Average model
fast simulation of gamma factory PoP@SPS

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Boundary conditions

Heavily inspired from Alexey spreadsheets especially for
  • proper accounting of saturation effect
  • purely longitudinal dynamics

Excitation rate of each ion in the 6D phase space is split into 2 contributions
  • Luminosity-like contribution related to the spatio-temporal average of laser intensity as seen by the considered ion
  • Cross-section like contribution related to the spectral average of laser intensity as seen by the considered ion – accounting for its linewidth

Product of these must account for saturation of the excitation probability

Further assume:
  • No phase-space correlations...
  • Simplifying density distributions
  • Lithium-like Lead
  • \( g_1=g_2 \)
Some (semi-)analytical expressions are available with Gaussian ion beam spectrum and flat laser-beam spectrum. The center ion-beam energy per Mic² is given by:

\[ \bar{\sigma} \approx \sigma_{\text{max}} \frac{\Gamma \pi}{4} \frac{\lambda_0^2}{hc \Delta \lambda \gamma_i (1 + \beta_i \cos \theta)} \left( \text{erf} \left( \frac{E_i (\gamma_+ - \gamma_i)}{\gamma_i \Delta E \sqrt{2}} \right) - \text{erf} \left( \frac{E_i (\gamma_- - \gamma_i)}{\gamma_i \Delta E \sqrt{2}} \right) \right) \]

where \( \gamma_{\pm} = \frac{E_t + \sqrt{E_t^2 + 2(h \nu_{\perp})^2 \cos \theta (1 + \cos \theta)}}{2h \nu_{\perp} (1 + \cos \theta)} \)

This is the average « cross section » over the ion beam spectrum if the laser beam spectrum were continuous...
Single ion interaction model

But laser-beam spectrum is not continuous! \[\rightarrow\] Comb spectrum

\[
f(\nu) = \sum_{n=N_0-N}^{N_0+N} \tilde{f}(\nu) \frac{1}{2\pi} \frac{\Delta \nu}{(\nu - nf_{\text{rep}} - f_{\text{ceo}})^2 + \Delta \nu^2 / 4}
\]

\[
\tilde{f}(\nu) = \frac{1}{2N+1} \sim \frac{\lambda_0^2}{\Delta \lambda} \frac{f_{\text{rep}}}{c}.
\]

Constant enveloppe distribution \[\rightarrow\] Any other distribution easy to implement

Linewidth of each comb’s peak is about 1kHz (determined by laser and optical cavity)

\[
2\gamma \hbar \Delta \nu \simeq 8.3 \cdot 10^{-10} \text{ eV}
\]

Contradictory with previous model

\[
\Gamma = \frac{\hbar}{\tau} \simeq 8.7 \cdot 10^{-6} \text{ eV}
\]

\[
2\gamma \hbar f_{\text{rep}} \simeq 3.3 \cdot 10^{-5} \text{ eV}
\]

0.3 peak within atomic resonance FWHM
(truncated) Laser comb

Center peak at $N_0F_{\text{rep}} + F_{\text{ceo}}$

Frep : separation between peaks
The interaction probability for a single ion reads 

\[
\sigma_i = \int_{E=0}^{+\infty} \int_{h\nu=0}^{+\infty} \sigma(E)\delta(E - h\nu\gamma(1 + \beta \cos \theta)) f(\nu) dE dh\nu.
\]

which simplifies, since \(\gamma(1 + \beta \cos \theta) h \Delta \nu \ll \Gamma\),

\[
\sigma_i = \sum_{n=N_0-N}^{N_0+N} \frac{\sigma_{\text{max.}}}{2N + 1} \frac{1}{1 + 4/\Gamma^2 (\gamma(1 + \beta \cos \theta) h (n f_{\text{rep.}} + f_{\text{ceo}}) - E_T)^2}.
\]

This can be written in a slightly different form

\[
\sigma_i(\gamma M_i c^2) = \sum_{n=N_0-N}^{N_0+N} \frac{\sigma_{\text{max.}}}{2N + 1} \frac{1}{1 + 4(\gamma M_i c^2 - \tilde{E}_T[n])^2 / \tilde{\Gamma}[n]^2}
\]

where

\[
\tilde{\Gamma}[n] = \Gamma \frac{M_i c^2}{(1 + \beta \cos \theta) h (n f_{\text{rep.}} + f_{\text{ceo}})} \sim \frac{\Gamma M_i c^2}{(1 + \beta \cos \theta) h (N_0 f_{\text{rep.}} + f_{\text{ceo}})} \sim \frac{\Gamma M_i c^2 \lambda_0}{2hc}
\]

and

\[
\tilde{E}_T[n] = \frac{E_T M_i c^2}{(1 + \beta \cos \theta) h (n f_{\text{rep.}} + f_{\text{ceo}})} \approx \frac{E_T M_i c^2}{(1 + \beta \cos \theta) h (N_0 f_{\text{rep.}} + f_{\text{ceo}})} (1 + (n-N_0) \frac{f_{\text{rep.}}}{N_0 f_{\text{rep.}} + f_{\text{ceo}}})
\]
\( \tau = 76 \text{ ps} \)

This anyway shall not be an 'issue' thanks to synchrotron oscillations.

