Spin-2 excitations in Gaiotto-Maldacena solutions

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Field theory dual:
$\mathcal{N}=2$ SCFTs realized as low energy limit of D4-branes ending on and intersecting NS5-branes and D6-branes $\rightarrow$ quivers with special unitary gauge groups.
Brane setup (IIA String Theory):

- D4-$(x^0,\ldots,x^3,x^6)$, NS5-$(x^0,\ldots,x^5)$, D6-$(x^0,\ldots,x^3,x^7,\ldots,x^9)$
- D4-D4 open strings give $\mathcal{N}=2$ SYM $SU(k_n)$
- D4-D4 strings between k_n,k_{n-1} branes ending on n-th NS5 give a massless hypermultiplet in the bifundamental $(k_{n-1},\bar{k}_n)$
- D4-D6 strings between k_n D4 and d_n D6 give d_n hypermultiplets in the fundamental of k_n

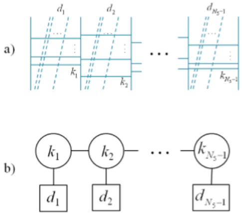

Figure: a) brane configuration. Vertical lines-NS5, horizontal lines-D4s, diagonal lines-D6s.
b) Corresponding quiver. Circular nodes $\rightarrow$ color $SU(k_n)$, horizontal lines $\rightarrow$ bifundamental hypermultiplets, square nodes $\rightarrow$ flavor $U(d_n)$, vertical lines d_n hypermultiplets in the fundamental of $SU(k_n)$. Picture taken from [1206.5916]

Gravity dual:

- Expectation: near horizon limits of the above mentioned brane configurations
- Dual Gaiotto-Maldacena geometries in type IIA are given in terms of a function solving the Laplace equation (Gaiotto, Maldacena [0904.4466])
- Each solution determines the dual quiver field theory (rules in [0904.4466] and Aharony, Berdichevsky, Berkooz [1206.5916]). Brane content determined by boundary conditions
- Connection with M-theory. Reduction of 11-dimensional LLM geometries (Lin, Lunin, Maldacena [0409174]) give 10-dimensional GM geometries
- 11-d geometry characterized by Toda equation. Requiring additional $U(1)$ isometry $\rightarrow$ 10-dimensional MG with Laplace equation

Type IIA SUGRA equations of motion:

$$\begin{aligned} 0 =& R_{MN} - \frac{1}{2}\partial_M\Phi\,\partial_N\Phi - \frac{e^{3\Phi/2}}{2}\Big(F_{MP}F_N{}^P - \frac{1}{16}g_{MN}F_2^2\Big) \\ &- \frac{e^{\Phi/2}}{12}\Big(F_{MPK\Lambda}F_N{}^{PK\Lambda} - \frac{3}{32}g_{MN}F_4^2\Big) - \frac{e^{-\Phi}}{4}\Big(H_{MPK}H_N{}^{PK} - \frac{1}{12}g_{MN}H_3^2\Big)\,, \end{aligned} \tag{1}$$

$$0 = \nabla^M\nabla_M\Phi - \frac{3}{8}e^{3\Phi/2}F_2^2 - \frac{e^{\Phi/2}}{96}F_4^2 + \frac{e^{-\Phi}}{12}H_3^2\,, \tag{2}$$

$$0 = \nabla^M\big(e^{-\Phi}H_{MNP}\big) - \frac{e^{\Phi/2}}{2}F_{NPK\Lambda}F^{K\Lambda} + \frac{1}{2\cdot 4!\cdot 4!}\varepsilon_{M_1\ldots M_8 NP}F^{M_1\ldots M_4}F^{M_5\ldots M_8}\,, \tag{3}$$

$$0 = \nabla^M\big(e^{3\Phi/2}F_{MN}\big) + \frac{e^{\Phi/2}}{6}F_{PK\Lambda N}H^{PK\Lambda}\,, \tag{4}$$

$$0 = \nabla^M\big(e^{\Phi/2}F_{MNPK}\big) - \frac{1}{144}\varepsilon_{M_1\ldots M_7 NPK}F^{M_1\ldots M_4}H^{M_5\ldots M_7}\,, \tag{5}$$

where $\varepsilon_{M_1\ldots M_{10}}$ is the totally antisymmetric Levi-Civita *tensor*.

