Integrable 2d sigma models:

RG flow, deformations and dualities

Arkady Tseytlin

Ben Hoare, Nat Levine, AT (to appear)

RG flow in bosonic 2d sigma model

$S = \frac{1}{4\pi\alpha'} \int d^2z\, G_{mn}(x)\, \partial x^m \partial x^n, \quad \alpha' = \hbar = 1$

∞-coupling theory: $\quad G_{mn}(x) = \delta_{mn} + \sum_r g_{mn;k_1 \dots k_r} x^{k_1} \dots x^{k_r}$

$\{g_{mn;k_1 \dots k_r}\} \leftrightarrow \{R_{mnkl}, D_k R_{mnkl}, \dots\}$

RG in ∞-coupling space $\{g_r(t)\}$ [Ecker, Honerkamp 71; Friedan 80]

$$\frac{d}{dt} G_{mn} = \beta_{mn} + D_{(m} X_{n)}$$

$$\beta_{mn} = R_{mn} + \tfrac{1}{2} R_{mpqr} R_n^{\ pqr} + \dots$$

find G_{mn} and X_m when flow is restricted to finite subspace?

example – manifest symmetry (isometry of metric, e.g. S^n)

what if symmetry is hidden?

Conjecture: integrability $\leftrightarrow$ invariance under RG flow

"sausage" model: deformed S^2 [Fateev, Onofri, Zamolodchikov 93] solves 1-loop Ricci flow; quantum S-matrix behind

same as YB (η-) deformation [Klimcik 02; Delduc, Magro, Vicedo 13]

of $\frac{SO(3)}{SO(2)}$ [Hoare, Roiban, AT 14]

$$ds^2 = \frac{h}{1+\varkappa^2 r^2}\Big[\frac{dr^2}{1-r^2} + (1-r^2)d\phi^2\Big]$$

$$\frac{d}{dt}h = (1+\varkappa^2) + \mathcal{O}(h^{-1})$$

$$\frac{d}{dt}\varkappa = h^{-1}\varkappa(1+\varkappa^2) + \mathcal{O}(h^{-2})$$

• classical integrability $\simeq$ inv. under 1-loop (i.e. Ricci) flow appears to be true also in higher $D > 2$ dim examples

[Fateev 96; Lukyanov 13; ....]

• why hidden charges may constrain RG flow?

• example: T-duality makes non-abel. symmetries hidden yet should remain symmetry of RG equations

$$S^2: \ \ ds^2 = h(dx^2 + \sin^2 x\, dy^2) \to \tilde{S}^2: \ \ \widetilde{ds^2} = h(dx^2 + \frac{1}{\sin^2 x} d\tilde{y}^2)$$

S^2: $SO(3)$ symmetry $\to$ one coupling h

$\tilde{S}^2$: only $SO(2)$ manifest but $SO(3)$ is hidden [Ricci, AT, Wolf 07]

still one-coupling?

how metric is deformed beyond 1-loop?

hidden symmetry: corrections to metric are correlated with corrections to T-duality transf. rule – only h runs

• similar questions about non-abelian duality

• hidden conserved charges expected to constrain RG flow but metric will be deformed beyond 1-loop

(cf. add local counterterms to maintain symmetry/Ward ident.)

• early example: complex SG model [Pohlmeyer; Lund, Regge]

$$L = \frac{\partial\zeta^*\partial\zeta}{1-\zeta\zeta^*} - m^2\zeta^*\zeta \ \to (\partial\phi)^2 + \tan^2\phi\,(\partial\theta)^2 - m^2\sin^2\phi$$

factorization of 1-loop S-matrix requires adding counterterm [de Vega, Maillet 8; Bonneau, Delduc 85]

• exists exact quantum S-matrix satisf. YBE [Dorey,Hollowood 95]

origin of counterterms: relation to $\frac{SU(2)}{U(1)}$ gWZW

appear automatically in gWZW formulation [Hoare, AT 10]

• generalized sine-Gordon models: integrable deformations of gauged WZW model [Hollowood, Miramontes et al 94, 97]

deformed as required by CFT [Dijkgraaf, Verlinde, Verlinde 91]

i.e. in order to satisfy 2-loop and higher conf. inv. cond. [AT 91]

• leading $\frac{1}{k}$ deformation of gWZW metric $\to$ counterterm required for factorization of 1-loop S-matrix [Hoare, AT 10]

Beyond leading 1-loop order:

classical σ-model will be non-trivially deformed
with number of RG running parameters preserved
motivation/reason: underlying quantum integrable S-matrix
depending on finite number of RG-invariant parameters
(RG scale/coupling hidden in dynamically generated mass)

• aim: study example of "sausage" model:
still 2-coupling $(h, \varkappa)$ theory beyond 1-loop?
[Fateev, Onofri, Zamolodchikov]: generalization of ZZ S-matrix for S^2
for deformed S^2: massive S-matrix depending on RG inv mass
and one quantized parameter:
suggests existence deformed-metric solution of RG eqs.
with just one running coupling

Construction of quantum deformation of η-model metric: aided by special limits of classical metric

$$ds^2 = h\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{1-r^2}{1+\varkappa^2 r^2} d\phi^2\Big]$$

