

Elements of statistics and data analysis

Purpose: become familiar with the treatment of correlated uncertainties, which are often a key ingredient of (neutrino) data analyses.

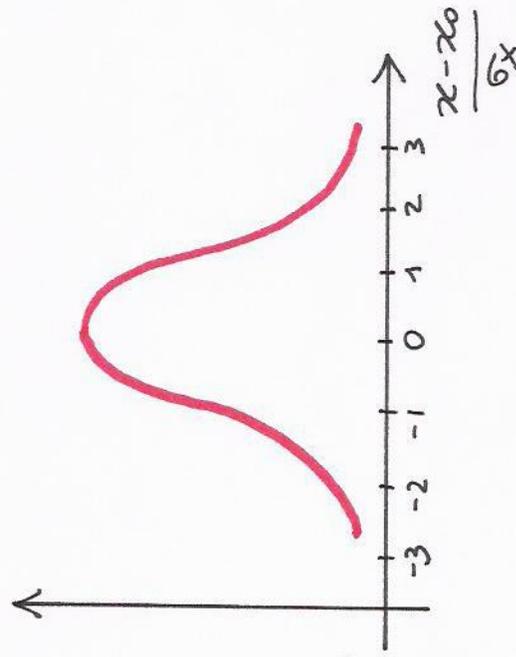
Error distribution for 1 variable

Here we consider only gaussian errors. Also Poisson fluctuations ($\propto \sqrt{N}$) are assumed to be nearly gaussian. [Some comments on small-N cases and on asymmetric errors will be presented at the end.]

Distribution for a single variable:

$$P(x, x_0) = \frac{1}{\sqrt{2\pi} \cdot \sigma_x} e^{-\frac{1}{2} \left(\frac{x-x_0}{\sigma_x} \right)^2},$$

corresponding to quote $x = x_0 \pm \sigma_x$



Area within: $\pm 1\sigma = 68.27\%$

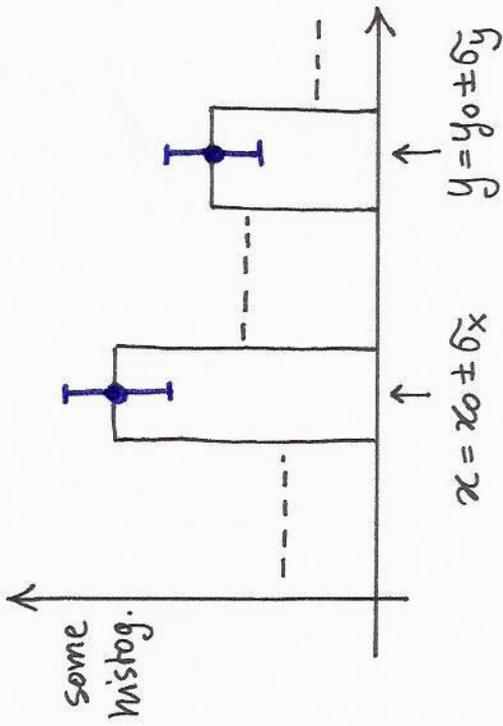
$\pm 2\sigma = 95.45\%$

$\pm 3\sigma = 99.73\%$

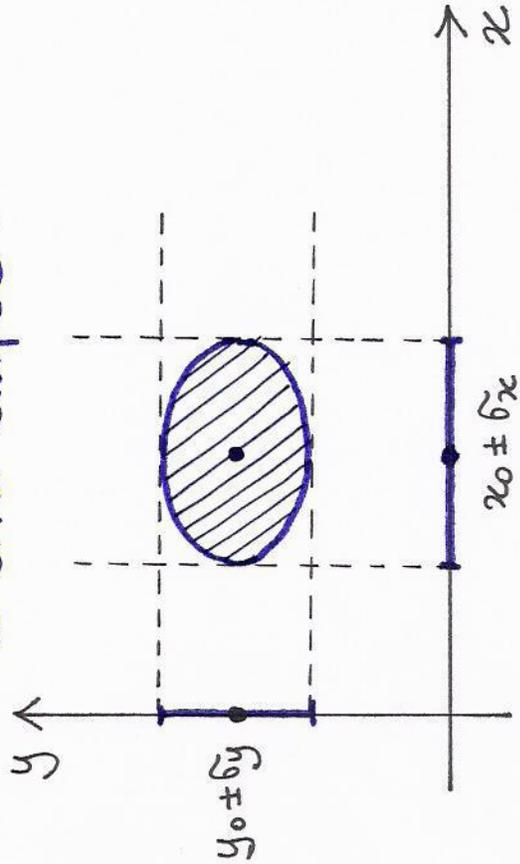
Error distribution for 2 variables (uncorrelated)

Let us consider 2 quantities x & y with errors which have no relation with each other (e.g. statistical errors of two bins):

$$P(x, y; x_0, y_0) = P(x, x_0) P(y, y_0)$$



1σ error ellipse :



Ellipse equation:

$$\Delta\chi^2 = 1$$

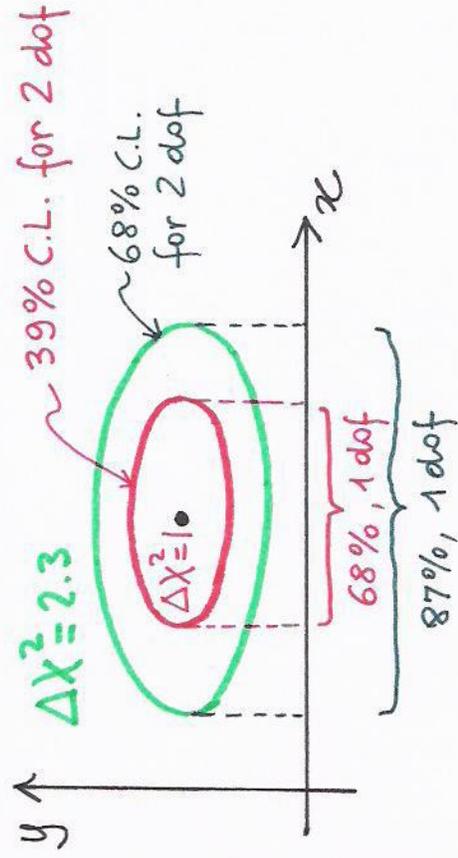
where
$$\Delta\chi^2 = \left(\frac{x-x_0}{\delta x}\right)^2 + \left(\frac{y-y_0}{\delta y}\right)^2$$

$$[\Delta\chi^2 = 0 \text{ at } (x, y) = (x_0, y_0)]$$

Note: The probability of finding (x,y) within the 16 error ellipse is not 68.27% : it is 39.35% !

- 68% = probability of finding x in $x_0 \pm 6x$, independently on y
- 68% = " " " y in $y_0 \pm 6y$, " " x
- 39% = joint probability of finding (x,y) within the 16 ellipse

If you really want an error ellipse containing 68% joint probab. (68% C.L. for 2 d.o.f.), then you should use $\Delta\chi^2 = 2.3$. Its projections define 87% C.L. for each variable (1 dof). This is not usually called a "16" ellipse.



Confusion may arise if a C.L. is quoted without the corresponding # d.o.f.

Error distribution for 2 variables (fully correlated)

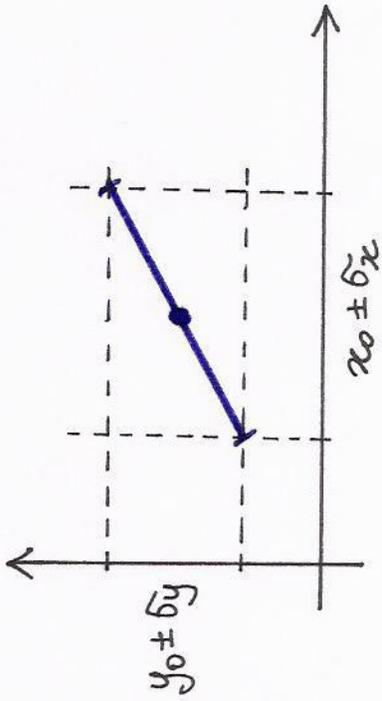
Consider two variables x and y with errors in one-to-one correspondence, e.g., two bins affected by a common normalization error:

Then the errors go both "up" or "down":

if $x = x_0 + \delta x$ then $y = y_0 + \delta y$

if $x = x_0 - \delta x$ then $y = y_0 - \delta y$

The error ellipse is degenerate \rightarrow
(fully correlated errors)

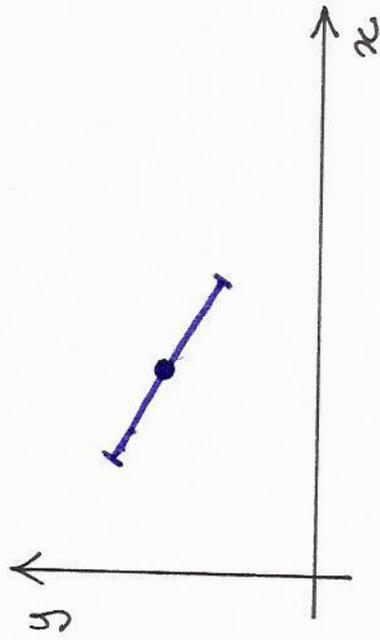


Analogously for full "anticorrelation":

E.g., two bins whose sum is constant;

then, $x = x_0 + \delta x$ implies $y = y_0 - \delta y$, \rightarrow

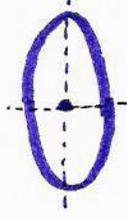
and the degenerate ellipse has a negative slope



Recap: Eqs. for 1σ error ellipses (limit cases)

- No correlation

$$1 = (x-x_0, y-y_0) \underbrace{\begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}^{-1}}_{\det \neq 0} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$



- Full correlation

$$1 = (x-x_0, y-y_0) \underbrace{\begin{pmatrix} \sigma_x^2 & \sigma_x \sigma_y \\ \sigma_x \sigma_y & \sigma_y^2 \end{pmatrix}^{-1}}_{\det = 0} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$



- Full anticorr.

$$1 = (x-x_0, y-y_0) \underbrace{\begin{pmatrix} \sigma_x^2 & -\sigma_x \sigma_y \\ -\sigma_x \sigma_y & \sigma_y^2 \end{pmatrix}^{-1}}_{\det = 0} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$



In general we expect: $1 = (x-x_0, y-y_0) \underbrace{\begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}^{-1}}_{\text{error matrix}} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$

More general 16 error ellipses

Let us consider two variables (x, y) and two sources of uncertainties:

- statistical (s_x, s_y) with no correlation,
- systematic (c_x, c_y) with full correlation,

$$\text{namely, } \begin{cases} x = x_0 \pm s_x(\text{stat}) \pm c_x(\text{syst}) \\ y = y_0 \pm s_y(\text{stat}) \pm c_y(\text{syst}) \end{cases}$$