Starting point: we compute the fluctuations of the eoms (Einstein, dilaton, F_2, F_4, H_3):
Fluctuation ansatz:
Background metric: $\bar{g}_{MN}$ in *Einstein frame*, conformally related to $\tilde{g}_{MN}$, through a warp factor:

$$\bar{g}_{MN} = e^{2A}\tilde{g}_{MN} \tag{6}$$

$$g_{MN} = \bar{g}_{MN} + \delta g_{MN} \tag{7}$$

$$\delta g_{MN} = e^{2A}h_{MN} \Rightarrow \delta g^{MN} = -e^{2A}h^{MN} \tag{8}$$

$$\Phi = \bar{\Phi} + \phi \tag{9}$$

$$H_3 = \bar{H}_3 + \delta H_3, \quad F_2 = \bar{F}_2 + \delta F_3, \quad F_4 = \bar{F}_4 + \delta F_4 \tag{10}$$

Notation:

$$\mathcal{A}_p \equiv (F_2, H_3, F_4), \alpha_p \equiv (\frac{3}{2}, -1, \frac{1}{2}), \beta_p = (\frac{1}{16}, \frac{1}{12}, \frac{3}{32}), \gamma_p \equiv (1, \frac{1}{2}, \frac{1}{6}) \tag{11}$$

$$(\mathcal{A}_p)_M \cdot (\bar{A}_p)_N = \mathcal{A}_{M\Sigma_1\ldots\Sigma_{p-1}}\bar{A}_N^{\Sigma_1\ldots\Sigma_{p-1}} \;\;\&\;\; (\mathcal{A}_p)\cdot(\bar{A}_p) = \mathcal{A}_{\Sigma_1\ldots\Sigma_p}\bar{\mathcal{A}}^{\Sigma_1\ldots\Sigma_p} \tag{12}$$

$$(\mathcal{A}_p)_M \cdot (\tilde{A}_p)_N = \mathcal{A}_{M\Sigma_1\ldots\Sigma_{p-1}}\tilde{A}_N^{\Sigma_1\ldots\Sigma_{p-1}} \;\;\&\;\; (\mathcal{A}_p)\cdot(\tilde{A}_p) = \mathcal{A}_{\Sigma_1\ldots\Sigma_p}\tilde{\mathcal{A}}^{\Sigma_1\ldots\Sigma_p} \tag{13}$$

Fluctuation of Einstein equation:

$$
\begin{aligned}
0 =& \frac{1}{2}\,\tilde{\nabla}^{\Sigma}\tilde{\nabla}_{M} h_{\Sigma N} + \frac{1}{2}\,\tilde{\nabla}^{\Sigma}\tilde{\nabla}_{N} h_{\Sigma M} - \frac{1}{2}\,\tilde{\nabla}^{2} h_{MN} - \frac{1}{2}\,\tilde{\nabla}_{N}\tilde{\nabla}_{M}\tilde{h} + 4\,\tilde{\nabla}^{\Sigma}A\,\tilde{\nabla}_{M} h_{\Sigma N} \\
&+ 4\,\tilde{\nabla}^{\Sigma}A\,\tilde{\nabla}_{N} h_{\Sigma M} - h_{MN}\,\tilde{\nabla}^{2}A - 8\,h_{MN}\,(\tilde{\nabla}A)^{2} - 4\,\tilde{\nabla}^{P}A\tilde{\nabla}_{P} h_{MN} - \frac{1}{2}\,\partial_{M}\varphi\,\partial_{N}\bar{\Phi} - \frac{1}{2}\,\partial_{M}\bar{\Phi}\,\partial_{N}\varphi \\
&- \frac{1}{2}\sum_{p=2}^{4}\gamma_{p}\,e^{2(1-p)A+\alpha_{p}\bar{\Phi}}\Big[(\delta\mathcal{A}_{p})_{M}\cdot(\tilde{\mathcal{A}}_{p})_{N} + (\tilde{\mathcal{A}}_{p})_{M}\cdot(\delta\mathcal{A}_{p})_{N} - \beta_{p}\,h_{MN}\tilde{\mathcal{A}}_{p}^{2} \\
&- (p-1)\,h_{PK}\,\tilde{\mathcal{A}}_{M\Sigma_{1}\ldots\Sigma_{p-2}}{}^{P}\,\tilde{\mathcal{A}}_{N}{}^{\Sigma_{1}\ldots\Sigma_{p-2}K}\Big] - \frac{\varphi}{2}\sum_{p=2}^{4}\alpha_{p}\,\gamma_{p}\,e^{2(1-p)A+\alpha_{p}\bar{\Phi}}\,(\tilde{\mathcal{A}}_{p}^{2})_{MN} + \tilde{g}_{MN}\,t\,.
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
t :=& \tilde{\nabla}^{\Sigma} h_{\Sigma P}\,\tilde{\nabla}^{P}A + h_{\Sigma P}\,\tilde{\nabla}^{\Sigma}\tilde{\nabla}^{P}A - \frac{1}{2}\,\tilde{\nabla}_{\Lambda}\tilde{h}\,\tilde{\nabla}^{\Lambda}A + 8\,h_{P\Sigma}\,\tilde{\nabla}^{P}A\,\tilde{\nabla}^{\Sigma}A \\
&+ \frac{\varphi}{2}\sum_{p=2}^{4}\alpha_{p}\,\beta_{p}\,\gamma_{p}\,e^{2(1-p)A+\alpha_{p}\bar{\Phi}}\,\tilde{\mathcal{A}}_{p}^{2} + \frac{1}{2}\sum_{p=2}^{4}\beta_{p}\,\gamma_{p}\,e^{2(1-p)A+\alpha_{p}\bar{\Phi}}\Big[2\,(\delta\mathcal{A}_{p})\cdot(\tilde{\mathcal{A}}_{p}) \\
&- p\,h_{PK}\,\tilde{\mathcal{A}}_{\Sigma_{1}\ldots\Sigma_{p-1}}{}^{P}\,\tilde{\mathcal{A}}^{\Sigma_{1}\ldots\Sigma_{p-1}K}\Big]\,.
\end{aligned}
\tag{15}
$$

