Yang-Baxter deformations of the AdS_3 string as marginal deformations of the WZW-model

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Motivation

The so-called Yang-Baxter (YB) deformation of string σ-models was first introduced by Klimčik almost 20 years ago

It has the very interesting property that it preserves integrability: If the original model is integrable so is its YB deformation

This leads to the exciting possibility of generating new integrable examples of gauge/gravity duality by starting from the duality

$$\text{Strings in } AdS_5 \times S^5 \leftrightarrow \text{(Planar) } \mathcal{N} = 4 \text{ Super-Yang-Mills}$$

and performing a YB deformation on the LHS

Motivation

So far, however, these deformations are only well understood at the classical level, but to generate new gauge/gravity dualities one must understand the deformations at the quantum level

One possible approach is to try to work out what these deformations do to the worldsheet CFT

For this reason we will to consider the string on

$$AdS_3 \times S^3 (\times M_4)$$

supported by NSNS flux, which is described by the

$$SL(2,\mathbb{R}) \times SU(2) \qquad \text{WZW-model}$$

and ask what the YB deformation looks like from the CFT point-of-view

Outline

- Yang-Baxter deformations
- Non-abelian T-duality realization
- Deformation rules for the SUGRA background
- Current-current ($J\bar{J}$) deformations
- Deformations of AdS_3 and $AdS_3 \times S^3$
- Conclusions

Yang-Baxter deformations

The simplest setting to explain the YB deformation is the Principal Chiral Model (PCM)

$$S_{PCM} = \int d^2x \, \mathrm{tr}(g^{-1}\partial_+ g g^{-1}\partial_- g)$$

with $g(x) \in G$ where G is some Lie group

The YB deformation of the PCM is simply

$$S_{YB} = \int d^2x \, \mathrm{tr}(g^{-1}\partial_+ g \frac{1}{1+\eta R} g^{-1}\partial_- g)$$

When the deformation parameter $\eta \to 0$ one recovers the PCM

The key ingredient is the map $R : \mathfrak{g} \to \mathfrak{g}$ which should be anti-symmetric

$$\mathrm{tr}(XRY) = \mathrm{tr}(R^T XY) = -\mathrm{tr}(RXY) \qquad X, Y \in \mathfrak{g}$$

Yang-Baxter deformations

This so-called R-matrix should also satisfy the Classical Yang-Baxter Equation (CYBE)

$$[RX, RY] - R([RX, Y] + [X, RY]) = c[X, Y]\,, \qquad \forall X, Y \in \mathfrak{g}$$

$$c = 0: \quad \text{CYBE} \qquad\qquad c \neq 0: \quad \text{modified CYBE}$$

The deformed model is also integrable just like PCM [Klimčik '08]

In the following we consider only the case $c = 0$. In that case the YB deformation has an interpretation in terms of non-abelian T-duality (NATD) [Hoare, Tseytlin; Borsato, LW '16]

Non-abelian T-duality realization

We start again from the PCM action

$$S_{PCM} = \int d^2x \, \mathrm{tr}(g^{-1}\partial_+ g g^{-1}\partial_- g)$$

Now imagine adding a B-field [Borsato, LW '16]

$$\int B \,, \qquad B = \zeta \omega_{IJ}(g^{-1}dg)^I \wedge (g^{-1}dg)^J$$

with a parameter ζ introduced for later convenience

If B is closed the equations of motion remain unchanged, in particular the model remains integrable. Now

$$dB = 0 \qquad \leftrightarrow \qquad \omega_{I[J} f^I{}_{KL]} = 0$$

i.e. ω is a Lie algebra 2-cocycle on $\mathfrak{g}$

Non-abelian T-duality realization

Now we perform non-abelian T-duality of this model

$$S_{PCM} + \int B$$

following the usual procedure:

1. Replace $g^{-1}\partial_\pm g \to A_\pm$ in the action
2. Enforce the vanishing of the field strength $F(A) = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$ by adding a Lagrange multiplier term $\nu F(A)$

Which leads to the first order action

$$S' = \int d^2x \, \mathrm{tr}(A_+ A_- + \zeta A_+ \omega A_- + \nu F_{+-}(A))$$

The equation of motion for ν just says that the curvature of A vanishes so that $A = g^{-1}dg$. Plugging this into S' gives back the original action

Non-abelian T-duality realization

Integrating out A instead gives the Non-abelian T-dual model

$$S_{NATD} = \int d^2x\, \partial_+ \nu^I \left(\frac{1}{\delta_{IJ} + \zeta \omega_{IJ} + \nu_K f^K{}_{IJ}} \right) \partial_- \nu^J$$

If ω is invertible it follows from the 2-cocycle condition that $R = \omega^{-1}$ satisfies the CYBE!

