Facets of λ-deformations

Integrability Duality and Beyond (Santiago de Compostella)

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Driezen, Sevrin, DT [1806.10712,1902.04142]

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Motivation

Dualities.....

....a catalyst for theoretical progress in diverse areas: statistical physics; QFT theory; condensed matter and of course String Theory.

- Target space T-duality – intrinsically stringy $\Rightarrow$ new geometric ideas e.g. generalised geometry or DFT
- More generally U-dualities $\Rightarrow$ M-theory?
- Gauge-gravity dualities or holography!

What other dualities?

What are their uses?

Motivation

A hierarchy of T-dualities

Bianchi–Conservation democracy ?

1. Abelian isometries ⇒ Abelian T-duality

$$K = \partial_\theta \ , \qquad [K, K] = 0 \ , \qquad d \star J = 0$$

2. Non-Abelian isometries ⇒ Non-Abelian T-duality Quevedo,De La Ossa

$$K_a = k_a^\mu \partial_\mu \ , \qquad [K_a, K_b] = f_{ab}{}^c K_c \ , \qquad d \star J_a = 0$$

3. Non-Abelian Non-isometries ⇒ Poisson-Lie T-duality Klimick,Severa

$$K_a = k_a^\mu \partial_\mu \ , \qquad [K_a, K_b] = f_{ab}{}^c K_c \ , \qquad d \star J_a = \tilde{f}^{bc}{}_a J_b \wedge J_c$$

Motivation

Reasons to be skeptical ...apologia

- Quantum g_s and α' status unclear ... **Holography; Talk of Tseytlin**
- Baroque or ugly geometries ... **wrong variables; Talk of Hassler**

Reasons to care

- Non-Abelian T-duality **holographic backgrounds** for exotic quiver QFTs
- η- and λ- **integrable deformations** of AdS_5 superstring
- Close connection to **gauged supergravity**
- A manifold structure for DFT

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Non-linear sigma model and principal chiral model

Strings in curved target space $\mathcal{M}$, $E_{ij} = G_{ij} + B_{ij}$:

$$S = \int \partial_+ X^i (G_{ij}(X) + B_{ij}(X)) \partial_- X^j$$

Suppose an isometry group G of vector field K_a then Noether currents

$$J_{\pm a} = K_a{}^i \left(G_{ij} \pm B_{ij} \right) \partial_\pm X^j$$

Useful example $\mathcal{M} = G$, a group manifold, and the PCM

$$S = \int \langle g^{-1} \partial_+ g, g^{-1} \partial_- g \rangle = \int L_+^a \kappa_{ab} L_-^b \ , \quad g = g(X) : \Sigma \to G$$

Left-invariant one-forms $L = g^{-1} dg$

Recap: the Principal Chiral Model

- **Classically (and Quantum) Integrable**: Lax formulation of e.q.m.

$$\mathcal{L}(z) = \frac{1}{1-z^2} g^{-1}dg + \frac{z}{1-z^2} \star g^{-1}dg \ , \quad d\mathcal{L} - \mathcal{L} \wedge \mathcal{L} = 0 \ ,$$

  $z \in \mathbb{C}$ an auxiliary parameter;

- **∞ of conserved charges** encoded in z-expansion of monodromy

$$T(z) = Pexp \int d\sigma \mathcal{L}_\sigma \ , \quad \partial_\tau T(z) = 0$$

Non-Abelian T-dual: The Buscher Procedure

Gauging procedure to obtain the non-Abelian T-dual geometry

1. **Gauge** G_L in PCM $\partial g \to Dg = \partial g - Ag$
2. **Double** the degrees of freedom with Lagrange multipliers

$$L_v = v_a F^a_{+-} \qquad F_{+-} = [D_+, D_-]$$

3. **Gauge Fix** $g = 1$ and integrate by parts
4. **Integrate out** non-propagating gauge fields to get new sigma model

$$S_{T-dual} = \frac{1}{\pi} \int \partial_+ v^a (\kappa^2 \delta_{ab} + F_{ab}{}^c v_c)^{-1} \partial_- v^b$$