Here by $\tilde{h}$ we mean the contraction $\tilde{h} := \tilde{g}^{MN} h_{MN}$

Gaiotto-Maldacena solutions in *Einstein frame*:

$$ds_E^2 = e^{-\frac{\Phi}{2}} f_0 \left(ds^2_{AdS_5} + f_1 d\Omega_2^2 + f_2 \left(d\eta^2 + d\sigma^2\right) + f_3 d\beta^2 \right) , \tag{16}$$

with $d\Omega^2 = d\chi^2 + \sin^2 \chi d\xi^2$.
$\Phi = \Phi(\eta, \sigma)$, $f_i = f_i(\eta, \sigma)$, expressed in terms of $V(\eta, \sigma)$ as:

$$\begin{gathered} e^{2\Phi} = 2^7 \sigma \frac{(\ddot{V} - 2\dot{V})^{3/2}}{\sqrt{\ddot{V}} \dot{V} \Delta} , \qquad \Delta = (\ddot{V} - 2\dot{V}) \frac{\ddot{V}}{\sigma^2} + \dot{V}'^2 , \qquad f_0 = \sigma \sqrt{\frac{\ddot{V} - 2\dot{V}}{\ddot{V}}} , \\ f_1 = -\frac{\ddot{V}\dot{V}}{2\sigma^2 \Delta} , \qquad f_2 = -\frac{\ddot{V}}{2\sigma^2 \dot{V}} , \qquad f_3 = \frac{\ddot{V}}{\ddot{V} - 2\dot{V}} . \end{gathered} \tag{17}$$

$$B_2 = \frac{1}{2} \left(\frac{\dot{V}\dot{V}'}{\Delta} - \eta \right) \mathrm{vol}_{\Omega_2} , \qquad C_1 = \frac{1}{8} \frac{\dot{V}\dot{V}'}{2\dot{V} - \ddot{V}} d\beta , \qquad C_3 = \frac{1}{16} \frac{\dot{V}^2 \ddot{V}}{\sigma^2 \Delta} d\beta \wedge \mathrm{vol}_{\Omega_2} , \tag{18}$$

with $\mathrm{vol}_{\Omega_2} = \sin \chi d\chi \wedge d\xi$,
$F_2 = dC_1$, $F_4 = dC_3 + C_1 \wedge H_3$, $H_3 = dB_2$.
Geometry specified by $V(\eta, \sigma)$, satisfying:

$$\ddot{V} + \sigma^2 V'' = 0 \tag{19}$$

Metric perturbations

The metric in *Einstein frame* is conformal to a direct product of AdS_5 with a 5-dimensional space $\mathcal{M}_5$

$$ds^2 = ds^2(AdS_5) + ds^2(\mathcal{M}_5) \tag{20}$$

Adopt the notation:

$$\begin{aligned} &M, N, P, K, \Lambda, \Sigma, \ldots : && \text{ten-dimensional indices,} \\ &\mu, \nu, \rho, \kappa, \lambda, \sigma, \ldots : && \text{indices in } AdS_5, \\ &m, n, k, p, q, s, \ldots : && \text{indices in } \mathcal{M}_5. \end{aligned}$$

Define: $X^M = (x^\mu, y^m)$

$$ds^2 = \tilde{g}_{MN} dX^M dX^N = \tilde{g}_{\mu\nu} dx^\mu dx^\nu + \tilde{g}_{mn} dy^m dy^n \tag{21}$$

with $\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(x)$, $\tilde{g}_{mn} = \tilde{g}_{mn}(y)$
In matrix notation, this is:

$$\tilde{g}_{MN}(x, y) = \begin{pmatrix} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & \tilde{g}_{mn}(y) \end{pmatrix} . \tag{22}$$

We will turn on only fluctuations of the metric along AdS_5

$$\delta g_{\mu\nu} = e^{2A} h_{\mu\nu} \tag{23}$$

The fluctuated metric reads:

$$ds_E^2 = e^{2A}[(\tilde{g}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + \tilde{g}_{mn} dy^m dy^n] \qquad (24)$$