(i) $\varkappa = 0$: S^2 – no deformation

(ii) $\varkappa^2 = -1$: flat – no deformation

(iii) $\varkappa^2 \to -1, \ r^2 = 1 - (1+\varkappa^2)\sinh^2 x$: $\frac{SO(1,2)}{SO(2)}$ gWZW

$$ds^2 = h(dx^2 + \tanh^2 x \, d\phi^2)$$

deformed by $\alpha' \sim h^{-1}$ corrections [Dijkgraaf, Verlinde, Verlinde 91]

similar deformation will be found for general $\varkappa$

Plan

• abelian T-duality

(path integral argument; quantum deformation)

• non-abelian duality

(deformation required for 2-loop consistency)

• gWZW: quantum deformation of effective σ-model

(corrections to 2d BH metric; generalized Sine-Gordon model)

• η-model

(quantum deformation of "sausage" $\frac{SO(3)}{SO(2)}$ or $\frac{SO(1,2)}{SO(2)}$ models)

• λ-model :

(2-loop deformation of $\frac{SO(1,2)}{SO(2)}$ model)

T-duality

$$ds^2 = dx^2 + M(x)dy^2$$

$$M\,\partial_m y \to \epsilon_{mn}\partial^n \tilde{y}, \quad y \to \tilde{y}, \qquad M \to M^{-1}$$

path integral transformation:

$$M(\partial y)^2 \;\to \epsilon^{mn}\partial_m \tilde{y} A_n + M A^m A_m$$

$$A_m = \partial_m y + \epsilon_{mn}\partial^n \bar{y}$$

integrate over $(y, \bar{y}, \tilde{y})$ with measures

$$\int d^2 z \sqrt{g} M (dy)^2; \;\int d^2 z \sqrt{g} M (d\bar{y})^2; \;\int d^2 z \sqrt{g} M^{-1} (d\tilde{y})^2$$

$$\int [dy][d\bar{y}] e^{-\int d^2 z \sqrt{g} M A^m A_m} = (\det Q)^{-1/2}$$

$$\int [d\bar{y}][d\tilde{y}] e^{-\int d^2 z \sqrt{g} \partial^a \tilde{y} \partial_a \bar{y}} = (\det H)^{-1/2}$$

$$Z[M] = \int [dx][dy] e^{-\frac{1}{4\pi\alpha'}\int d^2 z \sqrt{g}[(\partial x)^2 + M(x)(\partial_m y)^2]}$$

compare $Z[M]$ and dual-theory $Z[M^{-1}]$

compute det via anomaly (redefine to norm 1): [Schwarz, AT 92]

$$\frac{1}{2}\log\det Q = -\frac{1}{4\pi}\int d^2z\sqrt{g}\Big[\frac{1}{12}R^{(2)}\nabla^{-2}R^{(2)} + \frac{1}{2}R^{(2)}\log M + \frac{1}{2}(\partial_m \log M)^2\Big]$$

$$\frac{1}{2}\log\det H = -\frac{1}{4\pi}\int d^2z\sqrt{g}\Big[\frac{1}{12}R^{(2)}\nabla^{-2}R^{(2)} + \frac{1}{2}(\partial_m \log M)^2\Big]$$

$$\frac{Z[M]}{Z[M^{-1}]} = \frac{(\det H)^{1/2}}{(\det Q)^{1/2}} = e^{-\frac{1}{8\pi}\int d^2z\sqrt{g}R^{(2)}\log M(x)}$$

- dilaton shift found by [Buscher 88]; but missed det H and $(\partial \log M)^2$ contribution in det Q [AT 91; Schwarz, AT 92]
- on flat background with above definition of path integral

$$\int [dA]\, e^{-\int d^2z\, M A^m A_m} = (\det Q)^{-1/2} = e^{-\frac{1}{4\pi}\int d^2z\ \frac{1}{2}(\partial_m \log M)^2}$$

Corrections to T-duality transformation

- 1-loop beta-functions mapped to each other [Buscher 87]
- if G_{mn} solves 2-loop RG equation then $\tilde{G}_{mn}$

should be deformed to satisfy 2-loop RG

in string context: solve α'-corrected Weyl inv conditions

or effective eqs of motion [AT 91] (same for $O(d,d)$: [Panvel 92; ...])

- reason: undeformed $dx^2 + M(x)dy^2$

no longer solves 2-loop RG eqs (cf. 2d black hole metric)

- in general: $G_{mn} = (G_{ij}, G_{yy} = M)$

$$ds^2 = G_{ij}(x)dx^i dx^j + M(x)dy^2, \qquad M = e^{2\rho}$$

$$\tilde{\rho} = -\rho + \alpha'(\partial_i\rho)^2 + ..., \quad \tilde{G}_{ij} = G_{ij}\,, \quad \tilde{\phi} = \phi - \rho + \tfrac{1}{2}\alpha'(\partial_i\rho)^2 + ...$$

alternative (coordinate transformation):

$$\tilde{\rho} = -\rho\,, \qquad \tilde{G}_{ij} = G_{ij} + 2\alpha' D_i D_j \rho + ...$$

(to show: consider path integral with extra sources for x, y, etc.)