The errors sum up in quadrature at matrix level:

$$\sigma^2 = \begin{bmatrix} s_x^2 & 0 \\ 0 & s_y^2 \end{bmatrix} + \begin{bmatrix} c_x^2 & c_x c_y \\ c_x c_y & c_y^2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_x \sigma_y & \sigma_y^2 \end{bmatrix} \quad \leftarrow \text{Squared error matrix}$$

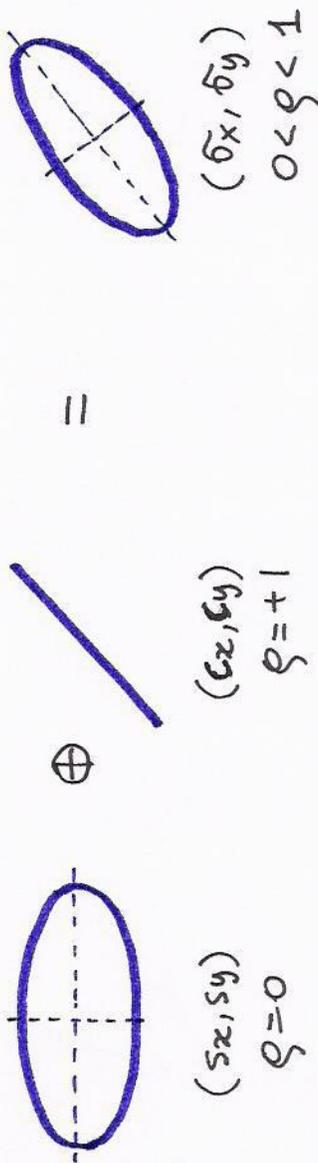
$$\text{where } \sigma_x^2 = s_x^2 + c_x^2$$

$$\sigma_y^2 = s_y^2 + c_y^2$$

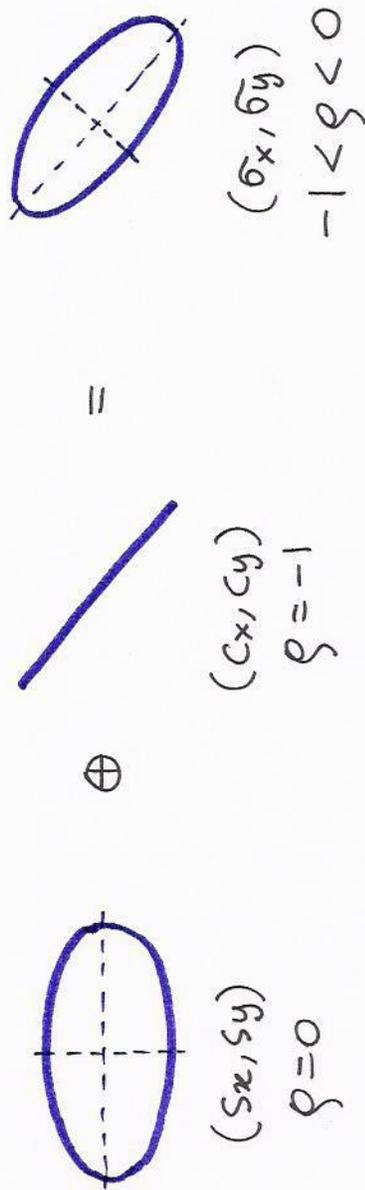
$$\rho = \frac{c_x c_y}{\sigma_x \sigma_y} = \begin{cases} 0 & \text{for } c_x \text{ or } c_y = 0 \quad (\text{no correlation}) \\ 1 & \text{for } s_x = s_y = 0 \quad (\text{full correlation}) \end{cases}$$

In general, the correlation ρ obeys: $0 \leq |\rho| \leq 1$

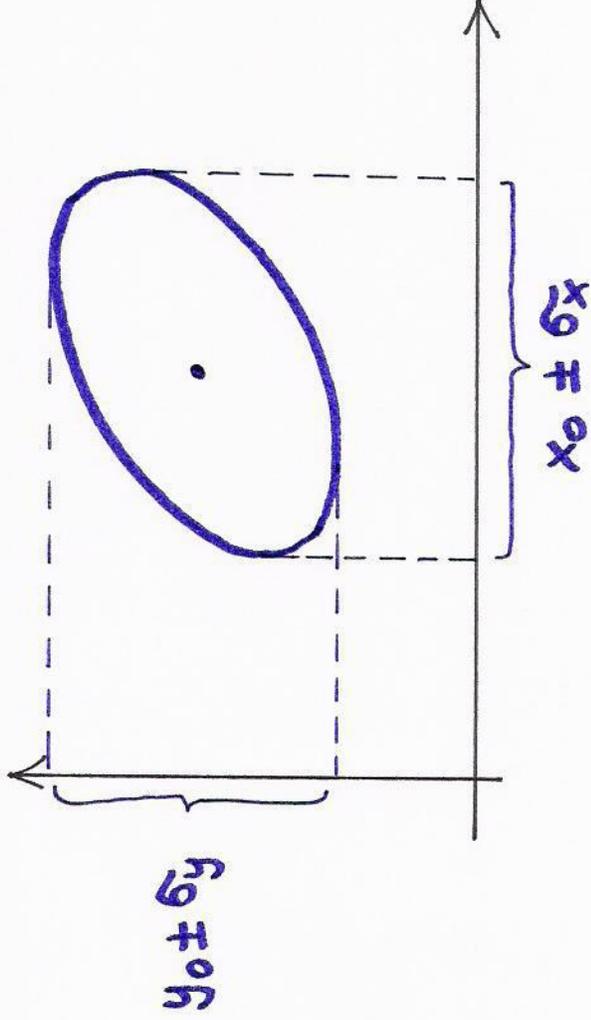
Qualitative shape of 1 σ error ellipse :



Analogously, adding a fully anticorrelated error source :



Projections of 1σ error ellipse coincide with $\pm 1\sigma$ ranges for the x and y variables:



Equation of 1σ ellipse:

$$\begin{aligned}
 1 &= \Delta\chi^2 \\
 &= (x-x_0, y-y_0) \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}^{-1} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \\
 &= \frac{1}{1-\rho^2} \left[\left(\frac{x-x_0}{\sigma_x} \right)^2 + \left(\frac{y-y_0}{\sigma_y} \right)^2 - 2\rho \frac{(x-x_0)(y-y_0)}{\sigma_x\sigma_y} \right]
 \end{aligned}$$

Probability distribution:
$$P = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2}\Delta\chi^2}$$

Positively correlated variables and their ratio

Let us consider two variables with positively correlated errors :

$$\begin{cases} x = x_0 \pm \delta x \\ y = y_0 \pm \delta y \end{cases} \quad \rho > 0$$

If the correlation is sizable, we expect a significant "cancellation" of errors in the ratio :

$$r = \frac{x}{y} = r_0 \pm \delta r \quad \text{with "small } \delta r \text{"}$$

The error of the ratio can be evaluated as :

$$\delta r^2 = \left(\frac{\partial r}{\partial x}\right)^2 \delta x^2 + \left(\frac{\partial r}{\partial y}\right)^2 \delta y^2 + 2\rho \left(\frac{\partial r}{\partial x}\right)\left(\frac{\partial r}{\partial y}\right) \delta x \delta y$$

→ $\frac{\delta r^2}{r_0^2} = \frac{\delta x^2}{x_0^2} + \frac{\delta y^2}{y_0^2} - 2\rho \frac{\delta x}{x_0} \frac{\delta y}{y_0}$; the "-2ρ..." term is responsible for error cancellation.

Note that, for $\rho = +1$ (full correlation) : $\frac{\delta r^2}{r_0^2} = \left(\frac{\delta x}{x_0} - \frac{\delta y}{y_0}\right)^2$; then, if $\frac{\delta x}{x_0} = \frac{\delta y}{y_0}$ (e.g., for a common normalization uncertainty) the cancellation is complete ($\delta r = 0$) as expected.

In general, it is preferable to use correlated variables (x, y) whenever possible, rather than their ratio $r = x/y$.

The main reason is that, if y is distributed as a Gaussian, then $1/y$ is distributed as a Lorentzian ("Breit-Wigner"), with a formally infinite variance. In practice, this may be problematic if σ_y is large and/or one is probing the distribution tails.

Therefore, if we measure $x = x_0 \pm \delta x$ and $y = y_0 \pm \delta y$, and if we know that $r = r_0 \pm \delta r$, it is convenient to keep (x, y) in the fit, together with the correlation

$$S = \left(\frac{\sigma_x^2}{x_0^2} + \frac{\sigma_y^2}{y_0^2} - \frac{\sigma_r^2}{r_0^2} \right) / \left(2 \frac{\delta x}{x_0} \frac{\delta y}{y_0} \right)$$

(instead of using r directly).

Historically, several "ratios" have been progressively abandoned in neutrino data fits.

Estimates of correlations and covariances

The squared error matrix $\sigma^2 = \begin{bmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{bmatrix}$

is also called "covariance matrix":

$$\sigma_{xy}^2 = \rho \sigma_x \sigma_y = \text{covariance of } (x, y)$$

$$\sigma_x^2 = \text{variance of } x$$

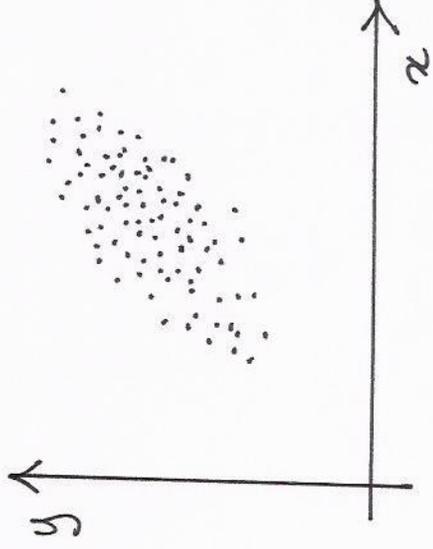
$$\sigma_y^2 = \text{variance of } y$$

If repeated measurements (or simulations) of the variables x and y are available, \rightarrow then:

$$x_0 = \frac{1}{n} \sum_{i=1}^n x_i, \quad \sigma_x^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - x_0)^2$$

$$y_0 = \frac{1}{n} \sum_{i=1}^n y_i, \quad \sigma_y^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - y_0)^2$$

$$\sigma_{xy}^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - x_0)(y_i - y_0); \quad \rho = \sigma_{xy}^2 / \sigma_x \sigma_y$$



$$\rightarrow \text{Get: } \begin{cases} x = x_0 \pm \sigma_x \\ y = y_0 \pm \sigma_y \end{cases} \rho$$

However, the ideal situation is to identify and break down all possible error sources to the two main categories of "uncorrelated" errors S and "fully correlated" errors C :

$$\sigma^2 = \underbrace{\begin{pmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} s_x^2 & 0 \\ 0 & s_y^2 \end{pmatrix} + \dots}_{\text{uncorrelated}} + \underbrace{\begin{pmatrix} c_x^2 & c_{xy} \\ c_{xy} & c_y^2 \end{pmatrix} + \dots}_{\text{fully correlated}}$$

[fully anticorrelated errors become fully correlated by changing the sign of one variable].

