Stochastic inflation in Ultra Slow Roll (USR).

Diego Cruces.

ICC-Universitat de Barcelona

17 May 2019

Based on arXiv:1807.09057 with C. Germani and T. Prokopec.
Recently published on JCAP 1903 (2019) no.03, 048.

Outline

1. Stochastic Inflation
   - Langevin equation
2. Regime of validity of stochastic inflation
3. Stochastic approach applied to USR
4. Conclusions

Separate universes approach

Stochastic formalism is based on the separate universes approach, which consists into assuming that, at super-Hubble scales:

1. Spatial gradients can be neglected.
2. The evolution of the field is well approximated by a KG equation in a local FRW space-time.

In this approach:

$$\left(\frac{\partial^2}{\partial N^2} + (3-\epsilon_1)\frac{\partial}{\partial N}\right)\phi_{IR}(N) + \frac{V'(\phi_{IR})}{H^2} = 0\,,$$

Stochastic formalism for inflation

The idea of stochastic inflation is to reduce the evolution of the full quantum inflation field dynamics to a much simpler. but almost equivalent, stochastic problem.
This is done by splitting the inflaton into two parts:

1. Quantum short-scale part, in which the field is fully quantum but for which perturbative methods apply.

$$\delta\phi_{UV}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \theta(k - \sigma a(t)H(t)) e^{-i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(t)\,,$$

2. Stochastic large-scale part, in which the field is influenced by the quantum sector by recieving kicks from an Markovian stochastic force.

$$\phi_{IR}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \theta(\sigma a(t)H(t) - k) e^{-i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(t)\,,$$

The Langevin equation

In the stochastic approach the IR modes recieve stochastic kicks from short-wavelength modes (UV modes), where the separate universe approach does not hold:

The UV modes are in the perturbative regime:

$$KG(\phi_{IR}) + MS(\delta\phi_{UV}) = 0$$

The Langevin equation

Using the definitions of ϕ_{IR} and $\delta\phi_{UV}$ introduced before and after some algebra we finally arrive to the Langevin equation:

$$\left[\frac{\partial^2}{\partial N^2} + (3-\epsilon_1)\frac{\partial}{\partial N}\right]\phi_{IR} + \frac{V'(\phi_{IR})}{H^2} = (3-\epsilon_1)\xi_1 + \frac{\partial \xi_1}{\partial N} + \xi_2 \, .$$

Where ξ_1 and ξ_2 are the stochastic forces

$$\begin{aligned} \xi_1 &= \sigma a H(1-\epsilon_1)\int \frac{d^3k}{(2\pi)^{3/2}}\delta(k-\sigma a H)e^{-i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}(N)\, , \\ \xi_2 &= \sigma a H(1-\epsilon_1)\int \frac{d^3k}{(2\pi)^{3/2}}\delta(k-\sigma a H)e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\partial \phi_{\mathbf{k}}(N)}{\partial N}\, , \end{aligned}$$

When can we use the stochastic approach to inflation?

- Separate universe approach $\rightarrow$ The IR field evolves approximately in a FRW universe. Is this "approximately" always valid?
- We need to compare:
  - Perturbations of the field in a FRW universe. $\rightarrow$ Linearization of Langevin equation with the noises switched off.
  - Perturbations of the field + perturbations of the metric $\rightarrow$ MS equation

Comparision

1. Linearized KG equation in FRW background.

$$\frac{\partial^2 \delta\phi_{IR}(N)}{\partial N^2} + \left(3 - \epsilon_1 + \frac{1}{3}\epsilon_1\epsilon_2\right)\frac{\partial \delta\phi_{IR}}{\partial N} + \left[-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3 + \frac{1}{3}\epsilon_1^2\epsilon_2 - \frac{1}{6}\epsilon_1\epsilon_2^2\right]\delta\phi_{IR} + \mathcal{O}(\delta\phi_{IR}^2) = 0\,.$$

2. MS equation.

$$\left[\frac{\partial^2}{\partial N^2} + (3-\epsilon_1)\frac{\partial}{\partial N} + \left(-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\right]\delta\phi(N,\mathbf{x}) + \mathcal{O}(\delta\phi^2) = 0$$

Comparision

The two equations differ by a damping term $\delta\gamma_{stoch} = \frac{1}{3}\epsilon_1\epsilon_2$ and by an effective mass term $\delta m_{stoch} = \frac{1}{3}\epsilon_1^2\epsilon_2 - \frac{1}{6}\epsilon_1\epsilon_2^2$.