T-duality: integrability vs RG flow: S^2 example

$$ds^2 = h(dx^2 + \sin^2 x\, dy^2) \to \widetilde{ds^2} = h(dx^2 + \frac{1}{\sin^2 x} d\tilde{y}^2)$$

• S^2: $SO(3)$ symm: $\to$ one coupling $h(t)$, metric not deformed
• $\tilde{S}^2$: only $SO(2)$ manifest but $SO(3)$ hidden
still one-coupling theory? how metric is deformed?
• hidden symmetry $\to$ corrections to metric are correlated
with corrections to T-duality transformation rule – only h runs

Non-abelian duality

$$L(g) = h\, J^a_m J^a_m\,, \qquad J^a_m = \mathrm{tr}(t^a g^{-1}\partial_m g)$$

$$L(g) \ \to\ L(A,v) = \epsilon^{mn}\, v^a F^a_{mn} + h\, A^a_m A^a_m \ \to\ \tilde{L}(v)$$

$$F^a_{mn} = \partial_m A^a_n - \partial_n A^a_m + f^{abc} A^b_m A^c_n$$

$$\tilde{L}(v) = \partial_m v^a (M^{-1})^{mn}_{ab}(v)\partial_n v^b$$

$$M^{ab}_{mn} = h\,\eta_{mn}\delta^{ab} + \epsilon_{mn} f^{abc} v^c$$

[Freedman, Townsend 80; Fridling, Jevicki 84; Fradkin, AT 84]

• $L(g)$ is 1-coupling theory, inv under RG due to symmetry

what about $\tilde{L}(v)$? hidden symmetry should constrain RG flow?

[no direct relation to abelian T-duality: appears only

if add extra dimensions x^r and $\mathcal{M}(x)\mathrm{tr}(g^{-1}\partial_m g)$]

• 2-loop problem found earlier for NAD of S^3 and S^2

[Subbotin, Tyutin 95; Balog, Forgacs, Horvath, Palla 96; Balazs et al 97

• attempted resolution [Bonneau, Casteill 01]: deform σ-model with some new parameters; but why RG flow still 1-coupling?

• logic: add local counterterms to preserve symmetry that became hidden under duality [cf. abelian T-duality in S^2] hidden symmetry $\rightarrow$ renormalizability with just one coupling

• explicit structure of required deformation:

path integral origin – integrate A^a_m: $\quad A^a_m = \partial y^a_m + \epsilon_{mn}\partial^n \bar{y}^a_m$

$\partial v \partial \bar{y} \rightarrow \det H$ here is trivial

$AM(v)A \rightarrow \det Q$ is non-trivial (v-dependent)

$$\frac{1}{2}\log\det Q \rightarrow -\frac{1}{4\pi}\int d^2z\sqrt{g}\Big[\frac{1}{2}R^{(2)}\log M + \frac{1}{2}(\partial_m \log M)^2\Big]$$

• get dilaton shift under NAD (in PV regularization [DeJaegher, Raeymaekers, Sevrin, Troost]) but also $(\partial_m \log M)^2$ term – here not cancelled by $\det H$

• similar $(\partial_m \log M)^2$ counterterm appears in λ-model below

gWZW: quantum deformation of effective σ-model

- CFT: geometry probed by point-like string (e.g. tachyon Φ)

$T = \frac{1}{\mathrm{k}+c_G} J_G^2 - \frac{1}{\mathrm{k}+c_H} J_H^2$ [Dijkgraaf, Verlinde, Verlinde 91]

$SL(2)/U(1)$: $L_0 + \bar{L}_0 = H, \quad H\Phi \to \frac{1}{e^{-2\phi}\sqrt{G}} \partial_m (e^{-2\phi}\sqrt{G}\, G^{mn}\partial_n)\Phi$

$$ds^2 = (\mathrm{k}-2)\big(dr^2 + \frac{\tanh^2 r}{1-\frac{2}{k}\tanh^2 r}\, dy^2\big)\,, \qquad e^{-2\phi}\sqrt{G} = \sinh r\, \cosh r$$

- alternative derivation: from local part of eff action in gWZW

[AT 93; Bars, Sfetsos 93]

$$S_{\rm eff} = -\frac{\mathrm{k}+c_G}{4\pi}\mathrm{Tr}\Big[\,\frac{1}{2}\int d^2x\ \ g^{-1}\partial_+ g\, g^{-1}\partial_- g - \frac{1}{3}\int d^3x\ (g^{-1}\partial_m g)^3$$
$$+\int d^2x\ \big(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g$$
$$-g^{-1}A_+ g A_- - \big[1 - \frac{2(c_G - c_H)}{\mathrm{k}+c_G}\big] A_+ A_-\big)\Big]$$

- equiv. leading $\frac{1}{k}$ correction: from $A_+ M A_- \to -\frac{1}{2}(\partial \log M)^2$
- same from preservation of conf inv at higher α' order [AT 91]

2d BH metric deformed by α' correction:

$$ds^2 = dx^2 + e^{2\rho(x)} dy^2, \qquad \phi = \phi(x)$$

$$\rho = \ln\tanh\gamma x\,, \qquad \phi = -\ln\cosh\gamma x\,, \qquad \alpha'\gamma^2 = \frac{1}{\mathrm{k}}$$

$$\beta_{mn} + 2D_m D_n \phi = 0\,, \qquad \beta_{mn} = \alpha' R_{mn} + \tfrac{1}{2}\alpha'^2 R_{mpqr} R_n{}^{pqr} + \ldots$$