In this case, one is really "summing in quadrature" all possible, independent (known) error sources.

Generalization to N variables $\{x_i\}_{1 \leq i \leq N}$

- $\Delta\chi^2 = 1$ error ellipsoid in N -dimensional space is defined by:

$$1 = \Delta x^T (\sigma^2)^{-1} \Delta x$$

where Δx is a column vector, $\Delta x = \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ \vdots \\ x_N - x_N^0 \end{pmatrix}$

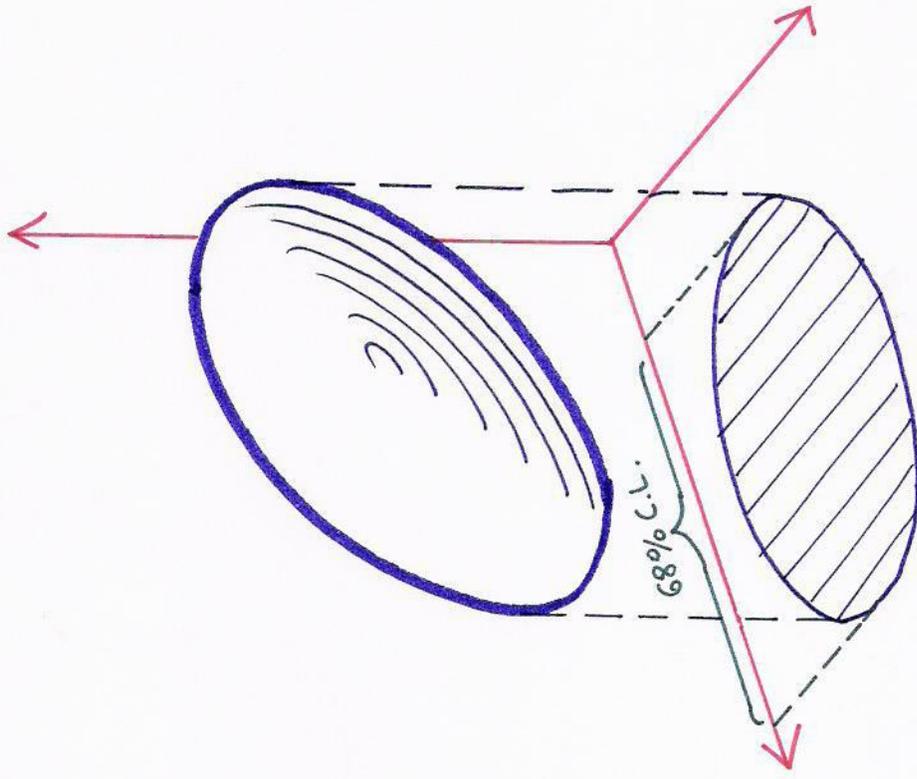
and $(\sigma^2)^{-1}$ is the inverse of the covariance matrix (symmetrical):

$$\sigma^2 = \begin{pmatrix} \sigma_1^2 & \xi_{12} \sigma_1 \sigma_2 & \xi_{13} \sigma_1 \sigma_3 & \dots \\ \sigma_2^2 & \xi_{23} \sigma_2 \sigma_3 & \dots & \dots \\ \sigma_3^2 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Probability distribution (multivariate Gaussian):

$$P = \frac{1}{(2\pi)^{N/2} \sqrt{\det \sigma^2}} e^{-\frac{1}{2} \Delta\chi^2}$$

- Projection of the $\Delta\chi^2=1$ ellipsoid onto one axis x_i gives the 1σ (68% C.L.) range on the x_i variable ($x_i = x_i^0 \pm \sigma_i$). This holds for any N . Variables $x_j \neq x_i$ are said to be "marginalized" or "projected away"



- However, the joint probability of (x_1, x_2, \dots, x_N) being inside the ellipsoid decreases with N :

$N=1$	68%
$N=2$	$39\% < (68\%)^2$
$N=3$	$20\% < (68\%)^3$
$N=4$	$9\% < (68\%)^4$
\vdots	\vdots

- Note: $n-\sigma$ ellipsoids defined by $\Delta\chi^2 = n^2$

Fitting data with a model

In this case we compare theoretical predictions x_i^{theo} with N experimental data x_i^{exp} . Predictions will depend on $N_p (< N)$ parameters \vec{p} : $x_i^{\text{theo}} = x_i^{\text{theo}}(\vec{p})$. Recipe:

- Build $\chi^2 = \Delta x^T (\sigma^2)^{-1} \Delta x$, $\Delta x = \text{column vector of } (x_i^{\text{theo}} - x_i^{\text{exp}})$
 $\sigma^2 = \sigma_{\text{theo}}^2 + \sigma_{\text{exp}}^2$

- Find $\chi_{\min}^2 = \min_{\vec{p}} \chi^2(\vec{p})$ at some $\vec{p} = \vec{p}_0$

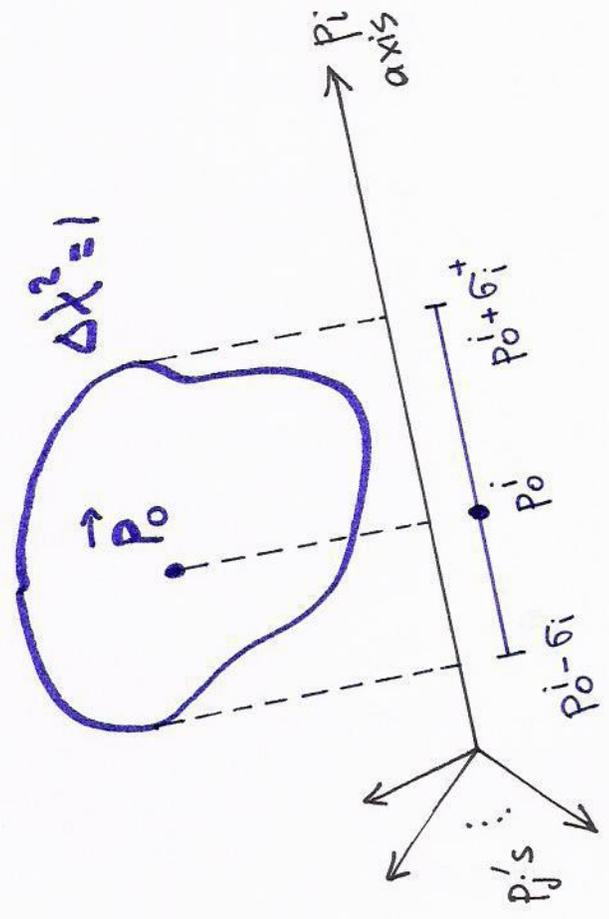
- Check that $\chi_{\min}^2 \sim N - N_p \pm \sqrt{2(N - N_p)}$ ← see, e.g., PDG
dof for test of hypothesis

- Check not ok: model wrong (χ_{\min}^2 too high) or "too good" (χ_{\min}^2 too low). Verify model, underestimated errors....

- Check OK. →

Parameter estimation

The N_p -dimensional manifold defined by:
 $\Delta\chi^2 \equiv \chi^2(\vec{p}) - \chi^2_{min} = 1$
 represents the "1 σ allowed region" of parameters. Projection onto one axis p_i provides $\pm 1\sigma$ ranges:
 $p = p_0^i \pm \sigma_i^{\pm}$ (generally asymmetric)

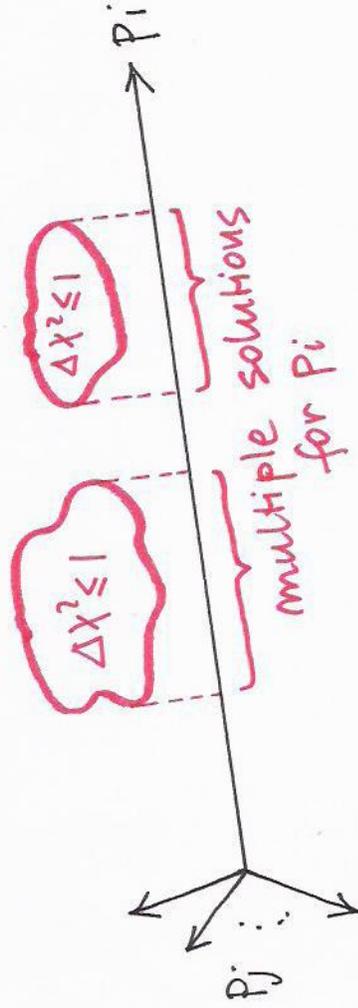
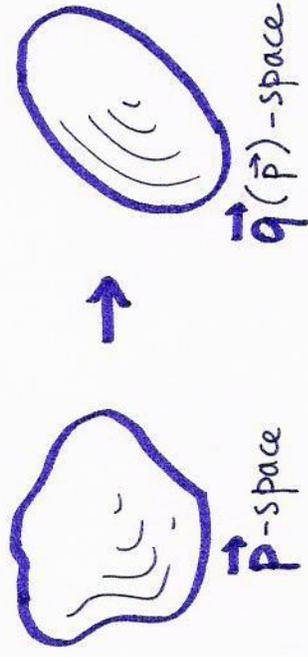


Projection onto p_i (\equiv marginalization over $p_j \neq p_i$) in practice means setting $\Delta\chi_i^2 = 1$, where the "reduced" $\Delta\chi_i^2$ is:

$$\Delta\chi_i^2 = \min_{p_j \neq p_i} (\chi^2(\vec{p}) - \chi^2_{min})$$

Analogously, $n\sigma$ ranges are defined by $\Delta\chi_i^2 = n^2$

- This procedure can be justified, as far as the allowed region is simply connected. The basic idea is that, through mapping, this volume can be transformed into an ellipsoid (\rightarrow gaussian machinery...)



- However, if there are several (disconnected) allowed regions, the statistical interpretation is problematic. No "consensus" approach in this case.

- In practice, most people keep using the $\Delta X^2 = \text{const}$ recipe also for multiple solutions, with some cautionary remarks.
- There is no other way than waiting for new experiments/data to solve the ambiguity ("degeneracy of solutions").

- Sometimes, one is interested not only in $\pm n\sigma$ limits on each variable separately ($\equiv \Delta\chi^2 = n^2$ projections), but on the joint probability of \vec{p} being in a volume defined by $\Delta\chi^2 = \text{const.}$

Relevant tables of $\Delta\chi^2$ level cuts (from PDG):

C.L.	N=1	N=2	N=3
68	1.00	2.30	3.53
90	2.71	4.61	6.25
95	3.84	5.99	7.82
99	6.63	9.21	11.34
99.73	9.00	11.83	14.16

E.g. the joint 95% C.L. region for two variables (p_i, p_j) is defined by $\Delta\chi^2_{ij} = 5.99$, where $\Delta\chi^2_{ij} = \min_{p_k \neq p_{ij}} [\chi^2(\vec{p}) - \chi^2_{\text{min}}]$.

Analyzing χ^2 contributions.