Classical equivalence (canonical transformation) to PCM

Non-Abelian T-dual: Example of S^3

Lag. multipliers in spherical coordinates

$$(v_1, v_2, v_3) \mapsto (r, \theta, \phi)$$

Extract T-dual geometry

$$\widehat{ds^2} = \frac{dr^2}{\kappa^2} + \frac{r^2\kappa^2}{r^2+\kappa^4}\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\widehat{B} = \frac{r^3}{r^2+\kappa^4}\sin\theta d\theta\wedge d\phi$$

$$\widehat{\Phi} = \phi_0 - \frac{1}{2}\log(r^2+\kappa^4)$$

Extends to RR sector and type II supergravity

λ-deformations: The Sfetsos Procedure

Rather similar to the Buscher procedure this recipe produces integrable λ deformations [Sfetsos 1312] as a regularisation of non-Abelian T-duality

1. **Double** the d.o.f.: $\kappa^2 S_{PCM}[\tilde{g}] + k S_{WZW}[g]$
2. **Gauge** G_L in PCM and G_{diag} in WZW
3. **Gauge Fix** $\tilde{g} = 1$
4. **Integrate out** non-propagating gauge fields

$$S_\lambda = k S_{WZW} + \frac{k\lambda}{2\pi} \int Tr(g^{-1}\partial_+ g \mathcal{O}_g \partial_- g g^{-1})$$

$$\mathcal{O}_g = (1 - \lambda \mathrm{ad}_g)^{-1} \qquad \lambda = \frac{k}{\kappa^2 + k}$$

Integrable model for all values of λ!

Interpolation between CFT and non-Abelian T-duals

Nice behaviour in limits of small and large deformations:

- $\lambda \to 0$: current bilinear perturbation

$$S_\lambda|_{\lambda\to 0} \approx kS_{WZW} + \frac{k}{\pi}\int \lambda J_+^a J_-^a + \mathcal{O}(\lambda^2)$$

- $\lambda \to 1$: non-Abelian T-dual of PCM

$$S_\lambda|_{\lambda\to 1} \approx \frac{1}{\pi}\int \partial_+ X^a(\delta_{ab} + f_{ab}{}^c X_c)^{-1}\partial_- X^b + \mathcal{O}(k^{-1})$$

In this limit the gauged WZW in the Sfetsos Procedure becomes a Lagrange multiplier term of the Buscher Procedure

λ Commentary

- λ deformations solve SUGRA with appropriate RR fields [Sfetsos DT, Borsato Wulff]
- Quantum group symmetry expected with $q = e^{\frac{i\pi}{k}}$ [Hollowood et al]
- Can be quantised on a light cone lattice as spin-k Heisenberg XXX spin-chain [Hollowood,Price,Appadu (+DT)]
- Also applied to cosets [Sfetsos], supercosets [Hollowood et al]
- One-loop marginal deformation in case of $PSU(2,2|4)$! [Appadu, Hollowoodl]

η, λ and Poisson-Lie

η and λ connected by generalised Poisson Lie T-duality

[Vicedo 1504; Hoare & Tseytlin 1504; Siampos Sfetsos DT 1506; Klimcik 1508]

- PL dualise η model + Analytic continue certain Euler angles and deformation parameters

$$\eta \to i\frac{1-\lambda}{1+\lambda} \,, \quad t \to \frac{\pi(1+\lambda)}{k(1-\lambda)}$$

- Acting on the parameter q we have

$$q = e^{\eta t} \leftrightarrow q = e^{\frac{i\pi}{k}}$$

Exciting question: quantum corrections? exact map?

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Hacking the λ model

A few different possibilities to find new λ type theories

- Change the PCM
  - $\mathbb{Z}_2$ graded coset$\Rightarrow \lambda - G/H$
  - Other integrable model $\Rightarrow$ multi-parameter deformations
- Change the WZW e.g. multiple factors
- Change the G_{diag} gauging

Multi-parameter λ-Model

- Sfetsos Procedure can be generalised by replacing PCM:

$$S[\tilde{g}] = \int Tr\left(\tilde{g}^{-1}\partial_+ g\Theta\tilde{g}^{-1}\partial_- g\right)$$