In fact the field redefinition

$$\nu = \eta^{-1} \frac{1 - Ad_g^{-1}}{\log Ad_g} \omega \log g$$

brings the action to that of the YB model with $\eta = \zeta^{-1}$

$$S = \int d^2x\, \mathrm{tr}(g^{-1}\partial_+ g \frac{1}{1+\eta R} g^{-1}\partial_- g)$$

This explains why the YB deformation preserves integrability

Deformation rules for the SUGRA background

Since we know how to carry out NATD for the full Green-Schwarz superstring we can now construct the YB deformation for a general superstring with isometries

Doing this one finds the deformed background [Borsato, LW '18]

$$\tilde{G} - \tilde{B} = (G - B)[1 + \eta\Theta(G - B)]^{-1}$$

a generalization of the open-closed string map of Seiberg and Witten with 'non-commutativity parameter'

$$\Theta^{mn} = k_I^m k_J^n R^{IJ}$$

in terms of the Killing vectors k_I^m of the original model

The dilaton is given by (provided that $f_{IJ}^J = 0$ so there is no anomaly)

$$\tilde{\phi} = \phi - \frac{1}{2} \ln \det[1 + \eta\Theta(G - B)]$$

Deformation rules for the SUGRA background

The bispinor encoding the RR fields is deformed as

$$\tilde{\mathcal{S}} = \Lambda \mathcal{S} \,, \qquad \Lambda^{mn} = G^{mn} - 2\eta([1 + \eta\Theta(G - B)]^{-1}\Theta)^{mn}$$

When R is abelian the YB deformation is equivalent to TsT, e.g. the β-deformation of $AdS_5 \times S^5$ [Osten, van Tongeren '16]

Now we can apply these general formulas to the case we are interested in: $AdS_3 \times S^3$ with only NSNS flux

Clearly no RR flux will be generated by the deformation

To find all possible deformations we must find all possible R-matrices on the isometry algebra

Deformations of AdS_3

Consider first the case of just $AdS_3 \sim SL(2,\mathbb{R})$. The isometry group is

$$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

We denote the generators $S_0,\ S_+,\ S_-$ and $\bar{S}_0,\ \bar{S}_+,\ \bar{S}_-$

Finding all possible R-matrices boils down to finding the subalgebras which admit a 2-cocycle and one finds

$$\begin{aligned}
R_1 &= S_0 \wedge S_+ \\
R_2 &= (S_0 \mp \bar{S}_-) \wedge S_+ \\
R_3 &= (S_0 - a\bar{S}_0) \wedge S_+ \\
R_4 &= (S_0 - \bar{S}_0) \wedge (S_+ \pm \bar{S}_-) \\
R_5 &= S_0 \wedge S_+ \pm \bar{S}_0 \wedge \bar{S}_- + \lambda S_+ \wedge \bar{S}_-
\end{aligned}$$

$+$ abelian ones, where the generators involved all commute

Deformations of AdS_3

The abelian case (i.e. TsT) has been studied a lot in the literature so we focus on the non-abelian examples

We can now write down the deformed action which takes the form

$$\int d^2\sigma \partial_+ x^m \partial_- x^n \left[(G-B) \left(1 + \eta\Theta(G-B)\right)^{-1} \right]_{mn}$$

However, this form of the action is not suitable for a CFT interpretation

To lowest order in η we have

$$L = L_{WZW} - \frac{\eta}{2} \mathcal{J}_{I+} R^{IJ} \mathcal{J}_{J-}$$

where $\mathcal{J}_I$ are the isometry Noether currents

Deformations of AdS_3

The WZW model is characterized by having chiral currents

$$\partial \bar{J}_a = \bar{\partial} J_a = 0$$

But taking the AdS_3 metric and B-field

$$ds^2 = \frac{-dx^+ dx^- + dz^2}{z^2} \qquad B = \frac{dx^+ \wedge dx^-}{2z^2}$$

the Noether currents differ from the chiral ones, e.g.

$$\mathcal{J}_0 = \left(J_0 - \tfrac{1}{2}\partial \log z, \tfrac{1}{2}\bar{\partial} \log z\right)$$

Since the natural objects from the WZW point-of-view are the chiral currents we want to express the deformed model in terms of those

Change of coordinates

Fortunately the change of coordinates

$$x^{\pm} \to z^{\eta R^{0\bar{0}}/2}\left(x^{\pm} - \frac{\eta}{2}(R^{0\pm} + R^{\bar{0}\pm})\log z\right) \qquad z \to z^{1+\eta R^{0\bar{0}}/2}$$

brings the Lagrangian to the form

$$L = L_{WZW} - \frac{\eta}{2} J'_a \left[(1+\eta R M)^{-1} R\right]^{a\bar{a}} \bar{J}'_{\bar{a}}$$

where $J'_a = J_a + \mathcal{O}(\eta)$ and similarly for $\bar{J}'$ while M is a certain matrix and expanding in η

$$L = L_{WZW} - \frac{\eta}{2} R^{a\bar{a}} J_a \bar{J}_{\bar{a}} + \mathcal{O}(\eta^2)$$

a $J\bar{J}$-deformation of the CFT

$J\bar{J}$-deformation

Such $J\bar{J}$-deformations have been studied a lot in the CFT literature

Since the conformal dimensions of J_a and $\bar{J}_a$ are $(1,0)$ and $(0,1)$ the $J\bar{J}$-deformation has dimension $(1,1)$ and is marginal at lowest order in η