Take the factorization:

$$h_{\mu\nu}(x, y) = h^{[tt]}_{\mu\nu}(x)\mathcal{Y}(y) \qquad (25)$$

with $h^{[tt]}_{\mu\nu}(x)$ transverse and traceless with respect to $\tilde{\nabla}^\mu$,

$$\tilde{\nabla}^\mu h^{[tt]}_{\mu\nu} = 0, \quad \tilde{g}^{\mu\nu} h^{[tt]}_{\mu\nu} = 0 \qquad (26)$$

$\Rightarrow$ Dilaton + Maxwell equations trivially satisfied!!!
For Einstein equation:

$$\begin{aligned} 0 = {} & \frac{1}{2}\,\tilde{\nabla}^\Sigma \tilde{\nabla}_M h_{\Sigma N} + \frac{1}{2}\,\tilde{\nabla}^\Sigma \tilde{\nabla}_N h_{\Sigma M} - \frac{1}{2}\,\tilde{\nabla}^2 h_{MN} - 4\,\tilde{\nabla}^P A \tilde{\nabla}_P h_{MN} - h_{MN}\,\tilde{\nabla}^2 A - 8\,h_{MN}\left(\tilde{\nabla} A\right)^2 \\ & + \frac{1}{2} h_{MN} \sum_{p=2}^{4} \beta_p \gamma_p\, e^{2(1-p)A + \alpha_p \bar{\Phi}}\, \tilde{\mathcal{A}}_p^2\,, \end{aligned} \qquad (27)$$

commuting 1st,2nd term covariant derivatives+using $\tilde{R}_{\mu\nu\rho\sigma} = \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\rho} - \tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma}$:

$$0 = \tilde{\nabla}^2 h_{\mu\nu} + 10\,h_{\mu\nu} + 8\,\tilde{\nabla}^P A \tilde{\nabla}_P h_{\mu\nu} + h_{\mu\nu}\left[2\,\tilde{\nabla}^2 A + 16\left(\tilde{\nabla} A\right)^2 - \sum_{p=2}^{4} \beta_p \gamma_p\, e^{2(1-p)A + \alpha_p \bar{\Phi}}\, \tilde{\mathcal{A}}_p^2\right]. \qquad (28)$$

All solutions in Gaiotto-Maldacena class satisfy:

$$\left[2\,\tilde{\nabla}^2 A + 16\left(\tilde{\nabla} A\right)^2 - \sum_{p=2}^{4} \beta_p \gamma_p\, e^{2(1-p)A+\alpha_p \bar{\Phi}}\, \tilde{\mathcal{A}}_p^2\right] = -8 \tag{29}$$

The equation to be solved is:

$$0 = \tilde{\nabla}^{\sigma}\tilde{\nabla}_{\sigma} h_{\mu\nu} + 2\, h_{\mu\nu} + \tilde{\nabla}^{m}\tilde{\nabla}_{m} h_{\mu\nu} + 8\,\tilde{\nabla}^{m} A \tilde{\nabla}_{m} h_{\mu\nu}\,. \tag{30}$$

Recalling

$$\tilde{\nabla}^{m}\tilde{\nabla}_{m} h_{\mu\nu} + 8\,\tilde{\nabla}^{m} A \tilde{\nabla}_{m} h_{\mu\nu} = e^{-8A}\tilde{\nabla}^{m}\left[e^{8A}\tilde{\nabla}_{m} h_{\mu\nu}\right] := \mathcal{L}(h_{\mu\nu})\,. \tag{31}$$

$h_{\mu\nu}$ is a scalar with respect to $\tilde{\nabla}_m$. The action of $\mathcal{L}$ on a scalar f can be written as:

$$\mathcal{L}(f) = \tilde{\nabla}^{m}\tilde{\nabla}_{m} f + 8\,\tilde{\nabla}^{m} A\,\tilde{\nabla}_{m} f = \frac{1}{\sqrt{\tilde{g}_{\mathcal{M}_5}}}\partial_m\left(\sqrt{\tilde{g}_{\mathcal{M}_5}}\,\tilde{g}^{mn}\,\partial_n f\right) + 8\,\tilde{g}^{mn}\,\partial_m A\,\partial_n f\,. \tag{32}$$

Moreover, (30) is the eom of a massive graviton of mass M propagating in AdS_5. This is given by the Pauli-Fierz equation:

$$0 = \tilde{\nabla}^{\sigma}\tilde{\nabla}_{\sigma} h_{\mu\nu} + \left(2 - M^2\right)\, h_{\mu\nu}\,. \tag{33}$$

Using this, the equation (30) reduces to an eigenvalue problem for the operator $\mathcal{L}$:

$$\mathcal{L}(h_{\mu\nu}) = -M^2 h_{\mu\nu}\,. \tag{34}$$

In terms of the coordinates of the metric (16) the operator $\mathcal{L}$ has the following form:

$$\mathcal{L}(f) = \frac{1}{f_1}\nabla^2_{(2)} f + \frac{1}{f_3}\partial_\beta^2 f + \frac{1}{\Delta f_0 f_1\sqrt{f_3}}\left[\partial_\eta\left(\frac{\Delta f_0 f_1\sqrt{f_3}}{f_2}\partial_\eta f\right) + \partial_\sigma\left(\frac{\Delta f_0 f_1\sqrt{f_3}}{f_2}\partial_\sigma f\right)\right], \tag{35}$$

where $\nabla^2_{(2)}$ is the Laplace operator on the two-sphere Ω_2. Quite complicated operator!!! Look for easy examples!!