- λ now a matrix Λ:

$$S_\lambda = kS_{WZW} + \frac{k}{2\pi}\int Tr(g^{-1}\partial_+ g\frac{1}{\Lambda^{-1} + ad_g}\partial_- gg^{-1})$$

$$\Lambda = 1 + k^{-1}\Theta$$

- **Idea**: if Θ defined integrable PCM, Λ can define an integrable theory

$\eta - \lambda$ Model

Start with η PCM

$$\Theta = (1 - \eta\mathcal{R})^{-1}$$

Rational Lax

$$\mathcal{L}_\sigma = (c_+(z) + d(z)\mathcal{R})\mathcal{J}_+ + (c_-(z) + d(z)\mathcal{R})\mathcal{J}_-$$

Second RG invariant

$$\Sigma = \frac{2\pi\eta\lambda}{k(1-\lambda)}$$

Quantum Group Symmetries

$$q_L = e^{\Sigma} \,, \quad q_R = e^{i\pi/k}$$

PL map to "bi-Yang-Baxter"

Asymmetric Gauging

Gauging in a WZW is constrained by 'anomaly' constraints (classical)

$$\langle [T_a^{(L)}, T_b^{(L)}], T_c^{(L)} \rangle = \langle [T_a^{(R)}, T_b^{(R)}], T_c^{(R)} \rangle$$
$$\langle T_a^{(L)}, T_b^{(L)} \rangle = \langle T_a^{(R)}, T_b^{(R)} \rangle$$

Some possibilities

- $U(1)_a$ and $U(1)_v$
- G_L 'null'
- G_{diag}
- $T^{(L)} = WT^{(R)}$ for some metric-preserving outer automorphism W

Using an automorphism allows a non-diagonal embedding of non-Abelian (sub)group

Asymmetric λ-model on symmetric space

Generalised construction includes λ deformation to axial gauged models

Start with a PCM on a G/H, append with $S_{WZW}[g]$, and we gauge the following G action:

$$g \to g_0^{-1} g W(g_0) \quad \tilde{g} \to g_0^{-1} \tilde{g}$$

Use minimal coupling in PCM and asymmetric gauged WZW

$$S_{\text{gWZW}}(g, A^A_\pm, W) = S_{\text{WZW}}(g) + \frac{k}{\pi} \int \langle A_-, \partial_+ g g^{-1} \rangle - \langle W(A_+), g^{-1} \partial_- g \rangle$$
$$+ \langle A_-, g W(A_+) g^{-1} \rangle - \frac{1}{2} \langle A_-, A_+ \rangle - \frac{1}{2} \langle W(A_-), W(A_+) \rangle .$$

Asymmetric λ-model on symmetric space

Completing the derivation we arrive at

$$S_\lambda(g, W) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi}\int \langle \partial_+ g g^{-1}, (\mathbf{1} - \mathrm{ad}_g W P_\lambda)^{-1}\, \partial_- g g^{-1}\rangle ,$$

$$e^{-2\Phi} = e^{-2\Phi_0}\ \det\left(ad_g W - \Omega\right)$$

$$P_\lambda(\mathfrak{g}) = \mathfrak{g}^{(0)} \oplus \frac{1}{\lambda}\mathfrak{g}^{(1)}$$

Residual gauge symmetry (fixed $\Rightarrow$ target space)

$$g \to hgW(h) \quad h \in H$$

Resulting theory still integrable; unclear what $\lambda \to 1$ vs. NABT or the PL map

$SL(2,R)/U(1)$: Of Cigars and Trumpets (undeformed)

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0;\end{pmatrix}, \quad \mathfrak{h} = T_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

$SL(2,R)$ Group element

$$g = e^{\frac{\tau-\theta}{\sqrt{2}}\,T_3}\, e^{\sqrt{2}\,\rho\,T_1}\, e^{\frac{\tau+\theta}{\sqrt{2}}\,T_3}$$

Axial gauging $g \to hgh$ gives **cigar** geometry ($k >> 1$):

$$ds_A^2 = k\left(d\rho^2 + \tanh^2\rho\, d\theta^2\right), \quad e^{-2\Phi_A} = e^{-2\Phi_0}\cosh^2\rho,$$

Vector gauging $g \to h^{-1}gh$ t gives **trumpet** geometry ($k >> 1$):

$$ds_V^2 = k\left(d\rho^2 + \coth^2\rho\, d\tau^2\right), \quad e^{-2\Phi_A} = e^{-2\Phi_0}\sinh^2\rho,$$