- The χ^2 is a global quantity. By itself, it does not necessarily "detect" single observables which might be badly fitted
 → need to split the total χ^2 into "pieces"

- One possibility is to look at the "residuals" or "pulls" after the fit:

$$(\text{pull})_i = \frac{x_i^{\text{theo}}(\vec{p}_0) - x_i^{\text{exp}}}{\sigma_i} \quad \left(\sigma_i = \sqrt{\sigma_{ii}^2} \right)$$

- Then, a large pull (say, $\gtrsim 3\sigma$) signals a potential problem in the data and/or in the model.

The Standard Model fit to LEP data is often shown in terms of such pulls.

- With the previous definitions, however,

$$\chi^2 \neq \Sigma(\text{pull})^2$$

since σ_{ij}^2 is not diagonal in general.

- It is possible to re-write the same χ^2 in a form $\Sigma(\text{pull})^2$ with a somewhat different approach, which also brings some technical advantages.

Previous χ^2 approach: "covariance method"

Alternative " " : "pull method"

Covariance method (recap)

- Consider N observables $\{R_n\}_{n=1\dots N}$

$\{R_n^{\text{theo}}\}$ = theoretical predictions

$\{R_n^{\text{exp}}\}$ = experimental measurements

$$(R_n^{\text{theo}} - R_n^{\text{exp}}) \pm \underbrace{u_n \pm c_n^1 \pm \dots \pm c_n^K}_{\substack{\text{Set of } K \text{ systematics} \\ \text{produced by independ. sources}}}$$

↑
uncorrel.
error

with $\rho(u_n, u_m) = 0$ (always uncorrelated)
 $\rho(c_n^k, c_m^k) = 1$ (fully correlated for the same k -th source)
 $\rho(c_n^k, c_m^h) = 0$ ($h \neq k$, uncorrelated from different sources)

- Then: Build $\sigma_{nm}^2 = \delta_{nm} u_n u_m + \sum_{k=1}^K c_n^k c_m^k$
 and evaluate $\chi_{\text{cov}}^2 = \sum_{n,m=1}^N (R_n^{\text{exp}} - R_n^{\text{theo}}) [\sigma_{nm}^2]^{-1} (R_m^{\text{exp}} - R_m^{\text{theo}})$

as discussed previously

Pull method

- Shift the theoretical predictions linearly in the systematics:

$$R_n^{\text{theor}} \rightarrow R_n^{\text{theo}} + \sum_{k=1}^K \xi_k C_n^k$$

where $\xi_k =$ univariate gaussian random variable ($\langle \xi_k \rangle = 0, \langle \xi_k^2 \rangle = 1$)

- Minimize over ξ_k the following sum of squared residuals:

$$\chi_{\text{pull}}^2 = \min_{\{\xi_k\}} \left[\sum_{n=1}^N \left(\frac{R_n^{\text{exp}} - (R_n^{\text{theo}} + \sum_{k=1}^K \xi_k C_n^k)}{u_n} \right)^2 + \sum_{k=1}^K \xi_k^2 \right]$$

Squared residuals \uparrow penalty term

- At minimum ($\xi_k \stackrel{\text{def}}{=} \bar{\xi}_k$): $\bar{R}_n^{\text{theo}} = R_n^{\text{theo}} + \sum_{k=1}^K \bar{\xi}_k C_n^k \leftarrow$ "shifted" predictions

$$\text{and } \chi_{\text{pull}}^2 = \sum_{n=1}^N \left(\frac{R_n^{\text{exp}} - \bar{R}_n^{\text{theo}}}{u_n} \right)^2 + \sum_{k=1}^K \bar{\xi}_k^2$$

$$= \sum_{n=1}^N \left(\text{pull of observable} \right)_n^2 + \sum_{k=1}^K \left(\text{pull of systematic} \right)_k^2 \leftarrow \text{"diagonal" form}$$

• It turns out that: $\chi^2_{\text{pull}} \equiv \chi^2_{\text{covariance}}$ (some algebra needed)

so the methods are numerically equivalent.

- The pull approach may be more convenient for large N . The inversion of a large $N \times N$ covariance matrix may be unstable, especially if systematics dominate. The pull method leads to k equations (linear) in the ξ_k 's, which is solved by a $k \times k$ matrix inversion, with $k \ll N$ usually.
- In addition, relatively large ξ_k 's may signal systematic "offsets" required to match data and theory.
- Several ν data analyses are now performed in terms of χ^2_{pull} .

See hep-ph/0206162 for details

Comment on low-statistic bins

- The fit to a histogram may become problematic if one (or more) bin contains a low number of events N_{exp} (or even none, $N_{exp}=0$).
- In this case, the Gaussian approximation

$$\chi^2 \ni \frac{(N_{exp} - N_{theo})^2}{N_{exp}} \quad \underline{\text{fails}}$$

- In this case, the PDG suggests an alternative form, which embeds more properly the Poisson nature of the fluctuations:

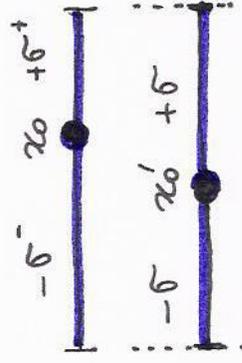
$$\chi^2 \ni 2 \left(N_{theo} - N_{exp} + N_{exp} \ln \frac{N_{exp}}{N_{theo}} \right)$$

[or $2 N_{theo}$ if $N_{exp}=0$]

- Additional systematics can still be embedded in the pull approach via: $N_{theo} \rightarrow N_{theo} + \sum_k \xi_k C_k^k$
- Gaussian limit recovered at large N .
Hint: expand logarithm at second order in $(N_{theo} - N_{exp})/N_{exp}$.

Comment on asymmetric errors

- If a variable x is affected by asymmetric errors: $x = x_0 \pm \sigma^+$
there is no "consensus recipe" to write a χ^2 contribution.
- Sometimes the range is conservatively symmetrized to the largest error: $x = x_0 \pm \sigma_{\max}$; $\sigma_{\max} = \max(\sigma^+, \sigma^-)$
- A better recipe has been argued in physics/0403086:
shift $x_0 \rightarrow x'_0$ so that the new $\pm 1\sigma$ range reproduces the old one.



$$x'_0 + \sigma \equiv x_0 + \sigma^+$$

$$x'_0 - \sigma \equiv x_0 - \sigma^-$$

Then the χ^2 contribution is: $\left(\frac{x - x'_0}{\sigma}\right)^2$.

Advanced exercises

on 3ν oscillation probabilities
in vacuum and matter

- Calculation of $P(\nu_\alpha \rightarrow \nu_\alpha)$ in vacuum at 1st order in the small parameter $\delta m^2 x / 4E$
[Discussion on sensitivity to hierarchy.]
- Calculation of $P(\nu_\mu \rightarrow \nu_e)$ in constant matter at 2nd order in the small parameters δm^2 and $\sin^2 \theta_{13}$
[Important for next-generation experiments. Results also reported in PDG ~ review.]
- Calculation of $P(\nu_\alpha \rightarrow \nu_\beta)$ in constant matter for $\delta m^2 \ll 0$.
[One-mass-scale dominance in matter.]

Calculation of $P_{\alpha\alpha}$ in vacuum

at 1st order in $\delta m^2 x / 4E$.

- let us consider normal hierarchy for definiteness:

$$\left\{ \begin{array}{l} m_2^2 - m_1^2 = \delta m^2 \\ m_3^2 - m_2^2 = \Delta m^2 - \delta m^2/2 \\ m_3^2 - m_1^2 = \Delta m^2 + \delta m^2/2 \end{array} \right.$$

Reminder: $\sin^2(y + \delta y) \cong \sin^2 y + \sin 2y \cdot \delta y + \mathcal{O}(\delta y^2)$

Then: $\text{Im}(J_{\alpha\beta}^i) = 0$

$$\text{Re}(J_{\alpha\beta}^i) = |U_{\alpha i}|^2 |U_{\beta j}|^2$$

$$\begin{aligned} P_{\alpha\alpha} &= 1 - 4 |U_{\alpha 1}|^2 |U_{\alpha 2}|^2 \sin^2 \left(\frac{\delta m^2 x}{4E} \right) \\ &\quad - 4 |U_{\alpha 2}|^2 |U_{\alpha 3}|^2 \sin^2 \left(\frac{\Delta m^2 - \delta m^2}{4E} x \right) \\ &\quad - 4 |U_{\alpha 1}|^2 |U_{\alpha 3}|^2 \sin^2 \left(\frac{\Delta m^2 + \delta m^2}{4E} x \right) \end{aligned}$$

↑
exact

At 1st order in δm^2 , the 1st term vanishes, while the other two become:

$$P_{\alpha\alpha} \simeq 1 - 4|U_{\alpha 2}|^2|U_{\alpha 3}|^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 4|U_{\alpha 1}|^2|U_{\alpha 3}|^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 4\left[|U_{\alpha 2}|^2|U_{\alpha 3}|^2\left(-\frac{\delta m^2 x}{4E}\right) + |U_{\alpha 1}|^2|U_{\alpha 3}|^2\left(+\frac{\delta m^2 x}{4E}\right)\right] \cdot \sin\left(\frac{2\Delta m^2 x}{4E}\right)$$

$$= 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$- 4|U_{\alpha 3}|^2(|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2) \left(\frac{\delta m^2 x}{4E}\right) \sin\left(\frac{2\Delta m^2 x}{4E}\right)$$

$$= 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$- 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2) \frac{|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2}{1 - |U_{\alpha 3}|^2} \left(\frac{\delta m^2 x}{4E}\right) \sin\left(\frac{2\Delta m^2 x}{4E}\right)$$

$$= 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2) \cdot \left[\sin^2\left(\frac{\Delta m^2 x}{4E}\right) + \frac{|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2}{|U_{\alpha 1}|^2 + |U_{\alpha 2}|^2} \left(\frac{\delta m^2 x}{4E}\right) \sin\left(\frac{2\Delta m^2 x}{4E}\right) \right]$$

$$\simeq 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2) \sin^2\left(\frac{\Delta m^2 + \frac{|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2}{|U_{\alpha 1}|^2 + |U_{\alpha 2}|^2} \cdot \frac{\delta m^2}{2} x}{4E}\right),$$

Similar to 0th order formula, but with $\Delta m^2 \rightarrow \Delta m^2 + \frac{|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2}{|U_{\alpha 1}|^2 + |U_{\alpha 2}|^2} \frac{\delta m^2}{2}$.

A sensitivity to hierarchy arises, since the above equation is not invariant for $\Delta m^2 \rightarrow -\Delta m^2$: the relative signs of δm^2 and $\pm \Delta m^2$ matter.