2d target space: $R_{mnkl} = \frac{1}{2} R(G_{mk}G_{nl} - G_{ml}G_{nk})$

$$\beta_{mn} = (\tfrac{1}{2}\alpha' R + \tfrac{1}{4}\alpha'^2 R) G_{mn} + \ldots\,, \qquad R = -2(\rho'' + \rho'^2)$$

leading coefficients universal (scheme-independent)

solution to order α'^2: [AT 91]

$$\rho = \ln\tanh\gamma x + \alpha'\gamma^2 \tanh^2\gamma x + \mathcal{O}(\alpha'^2)$$

$$\phi = -\ln\cosh\gamma x + \tfrac{1}{2}\alpha'\gamma^2 \tanh^2\gamma x + \mathcal{O}(\alpha'^2)$$

exact $SL(2)/U(1)$ gWZW metric [Dijkgraaf, Verlinde, Verlinde 91]

$$ds^2 = (\mathrm{k}-2)\Big(dr^2 + \frac{\tanh^2 r}{1-\frac{2}{\mathrm{k}}\tanh^2 r}\, dy^2\Big)$$

$$r \to \gamma x\,, \quad \tfrac{1}{\mathrm{k}-2} \to \alpha'\gamma^2$$

$$\rho = \ln\tanh\gamma x - \tfrac{1}{2}\ln\big(1 - \tfrac{2\alpha'\gamma^2}{1+2\alpha'\gamma^2}\tanh^2\gamma x\big)$$
$$= \ln\tanh\gamma x + \alpha'\gamma^2\tanh^2\gamma x + ...$$

agreement to α'^2 order: deformation of metric required to satisfy 2-loop conf. inv. condition (extended to 4-loop order: [Jack, Jones, Panvel 92])

Application: generalized sine-Gordon model

$\frac{G}{H}$ gWZW + integrable "mass term" [Hollowood, Miramontes et al 97]

$$S = -\frac{\mathrm{k}}{4\pi}\mathrm{tr}\Big[\frac{1}{2}\int d^2x\ \ g^{-1}\partial_+ g\, g^{-1}\partial_- g\ -\frac{1}{3}\int d^3x\ (g^{-1}\partial_m g)^3$$
$$+\int d^2x\ \ \big[A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-$$
$$+\, m^2\int d^2x\ \ (g^{-1}TgT - T^2)\Big]$$

• local part of quant. eff. action for gWZW [AT 93; Bars, Sfetsos 93]

$$S_{\rm eff} = -\frac{\mathrm{k}+c_G}{4\pi}\mathrm{Tr}\Big[\frac{1}{2}\int d^2x\ \ g^{-1}\partial_+ g\, g^{-1}\partial_- g\ -\frac{1}{3}\int d^3x\ (g^{-1}\partial_m g)^3$$
$$+\int d^2x\ \ \big(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g$$
$$-g^{-1}A_+ g A_- - \big[1-\frac{2(c_G-c_H)}{\mathrm{k}+c_G}\big]A_+ A_-\big)\Big]$$

axially gauged $\frac{SU(2)}{U(1)}$: in gauge $\quad g = e^{-\tau_3\theta}e^{2\tau_1\phi}e^{\tau_3\theta}$

solve for $A_\pm$:

$$\mathcal{L}_{\text{eff}} = \frac{\text{k}+2}{4\pi}\Big[\partial_+\phi\partial_-\phi + \frac{\tan^2\phi}{1-\frac{2}{k}\tan^2\phi}\,\partial_+\theta\partial_-\theta - m^2\sin^2\phi\Big]$$

rescaling ϕ and expanding for $k \gg 1$

$$L_{\text{eff}} = \tfrac{1}{2}\big(\partial_+\phi\partial_-\phi + \phi^2\partial_+\theta\partial_-\theta - m^2\phi^2\big) + \tfrac{2\pi}{\text{k}}\big(\tfrac{1}{3}\phi^4\partial_+\theta\partial_-\theta + \tfrac{m^2}{6}\phi^4\big) + \mathcal{O}\big(\tfrac{1}{k^2}\big)$$

resulting 1-loop S-matrix factorizes [Hoare, AT 10]

• alternative approach: direct integrating out of $A_\pm \equiv A_\pm\tau_3$:

$$S = \tfrac{\text{k}}{4\pi}\int d^2x\,\Big(\partial_+\phi\partial_-\phi + \sin^2\phi\,\partial_+\theta\partial_-\theta - m^2\sin^2\phi - A_+\sin^2\phi\,\partial_-\theta - A_-\sin^2\phi\,\partial_+\theta - \cos^2\phi\,A_+A_-\Big)$$

definition: $A_+ = \partial_+ u, \ A_- = \partial_- \bar{u}, \ [dA] \to [du d\bar{u}]$

results in non-trivial quantum correction [AT 91; Schwarz, AT 92]

$$\int [dA] \exp \left[i \int d^2x \sqrt{g}\, M\, A_+ A_- \right]$$

$$= \exp \left[\frac{i}{4\pi} \int d^2x \sqrt{g} \Big(-\tfrac{1}{2} R^{(2)} \ln M - \tfrac{1}{2} \partial_+ \ln M\, \partial_- \ln M \Big) \right]$$

i.e. in addition to dilaton shift $-\frac{1}{8\pi} \int d^2x \sqrt{g} R^{(2)} \ln \mathrm{M}$

get finite 1-loop correction $\Delta L = -\frac{1}{2} \partial_+ \ln M\, \partial_- \ln M$