Related by T-duality (+ $\mathbb{Z}_k$ action) or complex field redefinition

$SL(2,R)/U(1)$: Of λ Cigars and Trumpets

Automorphism choice:

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \lambda - \text{trumpet} \qquad W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \lambda - \text{cigar}$$

Like compact case parafermionic deformation giving λ-cigar geometry:

$$\begin{aligned} ds^2_{A,\lambda} &= k\left(\frac{1-\lambda}{1+\lambda}\left(d\rho^2 + \tanh^2\rho d\theta^2\right) + \frac{4\lambda}{1-\lambda^2}\left(\cos\theta d\rho - \sin\theta\tanh\rho d\theta\right)^2\right), \\ &= \frac{k}{1-\lambda^2}\frac{\left(\lambda\left(d\zeta^2 + d\bar{\zeta}^2\right) + (1+\lambda^2)d\zeta d\bar{\zeta}\right)}{1+|\zeta|^2}, \qquad (\zeta = \sinh(\rho)e^{i\theta}) \end{aligned}$$

Relation to λ-trumpet? PL-η relations?
k^{-1} corrections ? S-matrix? Spin-chain quantisation?

Some fun cigar speculation

FZZ Fateev, Zamolodchikov, Zamolodchikov duality maps cigar CFT to "Sine-Liouville model"
In turn to matrix model dual Kazakov, Kostov, Kutasov

$$\mathcal{Z} \sim \int d\Omega \int_{A(2\pi R)=\Omega A(0)\Omega^{-1}} [dA] e^{-\int Tr(\partial_x A \partial_x A)+V(A)}$$

Hidden integrability in this system (differential equation constrain free energy)!

We can match our λ-deformation to SL at large φ, and argue that is exact since parafermions commute with potential. What about the Matrix model dual - is there an integrable deformation?

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Integrable Boundaries

Boundaries break symmetries but b.c. that preserve integrability?

Technique: Conserved boundary Monodromy Cherednik 84, Sklyanin 88

Transport the Lax from $0 \to \pi$, and reflect $\pi \to 0$

$$T^b(z) = T^\Omega(0,\pi,-z)T(\pi,0,z)$$
$$= Pexp \int_0^\pi \Omega(\mathcal{L}_\sigma(-z)) \cdot Pexp \int_\pi^0 \mathcal{L}_\sigma(z)$$

$\Omega \in aut\, \mathfrak{g}$ automorphism encodes reflection at boundary.
Conserved charges $Q^{(n)} = Tr(T^b(z))^n$ if

$$\partial_\tau T^b(z) = [T^b(z), N(z)]$$

D-branes in the λ-model on coset

Return to symmetric space λ-model on group manifold

- Using explicit form of Lax we find integrable boundary conditions:

$$\mathcal{O}_{g^{-1}}[g^{-1}\partial_- g]|_{\partial\Sigma} = -\Omega \cdot \mathcal{O}_g[\partial_+ g g^{-1}]|_{\partial\Sigma}$$

- Interpret these as a mix of Dirichlet and Neumann b.c.

$$\partial_\tau X^D = 0 \ , \quad \widehat{G}_{ab}\partial_\sigma X^{bN} = \mathcal{F}_{ab}\partial_\tau X^{bN} = (\widehat{B}_{ab} + 2\pi\alpha' F_{ab})\partial_\tau X^{bN}$$

with gauge flux $F = dA$ on the brane.

- D-branes are twisted conjugacy classes – matching beautiful results in CFT

Alekseev Schomerus, Felder et al., Stanciu, Stanciu Figueroa-O'Farrill

$$C_\omega(g) = \{hg\omega(h^{-1})|h \in G\} \ , \quad \omega(e^{tX}) \sim e^{t\Omega X} \ .$$