The sensitivity to the hierarchy depends on the amplitude factor

$$|U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2)$$

and on the ratio

$$\frac{\delta m^2}{\Delta m^2} \frac{|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2}{|U_{\alpha 1}|^2 + |U_{\alpha 2}|^2}$$

For $\alpha=e$: weak sensitivity because amplitude is small ($|U_{e3}|^2 < \text{few \%}$)

For $\alpha=\mu$: weak sensitivity because, although $|U_{\mu 3}|^2 \sim 1/2$, the ratio is small:

$$\frac{\delta m^2}{\Delta m^2} \sim \frac{1}{30}; \quad \frac{|U_{\mu 1}|^2 - |U_{\mu 2}|^2}{|U_{\mu 1}|^2 + |U_{\mu 2}|^2} \sim \frac{\frac{1}{6} - \frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} \sim -\frac{1}{3}$$

$$\rightarrow \text{ratio is } \sim \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{30} \sim \frac{1}{180} \ll 1.$$

General reduction tools (in standard parametrization) in matter

- 3Y Hamiltonian in flavor basis for generic $N_e(x)$ profile :

$$\tilde{H} = U \frac{\mathcal{M}^2}{2E} U^\dagger + V$$

$$\mathcal{M}^2 = \text{diag}(m_1^2, m_2^2, m_3^2)$$

$$U = O_{23} \Gamma_\delta O_{13} \Gamma_\delta^\dagger O_{12}$$

$$\Gamma_\delta = \text{diag}(1, 1, e^{i\delta})$$

$$V = \text{diag}(\sqrt{2} G_F N_e, 0, 0)$$

$$N_e = N_e(x)$$

- It is easy to verify that:
$$\begin{cases} (O_{23} \Gamma_\delta)^\dagger V (O_{23} \Gamma_\delta) = V \\ \Gamma_\delta^\dagger O_{12} \mathcal{M}^2 O_{12}^\dagger \Gamma_\delta = O_{12} \mathcal{M}^2 O_{12}^\dagger \end{cases}$$

- Let's go from the flavor basis to a new "primed flavor" basis defined as :

$$\begin{bmatrix} (\nu e)^\prime \\ (\nu \mu)^\prime \\ (\nu \tau)^\prime \end{bmatrix} = (O_{23} \Gamma_\delta)^\dagger \begin{bmatrix} \nu e \\ \nu \mu \\ \nu \tau \end{bmatrix} \leftarrow \text{components}, \quad O_{23} \Gamma_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta} \\ 0 & -s_{23} & c_{23} e^{i\delta} \end{pmatrix}$$

- In the primed basis, the Hamiltonian is:

$$\tilde{H}' = (O_{23} \Gamma_\delta)^\dagger H (O_{23} \Gamma_\delta) = O_{13} O_{12} \frac{\mathcal{M}^2}{2E} (O_{13} O_{12})^\dagger + V$$

• The hamiltonian \tilde{H}' is simpler than \tilde{H} : it does not depend on δ (and is thus real symmetric) nor on θ_{23} .

• It is thus simpler to find the evolution operator \tilde{S}' in the primed basis, and then the evolution operator \tilde{S} in the flavor basis as:

$$\tilde{S}(x_f, x_i) = (O_{23} \Gamma_\delta) \tilde{S}'(x_f, x_i) (O_{23} \Gamma_\delta^\dagger)$$

• In terms of matrix components:

if $\tilde{S}' = \begin{pmatrix} \tilde{S}'_{ee} & \tilde{S}'_{e\mu} & \tilde{S}'_{e\tau} \\ \tilde{S}'_{\mu e} & \tilde{S}'_{\mu\mu} & \tilde{S}'_{\mu\tau} \\ \tilde{S}'_{\tau e} & \tilde{S}'_{\tau\mu} & \tilde{S}'_{\tau\tau} \end{pmatrix}$, then $\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta} \\ 0 & -s_{23} & c_{23} e^{i\delta} \end{pmatrix} \cdot \tilde{S}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} e^{i\delta} & c_{23} e^{-i\delta} \end{pmatrix}$

$$\tilde{S}'_{ee} = \tilde{S}_{ee}$$

$$\tilde{S}'_{\mu e} = \tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}$$

$$\tilde{S}'_{\tau e} = -\tilde{S}'_{\mu e} s_{23} + \tilde{S}'_{\tau e} c_{23} e^{i\delta}$$

$$\tilde{S}'_{\mu\mu} = \tilde{S}'_{\mu\mu} c_{23}^2 + \tilde{S}'_{\mu\tau} c_{23} s_{23} e^{-i\delta} + \tilde{S}'_{\tau\mu} c_{23} s_{23} e^{i\delta} + \tilde{S}'_{\tau\tau} s_{23}^2$$

$$\tilde{S}'_{\tau\mu} = -\tilde{S}'_{\mu\mu} c_{23} s_{23} - \tilde{S}'_{\mu\tau} s_{23}^2 e^{-i\delta} + \tilde{S}'_{\tau\mu} c_{23}^2 e^{i\delta} + \tilde{S}'_{\tau\tau} c_{23} s_{23}$$

$$\tilde{S}'_{\tau\tau} = \tilde{S}'_{\mu\mu} s_{23}^2 - \tilde{S}'_{\mu\tau} c_{23} s_{23} e^{-i\delta} - \tilde{S}'_{\tau\mu} c_{23} s_{23} e^{i\delta} + \tilde{S}'_{\tau\tau} c_{23}^2$$

with $\tilde{S}'_{e\mu}, \tilde{S}'_{e\tau}, \tilde{S}'_{\mu\tau}$ obtained from $\tilde{S}'_{\alpha\beta} \leftrightarrow \tilde{S}'_{\beta\alpha}$ and $+\delta \leftrightarrow -\delta$.

Note that $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha}$ for symmetrical matter density profiles \rightarrow

$$\text{(for } \bar{\nu}: \delta \rightarrow -\delta, V \rightarrow -V)$$

In general, $\tilde{S}'_{\alpha\beta} \neq \tilde{S}'_{\beta\alpha}$, even if $\tilde{H}'_{\alpha\beta} = \tilde{H}'_{\beta\alpha}$ (real symmetric).
 Indeed, let us divide a generic profile $N_e(x)$ into N steps $\{\Delta x_i\}_{i=1 \dots N}$,
 where $N_e \sim \text{const}$ in each step. Then:

$$\tilde{S}' = e^{-i\tilde{H}'_N \Delta x_N} e^{-i\tilde{H}'_{N-1} \Delta x_{N-1}} \dots e^{-i\tilde{H}'_2 \Delta x_2} e^{-i\tilde{H}'_1 \Delta x_1}.$$

Although $(\tilde{H}'_i)^T = \tilde{H}'_i$, the transpose of \tilde{S}' is not equal to \tilde{S}' , since
 the ordering of the steps is reversed from $1, \dots, N$ to $N, \dots, 1$

("reverse" density profile):

$$(\tilde{S}')^T = e^{-i\tilde{H}'_1 \Delta x_1} \dots e^{-i\tilde{H}'_N \Delta x_N}.$$

Therefore, although: $\tilde{S}'_{\alpha\beta}$ [direct profile] = $\tilde{S}'_{\beta\alpha}$ [reverse profile]
 in general it is: $\tilde{S}'_{\alpha\beta}$ [direct profile] \neq $\tilde{S}'_{\beta\alpha}$ [reverse profile]

unless the direct and reverse profiles are symmetrical (i.e., coincide
 upon reflection), so that: $\tilde{S}'_{\alpha\beta}$ [symmetric] = $\tilde{S}'_{\beta\alpha}$ [symmetric].

In particular, for constant density, $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha}$.

In this case: $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha} (\delta \rightarrow -\delta)$

\rightarrow osc. probabilities: $P_{\alpha\beta} = P_{\beta\alpha} (\delta \rightarrow -\delta)$ ($P_{\alpha\beta} = |\tilde{S}'_{\beta\alpha}|^2 = P(\gamma_\alpha \rightarrow \gamma_\beta)$)

Further reductions come from some peculiar symmetries of $\tilde{S}_{\alpha\beta}$ under the following substitutions, $S_{23} \rightarrow \pm C_{23}$ and $C_{23} \rightarrow \mp S_{23}$:

$$\tilde{S}_{\tau e} = \pm \tilde{S}_{\mu e} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right. \Rightarrow P_{\tau e} = P_{\mu e} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right. \stackrel{\text{det}}{=} P'_{\mu e}$$

$$\tilde{S}_{\mu\tau} = \mp \tilde{S}_{\tau\mu} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right. \Rightarrow P_{\tau\mu} = P_{\mu\tau} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right. \stackrel{\text{det}}{=} P'_{\mu\tau}$$

$$\tilde{S}_{\mu\mu} = \pm \tilde{S}_{\tau\tau} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right. \Rightarrow P_{\mu\mu} = P_{\tau\tau} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right. \stackrel{\text{det}}{=} P'_{\tau\tau}$$

The above relations, together with the unitarity of $P_{\alpha\beta}$, allow to express all the probabilities in terms of just two, e.g., $P_{\mu e}$ and $P_{\mu\tau}$, and their transformed

$$P'_{\mu e} = P_{\mu e} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right. \quad \text{and} \quad P'_{\mu\tau} = P_{\mu\tau} \left| \begin{array}{l} S_{23} \rightarrow \pm C_{23} \\ C_{23} \rightarrow \mp S_{23} \end{array} \right.$$

(It is equivalent to choose the upper or lower substitution).

Explicitly :

$$\left. \begin{aligned}
 P_{ee} &= 1 - P_{e\mu} - P_{e\tau} = 1 - P_{e\mu} - P'_{e\mu} \\
 P_{e\tau} &= P'_{e\mu} \\
 P_{\mu e} &= 1 - P_{\mu\mu} - P_{\mu\tau} = 1 - P_{\mu\mu} - P_{e\mu} + P_{e\mu} - P_{\mu\tau} = P_{e\mu} + P_{\tau\mu} - P_{\mu\tau} = P_{e\mu} + P'_{\mu\tau} - P_{\mu\tau} \\
 P_{\mu\mu} &= 1 - P_{e\mu} - P_{\tau\mu} = 1 - P_{e\mu} - P'_{\mu\tau} \\
 P_{\tau\mu} &= P'_{\mu\tau} \\
 P_{\tau\tau} &= 1 - P_{e\tau} - P_{\mu\tau} = 1 - P'_{e\mu} - P_{\mu\tau}
 \end{aligned} \right\}$$

Also : $P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) = P(\nu_\alpha \rightarrow \nu_\beta) \mid V \rightarrow -V, \delta \rightarrow -\delta$

and : $P_{\alpha\beta} = P_{\beta\alpha} (\delta \rightarrow -\delta)$ in constant matter ($V = \text{const}$)

These relations allow the reduction of $P_{\alpha\beta}^{(V)}$ calculations to a few independent probabilities.

Calculation of $P(\nu_e \rightarrow \nu_\mu)$ in matter

at 2nd order in the small parameters δm^2 and $\sin \theta_{13}$.