T-dual of $AdS_5 \times S^5$ (Fayyazudin, Smith [9902210]; Lozano, Nunez [1603.04440]) is an example of a GM geometry. (Not good boundary conditions for Laplace equation!) The corresponding potential is:

$$V^{ATD}(\eta, \sigma) = \ln \sigma - \frac{\sigma^2}{2} + \eta^2 \,. \tag{36}$$

$$\eta = 2\psi \,, \qquad \sigma = \sin \alpha \,. \tag{37}$$

ψ is T-dualized coordinate. As a result, we have for $\mathcal{L}$:

$$\mathcal{L}^{ATD}(f) = \partial_\alpha^2 f + \big(\cot \alpha - 3\ \tan \alpha\big)\partial_\alpha f + \frac{1}{\sin^2 \alpha}\partial_\beta^2 f + \frac{\cos^2 \alpha}{4}\ \partial_\psi^2 f + \frac{4}{\cos^2 \alpha}\nabla_{(2)}^2 f \ , \tag{38}$$

where $\nabla^2_{(2)}$ is the Laplace operator on the two-sphere $\Omega_2(\chi, \xi)$.
Ansatz:

$$Y(y) = \sum_{m,n,\ell} f_{m,n,\ell}(\alpha) e^{i(m\beta + n\psi)} \mathcal{Y}_\ell(\chi, \xi) \,, \qquad m, n \in \mathbb{Z} \,, \quad \ell = 0, 1, 2, \ldots \,, \tag{39}$$

where $\mathcal{Y}_\ell$ are the spherical harmonics on the two-sphere $\Omega_2(\chi, \xi)$ and $f_{m,n,\ell}(\alpha)$ are functions to be determined. The eigenvalue problem (34) translates to 2nd order DE for $f_{m,n,\ell}(\alpha)$:

$$0 = \partial_\alpha^2 f + \big(\cot \alpha - 3\ \tan \alpha\big)\partial_\alpha f + \Big[M^2 - \frac{m^2}{\sin^2 \alpha} - \frac{n^2}{4}\cos^2 \alpha - \frac{4\ell(\ell+1)}{\cos^2 \alpha}\Big] f \ , \tag{40}$$

If we perform the following change of variables:

$$z = \sin^2 \alpha \,, \qquad z \in [0,1] \,, \tag{41}$$

then the differential equation that we have to solve becomes:

$$0 = z(1-z)\,\partial_z^2 f + (1-3z)\,\partial_z f + \left[\frac{M^2}{4} - \frac{\ell(\ell+1)}{1-z} - \frac{m^2}{4\,z} - \frac{n^2}{16}(1-z)\right] f \,. \tag{42}$$

For $n = 0$ this equation can be brought into a hypergeometric form $\rightarrow$ analytically solved. For $n \neq 0$, this can be brought to a confluent Heun equation by setting:

$$f(z) = z^{\frac{|m|}{2}} (1-z)^{\ell}\, \mathfrak{f}(z) \,. \tag{43}$$

Indeed, if we do this the function $\mathfrak{f}$ has to satisfy the following DE:

$$0 = \partial_z^2 \mathfrak{f} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\right)\partial_z \mathfrak{f} + \frac{\alpha z - q}{z(z-1)}\mathfrak{f} \,, \tag{44}$$

with

$$\gamma = |m| + 1 \,, \quad \delta = 2(\ell+1) \,, \quad \varepsilon = 0 \,, \quad \alpha = -\frac{n^2}{16} \,, \quad q = \frac{M^2}{4} - \frac{n^2}{16} - \left(\frac{|m|}{2} + \ell + 1\right)^2 + 1 \,, \tag{45}$$

which is the confluent Heun equation.