1. SR
   $\epsilon_i \ll 1 \rightarrow$ **Stochastic formalism fails at second order in SR parameters**.
2. USR
   $\epsilon_1 \ll 1$ but $\epsilon_2 \simeq -6 \rightarrow$ **Stochastic formalism fails at first order in SR parameters**.

Faliure of the approach when $\epsilon_2 \sim \mathcal{O}(1)$

Imposing the new stochastic terms to be much smaller than γ and $\frac{m^2}{H^2}$ (terms appearing in MS equation).

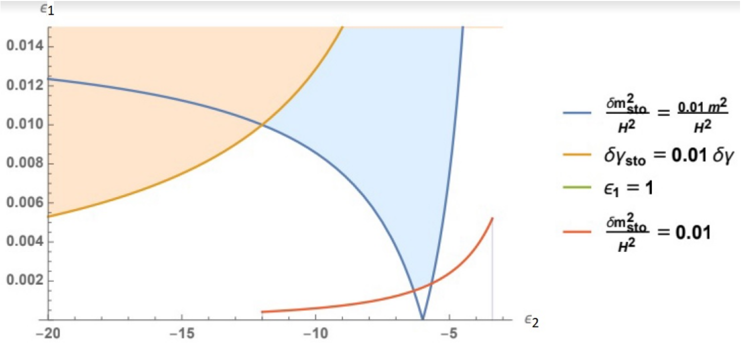

Figure: Stochastic approach fails in the dashed region

USR

A regime of USR during inflation → generation of a peak on the power spectrum → PBH.

Stochastic approach in USR is only reliable at zeroth order. One can calculate the correlators of the noises at zeroth order.

$$
\begin{aligned}
\langle \xi_1(N_1)\xi_1(N_2)\rangle &\simeq \left(\frac{H}{2\pi}\right)^2 \delta(N_1 - N_2)\,, \\
\langle \xi_2(N_1)\xi_2(N_2)\rangle &\simeq 0\,, \\
\langle \xi_2(N_1)\xi_1(N_2)\rangle &\simeq 0\,.
\end{aligned}
$$

And the Langevin equation that we have to solve is finally:

$$
\frac{\partial^2 \phi_{IR}}{\partial N^2} + 3\frac{\partial \phi_{IR}}{\partial N} = \frac{3H}{2\pi}\xi(N)\,,
$$

Solution to the Langevin equation in USR

We are interested in the fluctuactions around the homogeneous mean value $\phi_{IR}(N) - \langle \phi_{IR}(N) \rangle = \delta\phi_{IR}(N)$

The power spectrum of the scalar perturbations is proportional to the two-point correlation function $\langle \delta\phi_{IR}(N)\delta\phi_{IR}(\bar{N}) \rangle_{N \to \bar{N}}$

After some algebra, the result is

$$\langle \delta\phi_{IR}(N)\delta\phi_{IR}(N) \rangle = \frac{H^2}{4\pi^2}\left(N - \frac{1}{2}\right) + \text{Subleading terms}\,.$$

Power spectrum

Stochastic formalism allow us to calculate the correlator in real space and when $\lambda \to \infty$

This correlator is proportional to the power spectrum:

$$\langle \delta\phi_{IR} \delta\phi_{IR} \rangle = \int \mathcal{P}_{\delta\phi_{IR}}(k) d \ln k \,,$$

$\mathcal{P}_{\delta\phi_{IR}}(k)$ only depends on the value it acquires at Hubble crossing $\to d \ln k = dN$:

$$\mathcal{P}_{\delta_{IR}} \simeq \frac{d}{dN} \langle \delta\phi_{IR} \delta\phi_{IR} \rangle \simeq \frac{H^2}{4\pi^2}$$

Quantum diffusion at zeroth order in slow-roll parameters is irrelevant in USR inflation.

Conclusions

2 important results:

1. Separate universe approach generically fails at leading order in the slow-roll parameters and at any order whenever in the dashed region of the figure showed before $\Longrightarrow$ Generic faliure of the stochastic approach (Realistic?).
2. Keeping ourselves in the regime of validity of stochastic approach, we conclude that Quantum diffusion at zeroth order in slow-roll parameters is irrelevant in USR inflation.$\rightarrow$ PBH can be safely studied with MS approach for USR.