• here: $M = \cos^2 \phi \ \ \to$ extra $\frac{1}{k}$ quantum term in the action

$$\mathcal{L} = \frac{\mathrm{k}}{4\pi} \Big[\partial_+ \phi \partial_- \phi + \tan^2 \phi\, \partial_+ \theta \partial_- \theta - m^2 \sin^2 \phi - \frac{2}{\mathrm{k}} \tan^2 \phi\, \partial_+ \phi \partial_- \phi \Big]$$

$$\to \partial_+ \phi \partial_- \phi + \phi^2 \partial_+ \theta \partial_- \theta - m^2 \phi^2 + \frac{4\pi}{3\mathrm{k}} \big(\phi^4 \partial_+ \theta \partial_- \theta + \tfrac{1}{2} m^2 \phi^4 \big)$$

related to above by field redefinition:

gives same YBE-consistent 1-loop S-matrix [Hoare, AT 10]

η-model: quantum corrections to "sausage" metric

Recall special limits:

$$ds^2 = h\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{1-r^2}{1+\varkappa^2 r^2} d\phi^2\Big]$$

- $\varkappa = 0$ is S^2; $\quad \varkappa^2 = -1$ is flat
- $\varkappa^2 \to -1$ with $r^2 \to 1 - (1+\varkappa^2)\sinh^2 r$:

$\frac{SO(1,2)}{SO(2)}$ gWZW: $\qquad ds^2 = h(dr^2 + \tanh^2 r\, d\phi^2)$

- follow analogy with exact gWZW metric:

$$ds^2 = h\Big[dr^2 + \big(\coth^2 r - \frac{2}{\mathrm{q}}\big)^{-1} dy^2\Big]$$

$$\mathrm{q} = h + 2 = \mathrm{k}\,, \qquad h = \mathrm{k} - 2$$

• proposed exact solution of RG eqs.:

$$ds^2 = h\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \big(\frac{1+\varkappa^2 r^2}{1-r^2} + \frac{2}{\mathrm{q}}\big)^{-1} d\phi^2\Big]$$

$$\mathrm{q} = \frac{h+1-\varkappa^2}{\varkappa^2}\,, \qquad \mathrm{q}^{-1} = \varkappa^2 h^{-1}\big[1 - h^{-1}(1-\varkappa^2) + ...\big]$$

(modulo field/scheme and coordinate redefinitions)

Consistency checks:

• no deformation for $\varkappa = 0$ (S^2) and $\varkappa^2 = -1$ (flat space)

• in gWZW limit: $\varkappa^2 \to -1,\ r^2 \to 1-(1+\varkappa^2)\sinh^2 r$:

$$ds^2 = h\Big[dr^2 + \big(\coth^2 r - \frac{2}{h+2}\big)^{-1} dy^2\Big]$$

same as exact gWZW metric with $h = \mathrm{k} - 2$

• direct confirmation at 2-loop order:

$$\frac{d}{dt}G_{mn} = \beta_{mn} + D_{(m}X_{n)}$$

$$\beta_{mn} = R_{mn} + \tfrac{1}{2}R_{mpqr}R_n^{\ pqr} + ... = (\tfrac{1}{2}R + \tfrac{1}{4}R^2)G_{mn} + ...$$

solution for specific X_n with only h and $\varkappa$ running:

$$ds^2 = h\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{1-r^2}{1+\varkappa^2 r^2}\Big(1 - \frac{2\varkappa^2}{h}\frac{1-r^2}{1+\varkappa^2 r^2} + ...\Big)d\phi^2\Big]$$

$$\frac{d}{dt}h = (1+\varkappa^2) + h^{-1}(1+\varkappa^2)^2 + \mathcal{O}(h^{-2})$$

$$\frac{d}{dt}\varkappa = h^{-1}\varkappa(1+\varkappa^2) + \mathcal{O}(h^{-2})$$

matches $\frac{1}{h}$ term in exact metric

3-loop order:

$$\beta_{mn} = \left[\tfrac{1}{2}R + \tfrac{1}{4}R^2 + c_1 R^3 + c_2 (DR)^2 + c_3 R D^2 R\right] G_{mn} + c_4 D_m R D_n R$$

minimal subtraction scheme [Graham 87; Foakes, Mohammedi 87]

$$c_1 = \frac{5}{32}\,, \qquad c_2 = \frac{1}{16}\,, \qquad c_3 = 0\,, \qquad c_4 = -\frac{1}{16}$$

exact metric renormalizable in scheme ($G_{mn} \to (1 + \frac{3}{16}R^2) G_{mn}$)

$$c_1 = \frac{1}{4}\,, \qquad c_2 = -\frac{1}{8}\,, \qquad c_3 = 0\,, \qquad c_4 = \frac{1}{16}$$