D-branes in the $SU(2)$ λ-model

- DBI action

$$S_{DBI} = \int e^{-\Phi}\sqrt{\widehat{G}+\mathcal{F}}$$

- λ enters spectrum of D-branes. E.g. $SU(2)$, δ a scalar fluctuation and g a gauge flucuation

$$\frac{d^2}{dt^2}\begin{pmatrix}\delta\\ g\end{pmatrix} = -\frac{1}{k}\frac{1+\lambda^2}{1-\lambda^2}\begin{pmatrix} 2+\frac{(1+\lambda)^2}{1+\lambda^2}\Box & 2 \\ 2\,\Box & \frac{(1+\lambda)^2}{1+\lambda^2}\Box \end{pmatrix}\begin{pmatrix}\delta\\ g\end{pmatrix},$$

- Note δ not a moduli, D-branes are stabilised
- Flux quantisation $\Rightarrow$ D-branes stabilised to conjugacy classes of integrable highest weights Bachas, Petropolous; Stanciu Figueroa-O'Farrill
- e.g. $SU(2)_k$: 2 D0's and $k-1$ D2's wrapping S^2 whose size is a function of λ

D-branes and $SU(2)$ λ-model limits

Interesting to track the D-branes in the NABT limit and PL+analytic map

- Non-Abelian T-dual limit $\lambda \to 1$ scaling limit

$$g = 1 + i\frac{v^a T_a}{k} + O(k^{-2})$$

The D2 brane boundary conditions become

$$M^{-1}\partial_+ v = -M^{-T}\partial_- v \qquad M = (\delta_{ab} - f_{ab}{}^c v_c)$$

Immediately recognise the (reverse) T-dual of these

$$\tilde{g}^{-1}\partial_+\tilde{g} = \tilde{g}^{-1}\partial_-\tilde{g} \Rightarrow \text{D3 branes in PCM}$$

D-branes and $SU(2)$ λ-model limits

- Analytic Continuation + PL limit Sketch: write b.c.'s in terms of P, Q and use of canonical transformation to relate to p, q of η-model. Rewrite in terms auto-morphism gluing of right invariant forms:

$$R^i_+ = \mathbb{R}^i{}_k R^k_- \quad \mathbb{R} = \mathbb{O}_+^{-1}\mathbb{O}_- \quad \mathbb{O}_\pm = \frac{1}{1 \pm \eta \mathcal{R}}$$

D3 brane with world volume flux in η-PCM

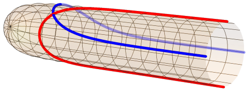
Asymmetric λ-model and D-branes

Powerful technique exposes subtle branes in non-rational CFT

$$ds^2 = k\frac{1+\lambda^2}{1-\lambda^2}\frac{d\xi d\bar{\xi}}{1+|\xi|^2} + \frac{\lambda}{1-\lambda^2}\frac{d\xi^2 + d\bar{\xi}^2}{1+|\xi|^2}$$

Find integrable D-branes:

- **D1** hairpins $\partial_\tau(\xi - \bar{\xi}) = 0$ and $\partial_\sigma(\xi - \bar{\xi}) = 0$



- **D0** living at the tip $\partial_\tau \xi = \partial_\tau \bar{\xi} = 0$, $\xi = \bar{\xi} = 0$
- **D2** with world volume gauge field

Exploit for superstring Miramontes,Hollowood,Schmidtt?

Conclusions

Conclusions and Open Questions

- Rich interplay between integrable models and generalised notions of duality
- Starting to flesh out a surprisingly wide landscape of integrable theories
- Elegant interplay of open-systems and integrability can be used to probe aspects of Duality

Open Questions

- Can we expand the integrable landscape using a duality web e.g. FZZ, Matrix model, level-rank, bosonisation
- Exhaust the full landscape of integrable NLSM? What is the overarching structure
- Continue progress in quantisation
- Integrable models to verify all orders validity of PL in α', g_s?
- More crunch about how to exploit this holography
- M-theoretic implications?

