lambda}(g_1, g_2) = S_{k_1}(g_1) + S_{k_2}(g_2) + \frac{\sqrt{k_1 k_2}}{\pi} \lambda \int d^2\sigma J_{1+}^a J_{2-}^a .$$

- ▶ That changes the β -function to

$$\beta^\lambda = -\frac{c_G}{2\sqrt{k_1 k_2}} \frac{\lambda^2(\lambda - \lambda_0)(\lambda - \lambda_0^{-1})}{(1 - \lambda^2)^2} ,$$

- ▶ A new fixed point in the IR at $\lambda = \lambda_0 = \sqrt{\frac{k_1}{k_2}} < 1$. Then

$$S_{k,\lambda_0}(g_1, g_2) = S_{k_1}(g_2 g_1) + S_{k_2 - k_1}(g_2) .$$

The theory in the IR is $G_{k_1} \times G_{k_2 - k_1}$.

- ▶ For composite operators in the case of **unequal levels**

$$\gamma_{\mathcal{O}(m,0)} = 0 ,$$

$$\gamma_{\mathcal{O}(2,1)} = \gamma_{J_1+J_2-} = \gamma_{\mathcal{O}_\lambda^{(1,1)}} ,$$

similarly to before.

Zamolodchikov's c -theorem [A.B. Zamolodchikov 86]

- ▶ There exist a function $C(\lambda) \geq 0$ such that

$$\frac{1}{2}\mu \frac{dC}{d\mu} = G_{ij}(\lambda)\beta^i\beta^j \geq 0 ,$$

where G_{ij} is a **positive definite metric** in coupling space.

- ▶ Near a **fixed point** λ^* in which $\beta^i(\lambda^*) = 0$ we have that

$$\beta^i = G^{ij}\partial_j C + \dots , \quad G^{ij} = G_{ij}^{-1} .$$

- ▶ At the **fixed points** λ^* of the **RG flow** $C(\lambda^*)$ equals the CFT **central charge**.

The exact C-function for the one coupling case

- ▶ **Symmetry** considerations, i.e. $k_{1,2} \mapsto -k_{2,1}$, $\lambda \rightarrow 1/\lambda$
- ▶ Demanding **finite limiting** behavior for $\lambda = \pm 1$ and $k \rightarrow \infty$
- ▶ Matching with the **perturbative** results
- ▶ The **exact in λ** (and leading in $1/k$) **C-function** is

$$C(\lambda, \lambda_0) = 2 \dim G - \frac{c_G \dim G}{\sqrt{k_1 k_2}} \frac{(\lambda_0 + \lambda_0^{-1})(1 - 3\lambda^2 - 3\lambda^4 + \lambda^6) + 8\lambda^3}{2(1 - \lambda^2)^3}$$

Properties and checks

- ▶ It is positive, i.e. $C(\lambda, \lambda_0) > 0$.
- ▶ For the UV CFT $G_{k_1} \times G_{k_2}$ and for the IR CFT $G_{k_1} \times G_{k_2-k_1}$ we obtain the correct for $k_{1,2} \gg 1$ expansion of the central charge.

Recall that for G_k this is exactly given by

$$c = \frac{2k \dim G}{2k + c_G} .$$

- ▶ In addition

$$\mu \frac{dC}{d\mu} = 12 \frac{c_G^2 \dim G}{k^2} \frac{\lambda^4 (\lambda - \lambda_0)^2 (\lambda - \lambda_0^{-1})^2}{(1 - \lambda^2)^6} ,$$

Hence, C is **monotonically increasing** from the IR to the UV.

Concluding remarks

Models of **interacting current** algebra theories.

- ▶ Focused on self- J_+J_- and mutually-interactions $J_{1+}J_{2-}$.
- ▶ There is generalization with **many** theories **self-** and **mutually interacting**, i.e. $J_{i+}J_{j-}$ [Georgiou-KS, 18]. New IR fixed points.
- ▶ IR theories correspond typically to **different CFTs** on the **left** and **right** sectors [Georgiou-KS-Pappas, to appear].
- ▶ Extension of the construction to **exact coset CFTs** [KS 13, Hollowood-Miramontes-Schmidt 14, KS-Siampos 17]

Computed exactly the **beta-function** and **anomalous dimensions**.

- ▶ Based on **leading order perturbative** results and **symmetries**.
- ▶ Based on the **effective action** and **geometry** in coupling space.

Many Future directions...., but the most relevant for this talk are:

- ▶ **Explain** the **high degeneracy** of the anomalous dim spectrum.
- ▶ Adopt? extend ? this work for **η -deformations** [Klimcik 02 & 08], [Delduc-Magro-Vicedo 13] and [Delduc-Lacroix-Magro-Vicedo 18]. A relation between λ - and η -defs via Poisson Lie-T-duality.