- The so-called "golden channel" $\nu_e \rightarrow \nu_\mu$ is particularly important in the context of future long-baseline accelerator experiments.
- We want to show that, for constant density N_e , and at 2nd order in the small parameters δm^2 and $\sin \theta_{13}$, $P(\nu_e \rightarrow \nu_\mu)$ takes the form:

$$P(\nu_e \rightarrow \nu_\mu) \simeq X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos \left(\delta - \frac{\Delta m^2 x}{4E} \right) + Z$$

$$\text{where } \begin{cases} X = \sin^2 \theta_{23} \left(\frac{\Delta m^2}{A - \Delta m^2} \right)^2 \sin^2 \left(\frac{A - \Delta m^2}{4E} x \right) \\ Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\delta m^2}{A} \right) \left(\frac{\Delta m^2}{A - \Delta m^2} \right) \sin \left(\frac{AX}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right) \\ Z = \cos^2 \theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A} \right)^2 \sin^2 \left(\frac{AX}{4E} \right) \end{cases} \quad (A = 2\sqrt{2} G_F N_e E)$$

- Note: sometimes one finds a further $\cos \theta_{13}$ factor in Y . This is, however, irrelevant at the stated 2nd order approximation.

- Note also that, once the "golden channel" probability $P(\nu_e \rightarrow \nu_\mu)$ is obtained, the so-called "silver channel" probability $P(\nu_e \rightarrow \nu_\tau)$ is immediately obtained via:

$$P(\nu_e \rightarrow \nu_\tau) = P(\nu_e \rightarrow \nu_\mu) \begin{cases} C_{23} \rightarrow \mp S_{23} \\ S_{23} \rightarrow \pm C_{23} \end{cases}$$

For antineutrinos:

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_{\mu,\tau}) = P(\nu_e \rightarrow \nu_{\mu,\tau} \mid A \rightarrow -A; \delta \rightarrow -\delta)$$

For inverted hierarchy:

$$\text{Just swap } \Delta m^2 \rightarrow -\Delta m^2$$

For swapped flavors:

$$P(\nu_{\mu,\tau} \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_{\mu,\tau} \mid \delta \rightarrow -\delta)$$

• We start the calculation by reminding that:

$$P(\nu_e \rightarrow \nu_\mu) = P_{e\mu} = |\tilde{S}'_{\mu e}|^2 \quad \text{with} \quad \tilde{S}'_{\mu e} = \tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{1e} s_{23} e^{i\delta}$$

$$\rightarrow P_{e\mu} = |\tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{1e} s_{23} e^{i\delta}|^2 = A_{e\mu} \cos^2 \delta + B_{e\mu} \sin \delta + C_{e\mu}$$

$$\text{with} \quad \begin{cases} A_{e\mu} = 2 \operatorname{Re} [\tilde{S}'_{\mu e} \tilde{S}'_{1e}] c_{23} s_{23} \\ B_{e\mu} = -2 \operatorname{Im} [\tilde{S}'_{\mu e} \tilde{S}'_{1e}] c_{23} s_{23} \\ C_{e\mu} = |\tilde{S}'_{\mu e}|^2 c_{23}^2 + |\tilde{S}'_{1e}|^2 s_{23}^2 \end{cases}$$

• The next "trick" is to reduce the evolution from 3ν to approximately $(2\nu) \oplus (1\nu)$, by exploiting the expansion in two phenomenologically "small" parameters: s_{13} and δm^2 (at the accelerator scale).

• A term T will be called "of 1st order" if proportional to δm^2 or s_{13} :

$$T \sim O_1 \quad \text{if} \quad T \propto s_{13} \quad \text{or} \quad T \propto \delta m^2$$

Analogously:

$$T \sim O_2 \quad \text{if} \quad T \propto s_{13}^2, (\delta m^2)^2, s_{13} \cdot \delta m^2$$

etc...

- We shall show that: $\tilde{S}'_{\mu e} \sim O_1$ and $\tilde{S}'_{\tau e} \sim O_1$.
Therefore, since $P_{\mu e}$ is a quadratic form in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$, it is $P_{\mu e} \sim O_2$ as desired.
- Let's remind that, in the primed basis, it is:

$$\tilde{H}' = O_{13} O_{12} \frac{\mathcal{U}^2}{2E} (O_{13} O_{12})^T + V$$

$$\mathcal{U}^2 = \text{diag} \left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \Delta m^2 \right)$$

$$V = \text{diag} (\sqrt{2} G_F N_e, 0, 0)$$

← in normal hierarchy
and up to a term $\propto \mathbb{1}$
- In the primed basis, the evolution decouples as $\exists \nu = (2\nu) \oplus (1\nu)$ in two limits:

$$S_{13} \rightarrow 0 \Rightarrow O_{13} = \mathbb{1}$$

$$\delta m^2 \rightarrow 0 \Rightarrow O_{12} \mathcal{U}^2 O_{12}^T = \mathcal{U}^2$$
- It is then convenient to define:

$$\tilde{H}^l = \lim_{S_{13} \rightarrow 0} \tilde{H}' \quad \text{and} \quad \tilde{H}^h = \lim_{\delta m^2 \rightarrow 0} \tilde{H}'$$

and to study the evolution operator components $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$ in \tilde{H}^l and \tilde{H}^h . The task is simple since both \tilde{H}^l and \tilde{H}^h have only one nontrivial 2×2 block submatrix, already solved in 2ν cases.

• limit $s_{13} \rightarrow 0$ in primed basis:

$$\begin{aligned} \tilde{H}^l &= \lim_{s_{13} \rightarrow 0} \tilde{H}^l = \frac{1}{2E} \left[O_{12} \begin{pmatrix} -\delta m^2/2 & \\ & +\delta m^2/2 \end{pmatrix} \Delta m^2 \right] O_{12}^T + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{A}{4E} \mathbb{1} + \frac{1}{2E} \left[O_{12} \begin{pmatrix} -\delta m^2/2 & \\ & +\delta m^2/2 \end{pmatrix} \Delta m^2 \right] O_{12}^T + \begin{pmatrix} A/2 & & \\ & -A/2 & \\ & & -A/2 \end{pmatrix} \\ &= \frac{A}{4E} \mathbb{1} + \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{12} \delta m^2 & \sin 2\theta_{12} \delta m^2 & 0 \\ \sin 2\theta_{12} \delta m^2 & \cos 2\theta_{12} \delta m^2 - A & 0 \\ 0 & 0 & 2\Delta m^2 - A \end{bmatrix} \end{aligned}$$

→ In the primed basis, for s_{13} , the (e, μ) flavors evolve separately from the (τ) one

→ $\tilde{S}_{ee}^l = \lim_{s_{13} \rightarrow 0} \tilde{S}'_{ee} = 0$ (no $\nu_e \rightarrow \nu_e'$ transitions)

→ $\tilde{S}'_{ee} = 0$ (s_{13}) = O_1 at least

Instead, $\tilde{S}_{\mu e}^l$ is nonzero. From the 2ν case (already worked out) we get:

$$\tilde{S}_{\mu e}^l = e^{-i\frac{A}{4E}x} \left[-i \sin 2\tilde{\theta}_{12} \sin \left(\frac{\delta m^2 x}{4E} \right) \right]$$

with $\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} / \sqrt{(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12}}$
 $\delta \tilde{m}^2 = \delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$

→ $\tilde{S}_{\mu e}^l = O(\delta m^2) = O_1$

• limit $\delta m^2 \rightarrow 0$ in primed basis:

$$\tilde{H}^h = \lim_{\delta m^2 \rightarrow 0} \tilde{H}' = \frac{1}{2E} \left(O_{13} \begin{bmatrix} 0 & 0 \\ 0 & \Delta m^2 \end{bmatrix} O_{13}^T + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \left(\frac{\Delta m^2}{4E} + \frac{A}{4E} \right) \mathbb{1} + \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{13} \Delta m^2 & 0 & \sin 2\theta_{13} \Delta m^2 \\ \sin 2\theta_{13} \Delta m^2 & -\Delta m^2 - A & 0 \\ 0 & 0 & \cos 2\theta_{13} \Delta m^2 - A \end{bmatrix}$$

→ In the primed basis, for $\delta m^2 \rightarrow 0$, the (e, μ) flavors evolve separately from the (ν) one.

→ $\tilde{S}_{\mu e}^h = \lim_{\delta m^2 \rightarrow 0} \tilde{S}'_{\mu e} = 0$ (no $\nu e' \rightarrow \nu \mu'$ transitions)

→ $\tilde{S}_{\mu e} = O(\delta m^2) = O_1$ at least

Instead, $\tilde{S}_{\tau e}^h$ is non-zero. From the 2 ν case (already worked out) we get:

$$\tilde{S}_{\tau e}^h = e^{-i \frac{A}{4E} x} e^{-i \frac{\Delta m^2}{4E} x} \left[-i \sin 2\tilde{\theta}_{13} \sin \left(\frac{\Delta \tilde{m}^2 x}{4E} \right) \right]$$

with $\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} / \sqrt{(\cos 2\theta_{13} - A/\Delta m^2)^2 + \sin^2 2\theta_{13}}$

$$\Delta \tilde{m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\tilde{\theta}_{13}$$

→ $\tilde{S}_{\tau e}^h = O(S_{13}) = O_1$

- Summarizing, at O_1 we have that:

$$\tilde{S}'_{\mu e} = O(\delta m^2) \simeq \tilde{S}'_{\mu e}{}^{\ell} = e^{-i \frac{A}{4E} x} \left[-i \sin 2\tilde{\theta}_{12} \sin \left(\frac{\delta m^2 x}{4E} \right) \right]$$

$$\tilde{S}'_{\tau e} = O(s_{13}) \simeq \tilde{S}'_{\tau e}{}^h = e^{-i \frac{A}{4E} x} e^{-i \frac{\Delta \tilde{m}^2}{4E} x} \left[-i \sin 2\tilde{\theta}_{13} \sin \left(\frac{\Delta \tilde{m}^2 x}{4E} \right) \right]$$

- One can drop the overall phase $e^{-i \frac{A}{4E} x}$ and get:

$$\tilde{S}'_{\mu e} = \left[-i \sin 2\tilde{\theta}_{12} \sin \left(\frac{\delta m^2 x}{4E} \right) \right] + O_2$$

$$\tilde{S}'_{\tau e} = e^{-i \frac{\Delta \tilde{m}^2}{4E} x} \left[-i \sin 2\tilde{\theta}_{13} \sin \left(\frac{\Delta \tilde{m}^2 x}{4E} \right) \right] + O_2$$

which is all is needed to calculate $P_{\mu\tau}$ as a quadratic form in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$.

- Further tricks involve a proper organization of terms and an expansion in the small parameter $\frac{\delta m^2}{A} = \frac{\delta m^2}{2\sqrt{2} G_F N e E}$

It is $\frac{\delta m^2}{A} \ll 1$ for $E \gtrsim 1 \text{ GeV}$ ("high-energy approximation") and typical crust/mantle densities $N e$.

- More precisely, so far we have:

$$A_{e\mu} = \sin 2\tilde{\theta}_{12} \sin 2\tilde{\theta}_{13} \sin 2\theta_{23} \sin\left(\frac{\tilde{\Delta m}^2 x}{4E}\right) \sin\left(\frac{\tilde{\Delta m}^2 x}{4E}\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} = \sin 2\tilde{\theta}_{12} \sin 2\tilde{\theta}_{13} \sin 2\theta_{23} \cdot \sin\left(\frac{\tilde{\Delta m}^2 x}{4E}\right) \sin\left(\frac{\tilde{\Delta m}^2 x}{4E}\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} = \cos^2 \theta_{23} \sin^2 2\tilde{\theta}_{12} \sin^2\left(\frac{\tilde{\Delta m}^2 x}{4E}\right) + \sin^2 \theta_{23} \sin^2 2\tilde{\theta}_{12} \sin^2\left(\frac{\tilde{\Delta m}^2 x}{4E}\right)$$

with $P_{e\mu} = A_{e\mu} \cos \delta + B_{e\mu} \sin \delta + C_{e\mu}$.