Special thanks to Riccardo, Yolanda, Falk and Luis for a terrific meeting

Appendix

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- Generalised λ models, symmetries, S-matrix and quantisation

`Appadu, Hollowood, Price, DT [1706.05322,1802.06016]`

Generalised λ & YB-λ Theories

- Sfetsos Procedure can be generalised by replacing PCM:

$$kS_{WZW}[g] + S[\tilde{g}] = \int Tr\left(\tilde{g}^{-1}\partial_+ g \Theta \tilde{g}^{-1}\partial_- g\right)$$

- λ now a matrix Λ:

$$S_\lambda = kS_{WZW} + \frac{k}{2\pi}\int Tr(g^{-1}\partial_+ g \frac{1}{\Lambda^{-1} + Ad_g}\partial_- g g^{-1})$$

$$\Lambda = 1 + k^{-1}\Theta$$

- **Idea**: if Θ defined integrable PCM, Λ can define an integrable theory

Generalised λ & YB-λ Theories for $SU(2)$

λ-XXZ Model

$$\Theta = \text{diag}(\xi^{-1}, \xi^{-1}, \lambda^{-1})$$

Trigonometric Lax

$$\mathcal{L}_\sigma = f_+[z]^a \mathcal{J}_+^a T^a - f_-[z]^a \mathcal{J}_-^a T^a$$

RG invariant

$$\gamma'^2 = \frac{k^2}{4} \frac{(1-\xi^2)(1-\lambda)^2}{\lambda^2 - \xi^2}$$

λ-YB Model

$$\Theta = I + \frac{1}{kt}(1 - \eta\mathcal{R})^{-1}$$

Rational Lax

$$\mathcal{L}_\sigma = (c_+ + d\mathcal{R})\mathcal{J}_+ + (c_- + d\mathcal{R})\mathcal{J}_-$$

RG invariant

$$\Sigma = \frac{2\pi\eta\lambda}{k(1-\lambda)}$$

"Non ultra-local" i.e. central term in current algebra

$$\{\mathcal{J}_\pm^a(x), \mathcal{J}_\pm^b(y)\} = f_{ab}{}^c \mathcal{J}_\pm^c(x)\delta_{xy} \pm \frac{k}{2\pi}\delta^{ab}\delta'_{xy}$$

Classical Symmetries

- Expand monodromy to find symmetries but need to determine expansion points!

$$T(z) = P\exp\left(-\int \mathcal{L}_\sigma(z)\right)$$

- Determine Maillet r/s algebra

$$\{\mathcal{L}_\sigma^{\underline{1}}, \mathcal{L}_\sigma^{\underline{2}}\} = [r(z_1, z_2), \mathcal{L}_\sigma^{\underline{1}} + \mathcal{L}_\sigma^{\underline{2}}]\delta_{12} + [s(z_1, z_2), \mathcal{L}_\sigma^{\underline{1}} - \mathcal{L}_\sigma^{\underline{2}}]\delta_{12} - 2s(z_1, z_2)\delta'_{12}$$

- Locate special points $z_\star$ where $\lim_{\epsilon\to 0} r(z_\star, z_\star + \epsilon) = \textit{finite}$

Charges and Symmetries

- Special points associated to Quantum Group Symmetries
- e.g. For $\lambda - YB$ model at $c(z_\star) = i\, d(z_\star)$ we find

$$Q^3 \sim \int \mathcal{J}_0^3 \,, \quad Q^\pm \sim \int (\mathcal{J}_0^1 \pm i\mathcal{J}_0^2) \exp\left[-i\Sigma \int_{-\infty}^{\pm x} \mathcal{J}_0^3(\pm y)dy\right]$$

$$q = \exp\left(\frac{2\pi\eta\lambda}{k(1-\lambda)}\right) = e^\Sigma \quad \textbf{Homogenous Gradation}$$

- For $\lambda - XXZ$ model similar with $q = \exp[\pi\sqrt{\gamma'^2}]$ **Principal Gradation**
- QG parameters are RG invariant
- Second quantum group point given by KM currents with

$$q'_{cl} = \exp\left(\frac{i\pi}{k}\right)$$

Exact S-Matrix

Based on symmetries, limits and RG behaviour, we find conjectured form for S-matrices using known blocks

- λ-XXZ Model in UV Safe Domain $\gamma'^2 < 0$ **Bernard LeClair**

$$\mathcal{S}_{\lambda-XXZ} = S_{SG}(\theta, \gamma') \otimes S^{(k)}_{RSOS}(\theta)$$

- λ-XXZ Model Other Domain (periodic in rapidity)

$$\mathcal{S}_{\lambda-XXZ} = S_{p}(\theta, \Sigma) \otimes S^{(k)}_{RSOS}(\theta)$$