The question whether it remains marginal beyond lowest order was analyzed by Chaudhuri and Schwartz in 1989

Deforming by

$$gO(\sigma,\bar{\sigma}) = gc^{ab} J_a(\sigma)\bar{J}_b(\bar{\sigma})$$

they found, by looking at the 2pt function of O in the deformed theory, that a necessary condition is

$$C \cdot C + \bar{C} \cdot \bar{C} = 0$$

where $C^{abc} \equiv c^{da} c^{eb} f_{de}{}^{c}$, and $\bar{C}^{abc} \equiv c^{ad} c^{be} f_{de}{}^{c}$

$J\bar{J}$-deformation

In the case of a compact group both terms are positive and one finds

$$C^{abc} = \bar{C}^{abc} = 0$$

This actually implies that the algebra of the currents involved in the deformation is abelian

In that case they further argued, using the fact that the calculation is isomorphic to that for free bosons, that O remains marginal to all orders in conformal perturbation theory in g

Similarly, for a compact group only abelian R-matrices, giving TsT transformations, are possible. The corresponding SUGRA solutions have been argued to give the completion of the lowest order deformation to all orders in η and lowest order in α'

[Hassan, Sen '92; Henningson, Nappi '93; Kiritsis '93]

Deformations of AdS_3

Since $SL(2,\mathbb{R})$ is non-compact the condition of CS does not require the algebra to be abelian

In fact one finds for the five non-abelian R-matrices of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

$R^{AB}\mathbf{T}_A \wedge \mathbf{T}_B$	Deformation?	$c^{ab} J_a \bar{J}_b$
$S_0 \wedge S_+$	No def.	0
$(S_0 \mp \bar{S}_-) \wedge S_+$	TsT	$\pm J_+ \bar{J}_-$
$(S_0 - a\bar{S}_0) \wedge S_+$	TsT	$aJ_+ \bar{J}_0$
$(S_0 - \bar{S}_0) \wedge (S_+ \pm \bar{S}_-)$	Not SUGRA	$J_+ \bar{J}_0 \pm J_0 \bar{J}_-$
$S_0 \wedge S_+ \pm \bar{S}_0 \wedge \bar{S}_- + \lambda S_+ \wedge \bar{S}_-$	TsT	$\lambda J_+ \bar{J}_-$

Deformations of AdS_3

We note that

- While all five R-matrices are non-abelian four of them give deformations which are equivalent to abelian ones (TsT)
- One even gives no deformation at all despite involving non-abelian T-duality!
- The fourth R-matrix involves a non-abelian set of currents but the corresponding background does not solve SUGRA (beyond lowest order in η)
- All involve non-abelian T-duality on a non-unimodular algebra and one would expect this to give a Weyl anomaly

  [Alvarez, Alvarez-Gaume, Lozano '94; Elitzur, Giveon, Rabinovici, Schwimmer, Veneziano '94]
- Instead the anomaly vanishes for four of them (They are 'trivial' solutions of generalized SUGRA) [LW '18]

Deformations of $AdS_3 \times S^3$

For $AdS_3 \times S^3$ there are more options since the R-matrix can now involve also one generator from each $\mathfrak{su}(2)$ (the left and right isometries of S^3)

One finds a list of ten R-matrices in this case

Many of them are just additional TsT transformations of the AdS_3 solutions already considered

However, now one also finds a deformation which involves a non-abelian set of currents and gives a SUGRA solution:

$$R = (S_0 + \bar{S}_0 + T) \wedge T' + S_+ \wedge \bar{S}_-$$

where T, T' are combinations of S^3 isometry generators. It is in fact unimodular, which guarantees that it gives a SUGRA solution.

Deformations of $AdS_3 \times S^3$

The corresponding $J\bar{J}$-deformation is

$$c^{ab} J_a \bar{J}_b = a J_0 \bar{J}_2 + b J_1 \bar{J}_0 + c J_1 \bar{J}_2 + J_+ \bar{J}_-$$

with $J_1 \in \mathfrak{su}_L(2)$ and $\bar{J}_2 \in \mathfrak{su}_R(2)$

It satisfies the condition of Chaudhuri and Schwartz while being non-abelian

Whether it remains marginal also to higher orders in η is an interesting question

Conclusions

- Yang-Baxter deformations based on R solving the CYBE can be realized via non-abelian T-duality
- This means that they can be carried out for a general SUGRA background with isometries
- When R is abelian this is equivalent to TsT, or $O(d,d)$ β-shifts
- In the case of the $AdS_3 \times S^3$ WZW model this corresponds to integrated marginal $J\bar{J}$-deformations
- Non-abelian R-matrices give either TsT, non-TsT SUGRA backgrounds or non-SUGRA backgrounds

Outlook

- Abelian YB deformations/TsT/$O(d, d)$ can be completed to all orders in α'
- What about the non-abelian deformation we found of $AdS_3 \times S^3$?
- It solves the one-loop beta function equations, what about the two-loop ones? [In progress...]