The high-energy expansion in $\tilde{\Delta m}^2/A$ will allow us to express $\tilde{\Delta m}^2$, $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ in terms of their vacuum values Δm^2 , θ_{12} and θ_{13} (together with an expansion in the small parameter s_{13}).

- For $\delta m^2/A \ll 1$:

$$\begin{aligned} \sin 2\tilde{\theta}_{12} &= \frac{\sin 2\theta_{12}}{\sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}} = \frac{\sin 2\theta_{12}}{\sqrt{\cos^2 2\theta_{12} - \frac{2A}{\delta m^2} \cos 2\theta_{12} + \left(\frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}} \\ &= \frac{\sin 2\theta_{12}}{\sqrt{\left(\frac{A}{\delta m^2}\right)^2 \left(1 - 2\frac{\delta m^2}{A} \cos 2\theta_{12} + \dots\right)}} \approx \frac{\sin 2\theta_{12}}{\frac{|A|}{\delta m^2} \left(1 - \frac{\delta m^2}{A} \cos 2\theta_{12}\right)} \approx \sin 2\theta_{12} \frac{\delta m^2}{|A|} + O_2 \end{aligned}$$

$$\frac{\delta m^2}{\tilde{\delta m}^2} = \frac{\sin 2\tilde{\theta}_{12}}{\sin 2\theta_{12}} = \frac{\delta m^2}{|A|} + O_2 \rightarrow \tilde{\delta m}^2 = |A| + O_2$$

$$\sin\left(\frac{\delta m^2 x}{4E}\right) \approx \sin\left(\frac{|A|x}{4E}\right) + O_2$$

- For $s_{13} \ll 1$:

$$\sin 2\tilde{\theta}_{13} = \frac{\sin 2\theta_{13}}{\sqrt{\left(\cos 2\theta_{13} - \frac{A}{\delta m^2}\right)^2 + \delta m^2 2\theta_{13}}} \approx \frac{\sin 2\theta_{13}}{\sqrt{\left(1 - \frac{A}{\delta m^2}\right)^2}} + O_2 = \frac{\sin 2\theta_{13}}{\left|1 - \frac{A}{\delta m^2}\right|} + O_2$$

$$\sin 2\tilde{\theta}_{13} = \left| \frac{\Delta m^2}{\Delta m^2 - A} \right| \sin 2\theta_{13} + O_2$$

$$\tilde{\Delta m}^2 = \Delta m^2 \frac{\sin 2\theta_{13}}{\sin 2\tilde{\theta}_{13}} \approx \Delta m^2 \left| \frac{\Delta m^2 - A}{\Delta m^2} \right|$$

• We have then:

$$A_{\mu} \simeq \sin \theta_{12} \left(\frac{\delta m^2}{|A|} \right) \sin \theta_{13} \left| \frac{\Delta m^2}{\Delta m^2 - A} \right| \sin \theta_{23} \sin \left(\frac{|A|x}{4E} \right) \sin \left(\Delta m^2 \left| \frac{\Delta m^2 - A}{\Delta m^2} \right| \frac{x}{4E} \right) \cos \left(\frac{\Delta m^2 x}{4E} \right)$$

$$B_{\mu} \simeq \sin \theta_{12} \left(\frac{\delta m^2}{|A|} \right) \sin \theta_{13} \left| \frac{\Delta m^2}{\Delta m^2 - A} \right| \sin \theta_{23} \sin \left(\frac{|A|x}{4E} \right) \sin \left(\Delta m^2 \left| \frac{\Delta m^2 - A}{\Delta m^2} \right| \frac{x}{4E} \right) \sin \left(\frac{\Delta m^2 x}{4E} \right)$$

$$C_{\mu} \simeq c_{23}^2 \sin^2 \theta_{12} \left(\frac{\delta m^2}{A} \right)^2 \sin^2 \left(\frac{Ax}{4E} \right) + s_{23}^2 \sin^2 \theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A} \right)^2 \sin^2 \left(\frac{|\Delta m^2 - A| x}{4E} \right)$$

• Absolute values can be eliminated by inspection of all relevant \neq cases.
 E.g., by changing sign of $(\Delta m^2 - A)$: $A_{\mu}, B_{\mu}, C_{\mu}$ do not change.
 By changing sign of Δm^2 : only A_{μ} changes. Etc. ...

Then:

$$A_{\mu} \simeq \sin \theta_{12} \sin \theta_{13} \sin \theta_{23} \left(\frac{\delta m^2}{A} \right) \frac{\Delta m^2}{A - \Delta m^2} \sin \left(\frac{Ax}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right) \cos \left(\frac{\Delta m^2 x}{4E} \right)$$

$$B_{\mu} \simeq \sin \theta_{12} \sin \theta_{13} \sin \theta_{23} \left(\frac{\delta m^2}{A} \right) \frac{\Delta m^2}{A - \Delta m^2} \sin \left(\frac{Ax}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right) \sin \left(\frac{\Delta m^2 x}{4E} \right)$$

$$C_{\mu} \simeq c_{23}^2 \sin^2 \theta_{12} \left(\frac{\delta m^2}{A} \right)^2 \sin^2 \left(\frac{Ax}{4E} \right) + s_{23}^2 \sin^2 \theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A} \right)^2 \sin^2 \left(\frac{\Delta m^2 - A}{4E} x \right)$$

- Terms in $P_{e\mu} = A e_{\mu} \cos \delta + B e_{\mu} \sin \delta + C e_{\mu}$ can be finally organized as:

$$P(\gamma_e \rightarrow \gamma_{\mu}) = X \sin^2 \theta_{13} + Y \sin 2\theta_{13} \cos \left(\delta - \frac{\Delta m^2 x}{4E} \right) + Z$$

$$\text{with } \begin{cases} X = \sin^2 \theta_{23} \left(\frac{\Delta m^2}{A - \Delta m^2} \right)^2 \sin^2 \left(A - \frac{\Delta m^2}{4E} x \right) \\ Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\delta m^2}{A} \right) \left(\frac{\Delta m^2}{A - \Delta m^2} \right) \sin \left(\frac{AX}{4E} \right) \sin \left(\frac{A - \Delta m^2}{4E} x \right) \\ Z = \cos^2 \theta_{23} \sin^2 \theta_{12} \left(\frac{\delta m^2}{A} \right)^2 \sin^2 \left(\frac{AX}{4E} \right) \end{cases}$$

as desired.

- Another way of organizing the terms is:

$$P_{e\mu} = x^2 f^2 + 2xyfg \cos(\Delta - \delta) + y^2 g^2$$

$$\text{with: } \begin{cases} x = \sin \theta_{23} \sin 2\theta_{13} \\ y = \frac{\delta m^2}{\Delta m^2} \cos \theta_{23} \sin 2\theta_{12} \\ f = \sin \left(\frac{\Delta m^2 - A}{4E} x \right) \frac{\Delta m^2}{\Delta m^2 - A} \\ g = \sin \left(\frac{AX}{4E} \right) \cdot \frac{\Delta m^2}{A} \end{cases}$$

Calculation of $P_{\alpha\beta}$ for $\delta m^2 = 0$ (constant matter).

This is the "matter version" of the one-mass-scale limit, previously examined in vacuum. In this case, the Hamiltonian in the primed basis is:

$$\tilde{H}'(\delta m^2 = 0) = \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{13} \Delta m^2 & 0 & \sin 2\theta_{13} \Delta m^2 \\ 0 & -\Delta m^2 - A & 0 \\ \sin 2\theta_{13} \Delta m^2 & 0 & \cos 2\theta_{13} \Delta m^2 - A \end{bmatrix} \quad (+ \text{const. } \mathbb{1})$$

corresponding to a $(2\nu) \oplus (1\nu)$ block form, which can be easily diagonalized using previous results:

$$\tilde{H}' = \tilde{U} \frac{\tilde{\mathcal{H}}^2}{2E} \tilde{U}^T \quad \text{with} \quad \tilde{U} = \begin{pmatrix} \cos \tilde{\theta}_{13} & 0 & \sin \tilde{\theta}_{13} \\ 0 & 1 & 0 \\ -\sin \tilde{\theta}_{13} & 0 & \cos \tilde{\theta}_{13} \end{pmatrix}$$

$$\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} / \sqrt{(\cos \theta_{13} - A/\Delta m^2)^2 + \sin^2 2\theta_{13}}$$

$$\tilde{\mathcal{H}}^2 = \text{diag} \left(-\frac{\Delta \tilde{m}^2}{2}, -\frac{\Delta m^2 - A}{2}, +\frac{\Delta \tilde{m}^2}{2} \right) \quad \text{with} \quad \Delta \tilde{m}^2 \sin 2\tilde{\theta}_{13} = \Delta m^2 \sin 2\theta_{13}$$

Note that:
$$\left\{ \begin{array}{l} \tilde{m}_3^2 - \tilde{m}_1^2 = \Delta \tilde{m}^2 \\ \tilde{m}_3^2 - \tilde{m}_2^2 = (\Delta m^2 + \Delta \tilde{m}^2 + A)/2 \\ \tilde{m}_2^2 - \tilde{m}_1^2 = (-\Delta m^2 - A + \Delta \tilde{m}^2)/2 \end{array} \right.$$

Also the evolution matrix $\tilde{S}' = e^{-i\tilde{H}'x}$ takes a block form:

$$\tilde{S}' = \begin{pmatrix} \tilde{S}'_{ee} & 0 & \tilde{S}'_{e\tau} \\ 0 & \tilde{S}'_{\mu\mu} & 0 \\ \tilde{S}'_{\tau e} & 0 & \tilde{S}'_{\tau\tau} \end{pmatrix} \quad \text{with } \tilde{S}'_{\mu\mu} \text{ trivially given by } e^{-i\tilde{H}'_{\mu\mu}x} :$$

$$\tilde{S}'_{\mu\mu} = e^{\frac{i}{4E}(\Delta m^2 + A)x} = \cos\left(\frac{\Delta m^2 + A}{4E}x\right) + i \sin\left(\frac{\Delta m^2 + A}{4E}x\right)$$

while the remaining 2x2 (e, τ) block has already been solved in 2v:

$$\tilde{S}'_{(e,\tau)} = \cos\left(\frac{\Delta \tilde{m}^2 x}{4E}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin\left(\frac{\Delta \tilde{m}^2 x}{4E}\right) \begin{pmatrix} -\cos 2\tilde{\theta}_{13} & \sin 2\tilde{\theta}_{13} \\ \sin 2\tilde{\theta}_{13} & \cos 2\tilde{\theta}_{13} \end{pmatrix}$$

Going from the primed to the physical flavor basis:

$$\tilde{S} = \begin{pmatrix} \tilde{S}'_{ee} & & & \\ & S_{13} \tilde{S}'_{e\tau} e^{-i\delta} & & C_{13} \tilde{S}'_{e\tau} e^{-i\delta} \\ & S_{13} \tilde{S}'_{\tau e} e^{i\delta} & & C_{13} \tilde{S}'_{\tau e} e^{i\delta} \\ & C_{13} \tilde{S}'_{\tau e} e^{i\delta} & -S_{13} C_{13} \tilde{S}'_{\mu\mu} + S_{13} C_{13} \tilde{S}'_{\tau\tau} & -S_{13} C_{13} \tilde{S}'_{\mu\mu} + S_{13} C_{13} \tilde{S}'_{\tau\tau} \\ & C_{13} \tilde{S}'_{\tau e} e^{i\delta} & -S_{13} C_{13} \tilde{S}'_{\mu\mu} + S_{13} C_{13} \tilde{S}'_{\tau\tau} & S_{13}^2 \tilde{S}'_{\mu\mu} + C_{13}^2 \tilde{S}'_{\tau\tau} \end{pmatrix}$$

with $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha}$ (constant matter).