- λ-YB Model (periodic in rapidity, parity broken)

$$\mathcal{S}_{\lambda-XXZ} = S_{h}(\theta, \Sigma) \otimes S^{(k)}_{RSOS}(\theta)$$

'Proving' S-matrix I

- Non-ultra-local *i.e.*δ' makes conventional techniques (QISM) inapplicable
- Alleviation Faddeev-Reshetikhin takes a limit, modifies UV but same IR properties

$$k \to 0 \,, \quad \frac{k}{\xi} \,, \frac{k}{\lambda} \text{ fixed}$$

- In this limit the Lax connection becomes ultra-local ($s(z,w) \to 0$) and can be regularised, and quantised, on a lattice
- Obtain a lattice theory, XXZ anisotropic spin chain.

$$H_{\frac{1}{2}} = \sum_{n=1}^{N} \left(\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \cos\gamma \sigma_n^3 \sigma_{n+1}^3\right)$$

- Actually need a spin $S = \frac{k}{2}$ chain and identify

$$\gamma = \frac{\pi}{\gamma'} - k$$

'Proving' S-matrix II

- Ground state using TBA Kirillov-Reshetikhin find Dirac Sea dominated by k-Bethe strings whose density $\rho(z)$ obeys integral equation

$$\rho(z) + \rho_h(z) + \frac{1}{\pi}\int K(z-y)\rho(y)dy = \epsilon(z)$$

- Holes with density ρ_h are excitations above the ground state
- **Amazing fact**, these excitations scatter relativistically with a kernel

$$\tilde{K}(z) = \frac{d}{dz} LogS(z) = \int_0^\infty \cos(z\omega)\left(\coth(k\omega) + \coth(\gamma'\omega)\right)\tanh \pi\omega$$

- This corresponds exactly to the S-matrix of the λ-XXZ Model

Appendix: S-matrix Technology

Rapidity

$$E = m \cosh\theta \ , \quad P = m \sinh\theta$$

Axioms:

1. *Factorization* 2-body factorisation, no particle production
2. *Analyticity*. Only poles along the imaginary axis $0 < Im\theta < \pi$ associated to stable bound states.
3. *Hermitian analyticity*

$$S_{ij}^{kl}(\theta^*)^* = S_{kl}^{ij}(-\theta) \ .$$

4. *Unitarity*

$$\sum_{kl} S_{ij}^{kl}(\theta) S_{mn}^{kl}(\theta)^* = \delta_{im}\delta_{jn} \ , \qquad \theta \in \mathbb{R} \ .$$

5. *Crossing*

$$S_{ij}^{kl}(\theta) = \mathcal{C}_{kk'} S_{k'i}^{lj'}(i\pi - \theta)\mathcal{C}_{j'j}^{-1} = S_{\bar{k}i}^{l\bar{j}}(i\pi - \theta) \ ,$$

where $\mathcal{C}$ is the charge conjugation matrix.

Appendix: Gradation I

$$[H_i, E_j] = a_{ij}E_j, \quad [H_i, F_j] = -a_{ij}F_j \ , \quad [E_i, F_j] = \delta_{ij}H_j$$

Generalised Cartan matrix a_{ij} has off diagonal elements equal -2.
$K = H_0 + H_1$ is central. $K = 0$, i.e. centreless representations $\widehat{\mathfrak{su}(2)}$ becomes the loop algebra. Reps are the tensor of an $\mathfrak{su}(2)$ rep and functions of a variable z. Gradation is the relative action in $\mathfrak{su}(2)$ space and z-space.

homogenous gradation

$$E_1 = T^+, \quad F_1 = T^-, \quad E_0 = z^2 T^-, \quad F_0 = z^{-2}T^+, \quad H_1 = -H_0 = T^3$$

. *principal* gradation

$$E_1 = zT^+, \quad F_1 = z^{-1}T^-, \quad E_0 = zT^-, \quad F_0 = z^{-1}T^+, \quad H_1 = -H_0 = T^3$$