Analitic case $n = 0$:
Heun equation reduces to a hypergeometric differential equation:

$$0 = z(1-z)\partial_z^2 \mathfrak{f} + \left[c - (a+b+1)z\right]\partial_z \mathfrak{f} - ab\mathfrak{f}\,, \tag{46}$$

with

$$a = 1 + \ell + \frac{|m|}{2} - \sqrt{1 + \frac{M^2}{4}}\,, \quad b = 1 + \ell + \frac{|m|}{2} + \sqrt{1 + \frac{M^2}{4}}\,, \quad c = |m| + 1\,. \tag{47}$$

Two linearly independent solutions. Regularity at $z = 0$ forces us to discard one. The solution is:

$$\mathfrak{f}(z) = {}_2F_1(a, b; c; z)\,. \tag{48}$$

But it behaves near $z = 1$ as:

$$\lim_{z\to 1^-} \frac{{}_2F_1(a, b; c; z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,. \tag{49}$$

Regularity at $z = 1$ forces us to require $a = -\nu$ with $\nu = 0, 1, 2, \ldots$ ..
The mass spectrum:

$$M^2 = \Big(2(\nu+\ell) + |m|\Big)\Big(2(\nu+\ell+2) + |m|\Big)\,, \qquad \Delta = 2(\nu+\ell+2) + |m|\,,$$
$$m \in \mathbb{Z}\,, \qquad \nu, \ell = 0, 1, 2, \ldots\,. \tag{50}$$

Taking $k = 2(\nu + \ell) + |m|$ we can write the above formula as:

$$M^2 = k\big(k+4\big)\,, \qquad k = 0, 1, 2, \ldots\,. \tag{51}$$

This matches a known result for the excitations of the metric along the AdS_5 directions in the case of $AdS_5 \times S^5$ (Kim, Romans, van Nieuwenhuizen). $n = 0$ modes are inert under T-duality.

Non analitic case:
We use WKB approximation and numerical techniques (shooting):
WKB method:

$$\partial_r\left(p(r)\partial_r\Psi\right)+\Big(M^2w(r)+q(r)\Big)\Psi=0\,, \tag{52}$$

$$p\approx p_1(r-r_*)^{s_1}\,,\qquad w\approx w_1(r-r_*)^{s_2}\,,\qquad q\approx q_1(r-r_*)^{s_3}\qquad \text{as}\quad r\to r_*\,, \tag{53}$$

$$p\approx p_2r^{t_1}\,,\qquad w\approx w_2r^{t_2}\,,\qquad q\approx q_2r^{t_3}\qquad \text{as}\quad r\to\infty\,, \tag{54}$$

Then, for large M, M is approximately:

$$M^2=\frac{\pi^2}{\xi^2}\,\nu\Big(\nu-1+\frac{\alpha_2}{\alpha_1}+\frac{\beta_2}{\beta_1}\Big)+\mathcal{O}(\nu^0)\,,\qquad \nu=1,2,\ldots.\,, \tag{55}$$

where ξ is given by:

$$\xi:=\int_{r_*}^{\infty}dr\sqrt{\frac{w}{p}}\,. \tag{56}$$

$$\alpha_1=s_2-s_1+2\,,\qquad \beta_1=t_1-t_2-2 \tag{57}$$

$$\begin{aligned}\alpha_2&=|s_1-1|\qquad \text{or}\qquad \alpha_2=\sqrt{(s_1-1)^2-4\frac{q_1}{p_1}}\quad (\text{if}\quad s_3-s_1+2=0)\,,\\ \beta_2&=|t_1-1|\qquad \text{or}\qquad \beta_2=\sqrt{(t_1-1)^2-4\frac{q_2}{p_2}}\quad (\text{if}\quad t_1-t_3-2=0)\,.\end{aligned} \tag{58}$$

Applied to our case, it gives:

$$M^2=4\nu(\nu+|m|+2\ell)\,,\qquad \nu=1,2,\ldots. \tag{59}$$

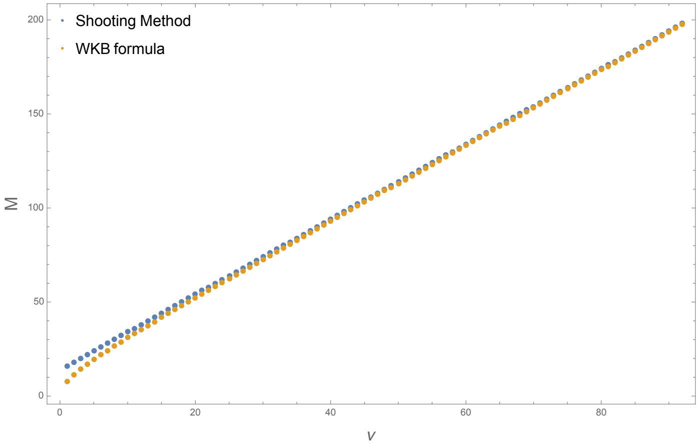

Figure: Comparison of the masses that are computed numerically with the ones computed using the WKB formula. For the computation we fix the quantum numbers as $(n, m, \ell) = (7, 4, 5)$.