RG running to 4-loop order:

$$\frac{d}{dt}h = (1 + \varkappa^2)\left[1 + h^{-1}(1 + \varkappa^2) + 2h^{-2}(1 + \varkappa^2)(1 - \varkappa^2) + \mathcal{O}(h^{-3})\right]$$

$$\frac{d}{dt}\varkappa = h^{-1}\varkappa(1 + \varkappa^2)\left[1 + \mathcal{O}(h^{-3})\right]$$

- $\varkappa^2 = -1$: exact fixed point (flat space)
- $\varkappa = 0$: RG eq for h reduces to S^2 case

Remarks:

- $\varkappa^2 = -1$: flat space

  $ds^2 = h[\frac{dr^2}{(1-r^2)^2} + (1 + 2h^{-1})d\phi^2]$, $\quad \mathrm{q}\big|_{\varkappa^2=-1} = -(h+2)$

- the flow is effectively 1-coupling one: exists RG invariant

$$\nu = \frac{h + 1 - \varkappa^2}{\varkappa} = \varkappa\, \mathrm{q}\,, \qquad \frac{d}{dt}\nu = 0$$

generalizes 1-loop relation $\frac{d}{dt}(\frac{h}{\varkappa}) = 0$

- relation of exact metric to quantum trigonometric S-matrix?

[Fateev, Onofri, Zamolodchikov 93]

massive S-matrix has perturbative expansion near

$$\nu^{-1} = \frac{\varkappa}{h+1-\varkappa^2} = \frac{1}{2}$$

describes scattering of free massive scalar and Dirac fermion or 2 massive scalars after bosonization

$\to$ strong-coupling dual of $\varkappa$-deformed S^2 σ-model
(cf. Sine-Liouville model vs 2d BH model)
can one reconstruct exact metric? (cf. [Litvinov, Spodyneiko 18])

• (1,1) susy σ-model case: first correction to 1-loop $\beta_{mn} = R_{mn}$
at 4 loops $\sim \zeta(3)R^4$ (in minimal scheme) $\to$ 0 in 2d case
no deformation in gWZW case (to all orders)
same expected here:
no deformation but h is running as in S^2 case

η-deformation of $H^2 = \frac{SO(1,2)}{SO(2)}$

relation to deformed S^2 by formal continuation:

$$r \to ir, \quad \phi \to i\phi, \quad \varkappa \to i\varkappa, \quad h \to -h$$

$$ds^2 = h\Big[\frac{dr^2}{(1+r^2)(1+\varkappa^2 r^2)} + \frac{1+r^2}{1+\varkappa^2 r^2}d\phi^2\Big]$$

- $\varkappa = 0$ is H^2; $\quad \varkappa = 1$ is flat
- $\varkappa \to 1$ with $r^2 \to -1 + (1-\varkappa^2)\sinh^2 r$:

$$ds^2 = h(-dr^2 + \tanh^2 r\, d\phi^2)$$

related to $\frac{SO(1,2)}{SO(2)}$ gWZW by continuation ($\phi \to i\phi$, $h \to -h$)

- like $\frac{SO(1,2)}{SO(2)}$ gWZW model is self T-dual: inv under

$$\phi \to \tilde{\phi}, \quad \tilde{\phi} \to \varkappa^{-1}\phi, \quad r \to \varkappa^{-1} r^{-1}$$

• conformally-flat coordinates: $r = \frac{q}{p}\,, \quad \phi = \frac{1}{2}\log(p^2+q^2)$

$$ds^2 = \frac{h}{p^2+\varkappa^2 q^2}(dp^2+dq^2)$$

ϕ-shift isometry became scaling $(p,q) \to \lambda(p,q)$

• exact metric (analytic continuation of $\frac{SO(3)}{SO(2)}$ case):

$\varkappa \to i\varkappa, \;\; h \to -h$: $\quad \mathrm{q} = \frac{h-1-\varkappa^2}{\varkappa^2}$

$$ds^2 = h\Big[\frac{dr^2}{(1+r^2)(1+\varkappa^2 r^2)} + \big(\frac{1+\varkappa^2 r^2}{1+r^2} + \frac{2}{\mathrm{q}}\big)^{-1} d\phi^2\Big]$$

• $\varkappa \to 1, \;\; r^2 \to -1 + (1-\varkappa^2)\sinh^2 r$: $\quad$ exact $\frac{SO(1,2)}{SO(2)}$ gWZW

- exact metric in "conformal" coordinates (p, q):

$$ds^2 = \frac{h(dp^2 + dq^2)}{p^2 + \varkappa^2 q^2} - \frac{\varkappa^2}{2} \frac{\left[d \log(p^2 + q^2)\right]^2}{1 - \frac{1-\varkappa^2}{h} \frac{p^2 - \varkappa^2 q^2}{p^2 + \varkappa^2 q^2}}$$

- quantum correction trivial for $\varkappa = 0$ (H^2) and $\varkappa^2 = 1$ (flat)
- leading 2-loop correction:

$$ds^2 = h \, \frac{dp^2 + dq^2}{p^2 + \varkappa^2 q^2} - \frac{\varkappa^2}{2} \frac{\left[d(p^2 + q^2)\right]^2}{(p^2 + \varkappa^2 q^2)^2}$$