The dependence on δ disappears in $|\tilde{S}_{\alpha\beta}|^2 = P_{\beta\alpha}$ as it should (since $\Delta m^2 \rightarrow 0$).

- Now we need to calculate only

$$P_{e\mu} = |\tilde{S}_{\mu e}|^2 = s_{23}^2 |\tilde{S}'_{1e}|^2$$

$$P_{\mu e} = |\tilde{S}_{e\mu}|^2 = s_{23}^2 c_{23}^2 |\tilde{S}'_{\mu\mu} - \tilde{S}'_{ee}|^2$$

The auxiliary $P'_{e\mu}$ and $P'_{\mu e}$ are given by

$$P'_{e\mu} = P_{e\mu} \begin{matrix} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{matrix} = c_{23}^2 |\tilde{S}'_{1e}|^2 = \frac{c_{23}^2}{s_{23}^2} P_{e\mu}$$

$$P'_{\mu e} = P_{\mu e} \begin{matrix} s_{23} \rightarrow \pm c_{23} \\ c_{23} \rightarrow \mp s_{23} \end{matrix} = s_{23}^2 c_{23}^2 |\tilde{S}'_{\mu\mu} - \tilde{S}'_{ee}|^2 = P_{\mu e}$$

so that

$$P_{ee} = 1 - P_{e\mu} - P'_{e\mu} = 1 - \frac{P_{e\mu}}{s_{23}^2}$$

$$P_{e\tau} = P'_{e\mu} = P_{e\mu} c_{23}^2 / s_{23}^2$$

$$P_{\mu\mu} = 1 - P_{e\mu} - P'_{\mu e} = 1 - P_{e\mu} - P_{\mu e}$$

$$P_{\tau\tau} = 1 - P'_{e\mu} - P_{\mu e} = 1 - P_{e\mu} c_{23}^2 / s_{23}^2 - P_{\mu e}$$

$$(P_{\alpha\beta} = P_{\beta\alpha})$$

- $\tilde{S}'_{te} = -i \sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right) \sin 2\tilde{\theta}_{13}$

$$|\tilde{S}'_{te}|^2 = \sin^2 2\tilde{\theta}_{13} \sin^2\left(\frac{\Delta\tilde{m}^2 x}{4E}\right) = 4 \tilde{s}_{13}^2 \tilde{c}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E} x\right)$$

$$P_{e\mu} = s_{23}^2 |\tilde{S}'_{te}|^2$$

$$= 4 \tilde{s}_{13}^2 \tilde{c}_{13}^2 s_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E} x\right)$$

$$= s_{23}^2 \sin^2 2\tilde{\theta}_{13} \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E} x\right)$$

← analogous to vacuum, but with $\left. \begin{matrix} \theta_{13} \rightarrow \tilde{\theta}_{13} \\ \Delta m^2 \rightarrow \Delta\tilde{m}^2 \end{matrix} \right\}$

- $\tilde{S}'_{\mu\mu} = \cos\left(\frac{\Delta\tilde{m}^2 + A}{4E} x\right) + i \sin\left(\frac{\Delta\tilde{m}^2 + A}{4E} x\right)$

$$\tilde{S}'_{\tau\tau} = \cos\left(\frac{\Delta\tilde{m}^2}{4E} x\right) - i \sin\left(\frac{\Delta\tilde{m}^2}{4E} x\right) \cos 2\tilde{\theta}_{13}$$

$$\tilde{S}'_{\mu\mu} - \tilde{S}'_{\tau\tau} = \left[\cos\left(\frac{\Delta\tilde{m}^2 + A}{4E} x\right) - \cos\left(\frac{\Delta\tilde{m}^2}{4E} x\right) \right] + i \left[\sin\left(\frac{\Delta\tilde{m}^2 + A}{4E} x\right) + \sin\left(\frac{\Delta\tilde{m}^2}{4E} x\right) \cos 2\tilde{\theta}_{13} \right]$$

$$\begin{aligned}
|\tilde{S}_{\mu\mu} - \tilde{S}_{cc}^1|^2 &= \cos^2\left(\frac{\Delta\tilde{m}^2 + A}{4E}x\right) + \cos^2\left(\frac{\Delta\tilde{m}^2}{4E}x\right) - 2\cos\left(\frac{\Delta\tilde{m}^2 + A}{4E}x\right)\cos\left(\frac{\Delta\tilde{m}^2}{4E}x\right) \\
&+ \sin^2\left(\frac{\Delta\tilde{m}^2 + A}{4E}x\right) + \cos^2 2\tilde{\theta}_{13} \sin^2\left(\frac{\Delta\tilde{m}^2}{4E}x\right) + 2\sin\left(\frac{\Delta\tilde{m}^2 + A}{4E}x\right)\sin\left(\frac{\Delta\tilde{m}^2}{4E}x\right)\cos 2\tilde{\theta}_{13} \\
&= \dots \text{ (trigonometry) } \dots \\
&= -4\tilde{S}_{13}^2 \tilde{C}_{13}^2 \sin^2\left(\frac{\Delta\tilde{m}^2}{4E}x\right) + 4\tilde{C}_{13}^2 \sin^2\left(\frac{\Delta\tilde{m}^2 + A + \Delta\tilde{m}^2}{8E}x\right) + 4\tilde{S}_{13}^2 \sin^2\left(\frac{\Delta\tilde{m}^2 + A - \Delta\tilde{m}^2}{8E}x\right) \\
&= -4\tilde{S}_{13}^2 \tilde{C}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E}x\right) + 4\tilde{C}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_2^2}{4E}x\right) + 4\tilde{S}_{13}^2 \sin^2\left(\frac{\tilde{m}_2^2 - \tilde{m}_1^2}{4E}x\right)
\end{aligned}$$

The final results for $P_{e\mu}$, P_{ee} and $P_{e\tau}$ resemble the vacuum expressions with $(\Delta\tilde{m}^2, \theta_{13}) \rightarrow (\Delta\tilde{m}^2, \tilde{\theta}_{13})$:

$$P_{e\mu} = 4\tilde{S}_{13}^2 \tilde{C}_{13}^2 \tilde{S}_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E}x\right) \quad (P_{e\mu} = P_{e\mu})$$

$$P_{ee} = 1 - 4\tilde{S}_{13}^2 \tilde{C}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E}x\right)$$

$$P_{e\tau} = 4\tilde{S}_{13}^2 \tilde{C}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E}x\right) \quad (P_{e\tau} = P_{e\tau})$$

whilst $P_{\mu\mu}$, $P_{\mu\tau}$ and $P_{\tau\tau}$ are more complicated :

$$\begin{aligned}
 P_{\mu\tau} = & -4 \tilde{S}_{13}^2 \tilde{C}_{13}^2 \tilde{S}_{23}^2 \tilde{C}_{23}^2 \sin^2 \left(\frac{\tilde{M}_3^2 - \tilde{M}_1^2}{4E} x \right) \\
 & + 4 \tilde{S}_{13}^2 \tilde{S}_{23}^2 \tilde{C}_{23}^2 \sin^2 \left(\frac{\tilde{M}_2^2 - \tilde{M}_1^2}{4E} x \right) \\
 & + 4 \tilde{C}_{13}^2 \tilde{S}_{23}^2 \tilde{C}_{23}^2 \sin^2 \left(\frac{\tilde{M}_3^2 - \tilde{M}_2^2}{4E} x \right)
 \end{aligned}
 \tag{P_{\tau\mu} = P_{\mu\tau}}$$

$$\begin{aligned}
 P_{\mu\mu} = & 1 - 4 \tilde{S}_{13}^2 \tilde{C}_{13}^2 \tilde{S}_{23}^4 \sin^2 \left(\frac{\tilde{M}_3^2 - \tilde{M}_1^2}{4E} x \right) \\
 & - 4 \tilde{S}_{13}^2 \tilde{S}_{23}^2 \tilde{C}_{23}^2 \sin^2 \left(\frac{\tilde{M}_2^2 - \tilde{M}_1^2}{4E} x \right) \\
 & - 4 \tilde{C}_{13}^2 \tilde{S}_{23}^2 \tilde{C}_{23}^2 \sin^2 \left(\frac{\tilde{M}_3^2 - \tilde{M}_2^2}{4E} x \right)
 \end{aligned}$$

$$\begin{aligned}
 P_{\tau\tau} = & 1 - 4 \tilde{S}_{13}^2 \tilde{C}_{13}^2 \tilde{C}_{23}^4 \sin^2 \left(\frac{\tilde{M}_3^2 - \tilde{M}_1^2}{4E} x \right) \\
 & - 4 \tilde{S}_{13}^2 \tilde{S}_{23}^2 \tilde{C}_{23}^2 \sin^2 \left(\frac{\tilde{M}_2^2 - \tilde{M}_1^2}{4E} x \right) \\
 & - 4 \tilde{C}_{13}^2 \tilde{S}_{23}^2 \tilde{C}_{23}^2 \sin^2 \left(\frac{\tilde{M}_3^2 - \tilde{M}_2^2}{4E} x \right)
 \end{aligned}$$

All these $P_{\alpha\beta}$'s are sensitive to $\pm \Delta m^2$ (hierarchy), provided that $\theta_{13} \neq 0$

Bottom line:

In order to be sensitive to the hierarchy ($\text{sign}(\pm \Delta m^2)$), one must "beat" Δm^2 -driven oscillations with Q -driven oscillations, where Q is a quantity having definite sign.

The previous exercises show that one can either use

$$Q = \delta m^2 > 0 \quad \leftarrow \text{"vacuum"}$$

$$\text{or } Q = A \geq 0 \quad (\nu/\bar{\nu}) \quad \leftarrow \text{"matter"}$$

but sensitivity to $\text{sign}(\Delta m^2)$ is anyway suppressed by small $\delta m^2/\Delta m^2$, or by small θ_{13} , or both.