Appendix: Homogenous Gradation

	$\vdots$	$\vdots$	$\vdots$	$\vdots$
↑	$+2$	$\mathfrak{Q}_2^+$	$\mathfrak{Q}_2^3$	$\mathfrak{Q}_2^-$
$z_* = +i\eta$	$+1$	$\tilde{\mathfrak{Q}}^+$	$\mathfrak{Q}_1^3$	$\mathfrak{Q}_1^-$
	0	$\mathfrak{Q}^+$	$\mathfrak{Q}^3 = -\tilde{\mathfrak{Q}}^3$	$\mathfrak{Q}^-$
$z_* = -i\eta$	-1	$\mathfrak{Q}_{-1}^+$	$\mathfrak{Q}_{-1}^3$	$\tilde{\mathfrak{Q}}^-$
↓	-2	$\mathfrak{Q}_{-2}^+$	$\mathfrak{Q}_{-2}^3$	$\mathfrak{Q}_{-2}^-$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure: The charges and their grades for the expansion of the monodromy around the pair of special points $z = \pm i\eta$. The blue/red and positive/negative graded charges are associated to $\pm i\eta$, respectively. The red and blue charges generate the affine quantum group in homogenous gradation and all the other charges are obtained by repeated Poisson brackets of these charges.

Appendix: Principal Gradation

$\widehat{\mathfrak{su}(2)}_p$.

	$\vdots$	$\vdots$		$\vdots$
$\uparrow$	$+2$		$\mathfrak{Q}^3_2$	
$z_* = \infty$	$+1$	$\mathfrak{Q}^+$		$\tilde{\mathfrak{Q}}^-$
	0		$\mathfrak{Q}^3 = -\tilde{\mathfrak{Q}}^3$	
$z_* = -\infty$	-1	$\tilde{\mathfrak{Q}}^+$		$\mathfrak{Q}^-$
$\downarrow$	-2		$\mathfrak{Q}^3_{-2}$	
	$\vdots$	$\vdots$		$\vdots$

Figure: The charges and their grades for the expansion of the monodromy around the pair of special points $z = \pm\infty$ (or $0, \infty$ with a multiplicative spectral parameter). The blue/red and positive/negative graded charges are associated to $\pm\infty$, respectively. The red and blue charges generate the affine quantum group in principal gradation and all the other charges are obtained by repeated Poisson brackets of these charges.

RG in YB-λ model

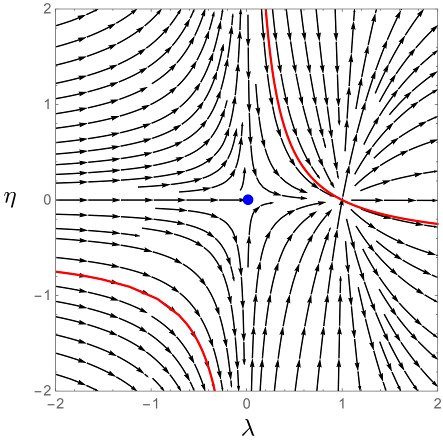

Figure: The RG flow of the YB lambda model (flows towards the IR). The WZW fixed point is the blue dot in the middle. The red curved is an example of a cyclic trajectory which has a jump from $\eta = +\infty$ to $-\infty$ at $\lambda = 0$ and a jump from $\lambda = -\infty$ to $\lambda = +\infty$.

RG in η-λ model

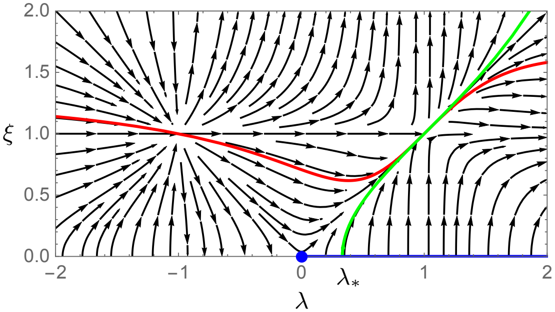

Figure: The RG flow (to the IR) of the XXZ lambda model. The WZW fixed point is identified by the blue blob. The blue line is a line of UV fixed points. The green curve is a UV safe trajectory that has $\gamma' \in \mathbb{R}$. The red curve is a cyclic RG trajectory with $\gamma' = i\sigma$, $\sigma \in \mathbb{R}$. The trajectory has a jump in the coupling λ from $-\infty$ to ∞, but is continuous in $1/\lambda$.

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Demulder, Dorigoni, DT [1604.07851]