\( \rightarrow \) Average spectral overlap recovered when averaging over several turns.
Spectral overlap contribution versus energy

\[ \tau = 760 \text{ ps} \]

Effect is totally washed out if lifetime is an order of magnitude larger.
Discussion about flat spectrum

I assume flat spectrum, this is motivated by what we usually get out of the laser amplifier typically implemented in the system we (LAL) are used of

This could ‘theoretically’ be compressed temporally to a sine cardinal-like shape that can be approximated by a gaussian

There will certainly be some residual chirp in a realistic situation, but its a good simplifying starting point
Spatial overlap contribution

\[ \tilde{\mathcal{L}} = 2n_L c \cos^2(\theta/2) \int \int \int \int n_{\text{laser}}(x_L, y_L, z_L - ct) \delta(x_i - \delta x, y_i - \delta y, z_i - \beta ct - \delta z) dx dy dz dt, \]

\[
\begin{align*}
x_L &= -x \cos \theta \cos \alpha - z \sin \theta \cos \alpha + y \sin \alpha \\
y_L &= x \cos \theta \sin \alpha + z \sin \theta \sin \alpha + y \cos \alpha \\
z_L &= x \sin \theta - z \cos \theta \\
x_i &= x \\
y_i &= y \\
z_i &= z
\end{align*}
\]

\[ \theta = 2\Phi: \text{crossing angle} \]

Account for arbitrary crossing plane and laser beam sizes.

\[ \tilde{\mathcal{L}}_\alpha = \frac{n_L}{2\pi \sigma_x \sigma_y \sqrt{1 + \tan^2 \phi \sigma_x^2} (\sigma_x^2 \sin^2 \alpha + \sigma_y^2 \cos^2 \alpha)} \]

\[ \times \exp \left( - \frac{(\delta y + \delta z \tan \phi \sin \alpha)^2 \sigma_x^2 + (\delta x + \delta z \tan \phi \cos \alpha)^2 \sigma_y^2 + (\delta y \cos \alpha - \delta x \sin \alpha)^2 \sigma_z^2 \tan^2 \phi}{2\sigma_x^2 \sigma_y^2 (1 + \frac{\tan^2 \phi \sigma_x^2}{\sigma_x^2 \sigma_y^2} (\sigma_x^2 \sin^2 \alpha + \sigma_y^2 \cos^2 \alpha))} \right) \]
Saturation effect

In order to properly account for saturation of excitation probability per ion:

- Remove the ion spectrum average from the « spectral overlap » contribution

For a given ion with boost $\gamma$, the maximum excitation probability reads

$$p_{\text{max}} = 0.5 \left( \text{erf} \left( \frac{E_i (\gamma_+ - \gamma_i)}{\gamma_i \Delta E \sqrt{2}} \right) - \text{erf} \left( \frac{E_i (\gamma_- - \gamma_i)}{\gamma_i \Delta E \sqrt{2}} \right) \right)$$

And the spatial overlap contribution is now independent on the choice of a particular ion

$$\bar{\sigma} \approx \sigma_{\text{max}} \frac{\Gamma \pi}{2} \frac{\lambda_0^2}{\hbar c \Delta \lambda \gamma_i (1 + \beta_i \cos \theta)}$$

While the spatio-temporal term does (see previous slide)

The single-ion interaction probability thus reads

$$p_{\text{max}} \left( 1 - \exp(-2 \mathcal{L} \sigma) \right) / 2$$

This probability is sampled over the ion bunch population to estimate via Monte-Carlo an ion-beam averaged excitation probability.
Generate 200 ions by propagating 1 ion at $E[0]=E_{\text{spread}}$, $\phi[0]=0$

Thus generate the 1-sigma ellipse of the ion beam. All these ions with $E[n,0]$ are equiprobable.

Each ion with $E[n,m]$ ($n=$ ion number, $m=$ turn number) radiates

If and only if $E[n,m]>E_{\text{min}}$ and $E[n,m]<E_{\text{max}}$ where $E_{\text{min}}$ and $E_{\text{max}}$ are driven by the laser bandwidth

\[ \bar{E} = \bar{N}_X / n_1 E_X / 2 \]

Results were shown yesterday, I will not reproduce them here!
Outlook

This average modeling have been cross-checked by Alexey, seems consistent with Monte-Carlo approaches

Possible improvements:
• the effect of laser chirping (needs much effort, forget about it for now)
• Gaussian laser spectrum (can be more easily done)
Backup from yesterday’s: 4-mirror cavity

6° crossing angle

2.6° crossing angle