Mass bound: We will now recast eq. (44) in the Sturm-Liouville fashion:

$$0 = S\mathfrak{f} + \lambda W(z)\mathfrak{f}\,, \qquad S := \frac{d}{dz}\left(P(z)\frac{d}{dz}\right) + Q(z)\,. \tag{60}$$

This is done for:

$$P(z) = z^{|m|+1}(1-z)^{2\ell+2}\,, \qquad Q(z) = -\frac{n^2}{16}\,z^{|m|}(1-z)^{2\ell+2}\,, \qquad W(z) = z^{|m|}(1-z)^{2\ell+1} \tag{61}$$

and

$$\lambda = \frac{1}{4}\Big[M^2 - \big(2\ell + |m|\big)\big(2\ell + |m| + 4\big)\Big]\,. \tag{62}$$

Let us also introduce the following inner product with respect to the weight function $W(z)$:

$$\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle_W := \int_0^1 dz\, W(z)\, \mathfrak{f}_1(z)\, \mathfrak{f}_2(z)\,. \tag{63}$$

We will impose boundary conditions such that two eigenfunctions of different eigenvalues are orthogonal. Then we have:

$$\begin{aligned} 0 &= S\mathfrak{f}_1 + \lambda_1 W(z)\mathfrak{f}_1\,, \\ 0 &= S\mathfrak{f}_2 + \lambda_2 W(z)\mathfrak{f}_2\,. \end{aligned} \tag{64}$$

If we multiply the first by $\mathfrak{f}_2$ and the second by $\mathfrak{f}_1$ and then subtract them we find:

$$0 = \mathfrak{f}_2\frac{d}{dz}\left(P(z)\frac{d\mathfrak{f}_1}{dz}\right) - \mathfrak{f}_1\frac{d}{dz}\left(P(z)\frac{d\mathfrak{f}_2}{dz}\right) + \big(\lambda_1 - \lambda_2\big)W(z)\,\mathfrak{f}_1\,\mathfrak{f}_2\,. \tag{65}$$

Integrating the last from 0 to 1 we get:

$$(\lambda_1 - \lambda_2)\langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle_W = P(z)\left(\mathfrak{f}_1 \frac{d\mathfrak{f}_2}{dz} - \mathfrak{f}_2 \frac{d\mathfrak{f}_1}{dz} \right)\Bigg|_{z=0}^{z=1} . \tag{66}$$

Hence we impose:

$$P\, \mathfrak{f}\, \frac{d\mathfrak{f}}{dz}\Bigg|_{z=0} = P\, \mathfrak{f}\, \frac{d\mathfrak{f}}{dz}\Bigg|_{z=1} = 0\,, \tag{67}$$

which in our case is satisfied as long as $\mathfrak{f}$ and $d\mathfrak{f}/dz$ are finite at the endpoints of the interval $[0, 1]$, or if they diverge they do it slow enough.

We derive a lower bound for the mass spectrum. From the Sturm-Liouville equation (60) we have:

$$\lambda\, \langle \mathfrak{f}, \mathfrak{f} \rangle_W = -\int_0^1 dz\, \mathfrak{f}\, S\mathfrak{f} = \int_0^1 dz\, P \left(\frac{d\mathfrak{f}}{dz} \right)^2 - \int_0^1 dz\, Q\, \mathfrak{f}^2 \geq 0\,, \tag{68}$$

which implies that $\lambda \geq 0$ or:

$$M^2 \geq \left(2\ell + |m|\right)\left(2\ell + |m| + 4\right) . \tag{69}$$

From the numerical analysis we get strong evidence that the bound is not violated for non-zero values of n, as it will be seen in the figures.

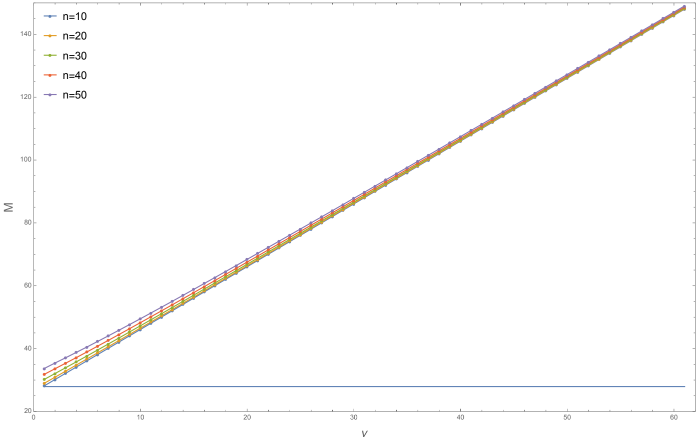

Figure: Mass spectra for fixed m and ℓ ($m = 10$, $\ell = 8$) and different values of n. The horizontal line represents the lower bound for the masses given in eq. (69).

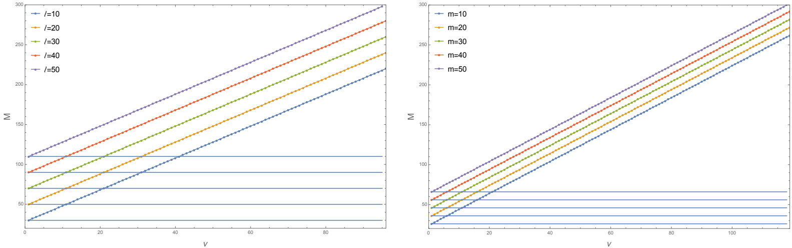

Figure: The figure on the left represents the mass spectra for $n = 10$, $m = 8$ and different values of ℓ. The figure on the right shows the mass towers for $n = 4$, $\ell = 7$ and different values of m. The horizontal lines represent the lower bounds for the masses given in eq. (69).