Quantum deformation of λ-model

$$L = gWZW_G + APA, \qquad P = P_{G/H}$$

[Sfetsos 13; Hollowood, Miramontes, Schmidt 14]

$$L = \mathrm{k}\,\mathrm{tr}\big[\tfrac{1}{2}(g^{-1}\partial g)^2 + A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g$$

$$-g^{-1}A_+ g A_- + A_+A_- + b^{-2}A_+PA_-\big] + \mathrm{k}\,\mathrm{WZ}(g)$$

- $b \to 0$: $PA_\pm = 0 \to$ G/H gWZW model
- $\mathrm{k} \to \infty$, $b \to \infty$, $g = e^{v/\mathrm{k}}$: NAD model

$$L = \mathrm{tr}(vF_{+-} + hA_+A_-), \qquad h = \mathrm{k}\,b^{-2}$$

- $G = SO(1,2)$, $H = SO(2)$: integrate out $A_\pm$

$$g = \exp(\alpha\sigma_3)\exp(i\beta\sigma_2), \quad \cosh\alpha = \sqrt{p^2+q^2}, \ \tan\beta = \tfrac{p}{q}$$

$$\mathcal{L} = \frac{\mathrm{k}}{p^2+q^2-1}(\kappa\,\partial_+ p\partial_- p + \kappa^{-1}\partial_+ q\partial_- q), \qquad \kappa \equiv (1+2b^2)^{-1}$$

- alternative form: $p \to a^{-1}p, \;\; q \to a^{-1}\kappa\,\bar{q}$

$$ds^2 = \frac{h}{p^2 + \kappa^2\bar{q}^2 - a^2}(dp^2 + d\bar{q}^2)\,, \qquad\qquad h \equiv \mathrm{k}\,\kappa$$

- $a = 0$: $\frac{SO(1,2)}{SO(2)}$ η-model (cf. T-self-dual) [Hoare, AT 15]

$$ds^2 = \frac{h}{p^2 + \varkappa^2\bar{q}^2}(dp^2 + d\bar{q}^2)\,, \qquad\qquad h \equiv \mathrm{k}\,\kappa\,, \quad \varkappa \equiv \kappa$$

- $\kappa = 1,\; a = 1$: $\qquad (p, q) = \cosh\alpha(\cos\beta, \sin\beta)$

  $\frac{SO(1,2)}{SO(2)}$ gWZW: $\qquad ds^2 = \mathrm{k}(d\alpha^2 + \coth^2\alpha\, d\beta^2)$

- $\kappa \to 0,\; \mathrm{k} \to \infty,\; a = 1,\;\; p = \kappa X,\;\; \bar{q} = \kappa^{-1} + \kappa Y$:

  NAD of H^2: $\qquad ds^2 = \frac{h}{X^2 + 2Y}(dX^2 + dY^2)$

- $\kappa \to 0,\; \mathrm{k} \to \infty,\; a = 1,\;\; p = \cosh\alpha$: special NAD (abelian)

  T-dual of H^2: $\qquad ds^2 = h(d\alpha^2 + \frac{1}{\sinh^2\alpha}d\bar{q}^2)$

Leading quantum correction to λ-model metric:

- 1-loop renormalizable [Isios,Sfetsos,Siampos 14; Appadu,Hollowood 15]
  what happens at 2-loop level?
- deformation from integral over $A_\pm$ as in gWZW case:

$$L = ... + A_+ M A_- \ , \qquad M = I - \mathrm{Ad}_{g^{-1}} + b^{-2} P$$

$$\Delta L = -\tfrac{1}{2}(\partial_m \log \det M)^2 \ \to \ -\tfrac{1}{2}\left[\partial_m \log(p^2+q^2-1)\right]^2$$

one-loop corrected metric:

$$ds^2 = \mathrm{k}\,\frac{\kappa\, dp^2 + \kappa^{-1} dq^2}{p^2+q^2-1} - \frac{1}{2}\frac{\left[d(p^2+q^2)\right]^2}{(p^2+q^2-1)^2}$$

- manifest Z_2 invariance: $\quad p \leftrightarrow q\ , \quad \kappa \to \kappa^{-1}$

Direct check: corrected metric solves 2-loop RG eqs

$$\frac{d}{dt}G_{mn} = (\tfrac{1}{2}R + \tfrac{1}{4}R^2 + ...)G_{mn} + D_m X_n + D_n X_m$$

$$X_p = \frac{p}{p^2+q^2-1}\left[1 + \frac{1}{\mathrm{k}}\frac{\kappa(p^2-1)-\kappa^{-1}p^2}{p^2+q^2-1}\right]$$

$$X_q = \frac{q}{p^2+q^2-1}\left[1 + \frac{1}{\mathrm{k}}\frac{\kappa^{-1}(q^2-1)-\kappa q^2)}{p^2+q^2-1}\right]$$

$$\tfrac{d}{dt}\mathrm{k} = (\kappa^{-1} - \kappa)^2\mathrm{k}^{-1} + \mathcal{O}(\mathrm{k}^{-2})$$

$$\tfrac{d}{dt}\kappa = -(1 - \kappa^2)\mathrm{k}^{-1} + \mathcal{O}(\mathrm{k}^{-3})$$

• exists RG invariant (as in η-model)

$$\nu = \mathrm{k} - (\kappa^{-1} + \kappa) + \mathcal{O}(\mathrm{k}^{-1}), \qquad\qquad \frac{d}{dt}\nu = 0$$

1-coupling theory in disguise; consistency with k=integer ?

Consistency checks:

- gWZW limit: $\kappa \to 1, \quad (p,q) = (1-\mathrm{k}^{-1})\cosh\alpha(\cos\beta, \sin\beta)$

$$ds^2 = \mathrm{k}(d\alpha^2 + \coth^2\alpha d\beta^2) - 2(d\alpha^2 - \tfrac{\coth^2\alpha}{\sinh^2\alpha}d\beta^2) + \mathcal{O}(\mathrm{k}^{-1})$$

matches large k expansion of exact gWZW metric

$$ds^2 = (\mathrm{k}-2)\big(d\alpha^2 + \tfrac{\coth^2\alpha\ d\beta^2}{1-\frac{2}{\mathrm{k}}\coth^2\alpha}\big)$$

- η-model limit: $\quad (\kappa, \mathrm{k}) \to (\varkappa, h)$ and scaling of coordinates

$$\kappa = \varkappa\big[1 + \tfrac{2(1-\varkappa^2)}{h}\big]\,, \qquad\qquad h = \mathrm{k}\,\kappa$$

$$p \to a^{-1}p\,, \qquad q \to a^{-1}\varkappa\big(1+\tfrac{1-\varkappa^2}{h}\big)\,q\,, \qquad a \to 0$$

$$ds^2 = h\,\tfrac{dp^2+dq^2}{p^2+\varkappa^2q^2} - \tfrac{\varkappa^2}{2}\,\tfrac{[d(p^2+q^2)]^2}{(p^2+\varkappa^2q^2)^2} + \mathcal{O}(h^{-1})$$

$$\tfrac{d}{dt}h = -(1-\varkappa^2) + (1-\varkappa^2)^2h^{-1} + \mathcal{O}(h^{-2})$$

$$\tfrac{d}{dt}\varkappa = -\kappa(1-\varkappa^2)h^{-1} + \mathcal{O}(h^{-3}) \quad \to$$

$$\tfrac{d}{dt}\mathrm{k} = (\kappa^{-1}-\kappa)^2\mathrm{k}^{-1} + \mathcal{O}(\mathrm{k}^{-2}), \quad \tfrac{d}{dt}\kappa = -(1-\kappa^2)\mathrm{k}^{-1} + \mathcal{O}(\mathrm{k}^{-3})$$

- RG invariant

$$\nu \equiv \frac{h}{\varkappa} - (\varkappa^{-1} + \varkappa) + \mathcal{O}(h^{-1}) = \mathrm{k} - (\kappa^{-1} + \kappa) + \mathcal{O}(\mathrm{k}^{-1})$$

$\hat{\mathrm{q}} = \exp\left(\frac{i\pi}{\nu}\right)$ is parameter in exact S-matrix?

- gWZW limit:

$$\kappa = 1 = \varkappa, \ \nu = h - 2 = k - 2, \quad \hat{\mathrm{q}} = \exp\left(\frac{i\pi}{\mathrm{k}-2}\right)$$

consistent with quantization of k

- general κ: need extra counterterm $\sim (\varkappa^{-1} + \varkappa)$

to have k as RG invariant?

NAD limit:

$\kappa \to 0, \ \mathrm{k} \to \infty, \quad h \equiv \mathrm{k}\,\kappa$

$p = \kappa\, X, \quad q = 1 + \kappa^2\, Y$:

$$ds^2 = h\, \frac{dX^2 + dY^2}{X^2 + 2Y} - \frac{1}{2}\, \frac{\big[d(X^2 + 2Y)\big]^2}{(X^2 + 2Y)^2}$$

• 2-loop renormalizable with same 2-loop β-function as for H^2

$$\frac{d}{dt} h = -1 + h^{-1} + \mathcal{O}(h^{-2})$$

• NAD is consistent with 2-loop RG once metric is deformed: resolves earlier problem with NAD at 2 loops –
NAD should commute with α' pert. theory or string eff. action

• same should be for Poisson-Lie duality at 2-loop level (λ-model and η-model are PL dual [Hoare, AT 15; Vicedo 15])

Summary:

- like in conformal theories RG flow in integrable σ-models requires specific quantum deformation of the metric, etc.
- exact metric for η-deformation of S^2 or H^2 which solves 3-loop RG flow equations and consistent with gWZW limit
- leading-order deformation of λ-model consistent with 2-loop RG flow, gWZW and η-model limits
- key role of $(\partial \log M)^2$ finite counterterm from integration over 2d auxiliary gauge field
- resolution of problem with non-abelian duality at 2-loop level
- same should be true for general (in) homog. YB deformations
- as η-model and λ-model are formally related by PL duality same should apply to PL duality: should commute with RG once backgrounds are properly modified

Some open problems:

• exact metric in λ-model ?
(non-conformal, eff action approach used in gWZW case does not appear to apply)

• generalization to 3-dim target space: deformations of S^3
(non-zero B-field, NAD of S^3 at 2-loop level – in progress)

• interpretation of exact metric of η-model?
(deformation of integrable structure, Hamiltonian in phase space)

• connection to massive non-perturbative S-matrix [Fateev et al]
(cf. derivation of σ-model from dual massive theory [Litvinov, Spodyneiko 18])