The non-Abelian T-dual of $AdS_5 \times S^5$ (Sfetsos, Thompson [1012.1320]) is another GM geometry. (A singular one!, but integrable (Nunez, Roychowdhury, Thompson [1804.08621]) as opposed to generic GM geometries (Borsato, Wulff [1706.10169]; Wulff [1903.08660])). Completion (Lozano, Nunez [1603.04440]). The potential $V(\eta,\sigma)$ reads:

$$V^{NATD}(\eta,\sigma) = \eta\Big(\ln\sigma - \frac{\sigma^2}{2}\Big) + \frac{\eta^3}{3}\,. \tag{70}$$

The change of coordinates that gives the NATD solution is the same as in eq. (37) where now we are going to rename ψ by ρ. The operator $\mathcal{L}$ in this case is:

$$\begin{aligned}\mathcal{L}^{NATD}(f) = \partial_\alpha^2 f + \ (\cot\alpha - 3\ \tan\alpha)\partial_\alpha f + \frac{1}{\sin^2\alpha}\partial_\beta^2 f + \frac{\cos^2\alpha}{4}\ \Big(\partial_\rho^2 f + \frac{2}{\rho}\ \partial_\rho f + \frac{1}{\rho^2}\ \nabla^2_{(2)} f\Big)\\ + \frac{4}{\cos^2\alpha}\nabla^2_{(2)} f\ .\end{aligned} \tag{71}$$

Ansatz:
ISSUE: range of the coordinate ρ? (NATD problem)
1st case: Allow ρ to run in the interval $[0,+\infty)$:

$$Y(y) = \sum_{m,\ell}\int_0^\infty dn\, f_{m,n,\ell}(\alpha)e^{im\beta}j_\ell(n\rho)\mathcal{Y}_\ell(\chi,\xi)\,, \qquad m\in\mathbb{Z}\,, \qquad n\in\mathbb{R}\,, \quad \ell = 0,1,2,\ldots\,, \tag{72}$$

For $n=0$ we get the spectrum of (50). When $n\neq 0$, one has to resort to numerics. Since $f_{m,n,\ell}(\alpha)$ satisfy the DE (40), we get the same WKB analysis as in the ATD case (59).

2nd case: ρ takes values in the interval $[0, \rho_*]$
$\rightarrow$ (hard cut-off in the geometry)
Boundary conditions of $Y(y)$ at $\rho = \rho_*$: assume that $Y|_{\rho=\rho_*} = 0$
$\rightarrow$ n can take only those values where:

$$n_{\ell s} = \frac{\rho_{\ell s}}{\rho_*} , \qquad s = 1, 2, \ldots , \tag{73}$$

with $\rho_{\ell s}$ being the roots of $j_\ell(\rho)$. The separation of variables scheme now reads:

$$Y(y) = \sum_{m,\ell,s} f_{m,\ell,s}(\alpha) e^{im\beta} j_\ell(n_{\ell s}\rho) \mathcal{Y}_\ell(\chi, \xi) , \qquad m \in \mathbb{Z} , \quad s = 1, 2, \ldots , \quad \ell = 0, 1, 2, \ldots . \tag{74}$$

n is replaced by $n_{\ell s}$.
$\rightarrow$ $n_{\ell s} = 0$ is analytic. Same mass spectrum as in eq. (50) where now $\ell \neq 0$ ($j_0(0) \neq 0$).
Case where $n_{\ell s} \neq 0$ the DE (40) can only be solved numerically. Replace n by $n_{\ell s}$. Same WKB as in eq. (59).

$$Y(y) = \sum_{m,\ell,s} f_{m,\ell,s}(\alpha) e^{im\beta} j_\ell(n_{\ell s}\rho) \mathcal{Y}_\ell(\chi, \xi) , \qquad m \in \mathbb{Z} , \quad s = 1, 2, \ldots , \quad \ell = 0, 1, 2, \ldots . \tag{75}$$

What we did:

- Whole set of linearized equations for warped geometries in type IIA with AdS_5
- Generic wave operator for Maldacena-Gaiotto geometries
- Complete study of Abelian and Non-Abelian T-dual of $AdS_5 \times S^5$
- $n = 0$ analytic case. $n \neq 0$ shooting + WKB. Agreement
- Mass bound

Outlook:

- Study of marginally deformations of Gaiotto-Maldacena (Nunez, Roychowdhury, Speziali, Zacarias [1901.02888])? (also characterized by $V(\eta, \sigma)$)

Thank you!