Lecture Notes

SHEAP2019

Pitayuth Wongjun

The Institute for Fundamental Study,
Naresuan University, Phitsanulok 65000, Thailand

June 21, 2019

Chapter 1

Kruskal-Szekeres coordinates

From Eddington-Fankelstein coordinates, It is found that there are two regions of the solutions, one corresponds to Schwarzschild black hole and the other corresponds to white hole. These two regions are described by two spacetime diagrams. Therefore, it is worthwhile to find the new coordinates which can combine these two regions in one diagram. This is the main purpose of this section and it corresponds to the Kruskal-Szekeres coordinates. Other motivations come from the fact that, in Eddington-Fankelstein coordinates, one of the coordinates in spacetime diagram is still singular at $r = 2\mu$ and the lightcone structure is not easy to sketch in the spacetime diagram. In any conformally flat coordinates, the lightcone structure is still the same with the lightcone in Minkowski spacetime. Therefore, it is easy the sketch the lightcone since the angle of the lightcone is fixed to 45° in all regions of the diagram. Therefore, let us begin with introducing the coordinates which are conformally flat as follow

$$\tilde{r} = r + 2\mu \ln\left|\frac{r}{2\mu} - 1\right|, \tag{1.1}$$

$$d\tilde{r} = \left(1 - \frac{2\mu}{r}\right)^{-1} dr \Rightarrow dr^2 = \left(1 - \frac{2\mu}{r}\right)^2 d\tilde{r}^2. \tag{1.2}$$

Substituting into the Schwarzschild line element in equation, one obtains

$$ds^2 = \left(1 - \frac{2\mu}{r}\right)(-c^2dt^2 + d\tilde{r}^2). \tag{1.3}$$

Conveniently, we omit the contribution from the solid angle by fixing $\theta = \pi/2$ and $\phi =$ constant. If we include it, the lightcone structure will not change. Each point in the spacetime diagram will represent the two sphere of θ and ϕ. Note that r is no longer be the coordinate. It plays the role of function of the coordinate $r(\tilde{r})$. We see that the coordinate $\tilde{r}$ still has singularity at $r = 2\mu$. This implicitly leads to the singularity of the metric at $r = 2\mu$. In order to see how the singularity emerges in the metric, one can investigate the metric at $r \sim 2\mu$. This leads to the relation

$$\tilde{r} \simeq 2\mu \ln\left|\frac{r}{2\mu} - 1\right| \Rightarrow \frac{r}{2\mu} \simeq 1 \pm e^{\frac{\tilde{r}}{2\mu}} \Rightarrow \left(1 - \frac{2\mu}{r}\right) \simeq \pm e^{\frac{\tilde{r}}{2\mu}}, \tag{1.4}$$

where the upper sign denotes the region $r > 2\mu$ and the lower sign denotes the region $r < 2\mu$. Thus the line element in equation (1.3) at $r \sim 2\mu$ can be approximated as

$$ds^2 \simeq \pm e^{\frac{\tilde{r}}{2\mu}}(-c^2dt^2 + d\tilde{r}^2). \tag{1.5}$$

From this line element, one can see that the metric blow up at $r = 2\mu$ since the coordinate $\tilde{r}$ will be infinity. In order to find the new coordinates to get rid of this factor, one can firstly transform the null coordinates such that

$$p = ct + \tilde{r},\ q = ct - \tilde{r} \ \Rightarrow\ d\tilde{r} = \frac{1}{2}(dp - dq),\ c\,dt = \frac{1}{2}(dp + dq) \ \Rightarrow\ -c^2dt^2 + d\tilde{r}^2 = -dp\,dq. \tag{1.6}$$

Substituting the results of the coordinate transformation into the approximated line element above, one obtains

$$ds^2 \simeq \mp e^{\frac{\tilde{r}}{2\mu}} dp\, dq = \mp e^{\frac{p-q}{4\mu}} dp\, dq. \tag{1.7}$$

Now we see that, in order to get rid of the divergent factor, one can find the new coordinates which satisfy the relation

$$d\tilde{p} \propto e^{\frac{p}{4\mu}} dp, \;\; \text{and} \;\; d\tilde{q} \propto e^{-\frac{q}{4\mu}} dq. \tag{1.8}$$

Therefore, one of the simple choices is that

$$\tilde{p} = e^{\frac{p}{4\mu}}, \;\; \text{and} \;\; \tilde{q} = \mp e^{\frac{-q}{4\mu}}. \tag{1.9}$$

This leads to the relation

$$\begin{aligned} dp\, dq &= \pm 16\mu^2 e^{-\frac{p-q}{4\mu}} d\tilde{p}\, d\tilde{q} = \pm 16\mu^2 e^{-\frac{\tilde{r}}{2\mu}} d\tilde{p}\, d\tilde{q}, \\ &= \pm 16\mu^2 e^{-\frac{r}{2\mu} - \ln\left|\frac{r}{2\mu} - 1\right|} d\tilde{p}\, d\tilde{q}, \\ &= \pm 16\mu^2 e^{-\frac{r}{2\mu}} \left(\left| \frac{r}{2\mu} - 1 \right| \right)^{-1} d\tilde{p}\, d\tilde{q}, \\ &= 16\mu^2 e^{-\frac{r}{2\mu}} \left(\frac{2\mu}{r - 2\mu} \right) d\tilde{p}\, d\tilde{q}. \end{aligned} \tag{1.10}$$

Substituting this results back into the line element (1.3) with using the relation in equation (1.6), one obtains

$$\begin{aligned} ds^2 &= -\left(\frac{r-2\mu}{r} \right) 16\mu^2 e^{-\frac{r}{2\mu}} \left(\frac{2\mu}{r-2\mu} \right) d\tilde{p}\, d\tilde{q}, \\ &= -32\mu^3 \frac{e^{-\frac{r}{2\mu}}}{r} d\tilde{p}\, d\tilde{q}. \end{aligned} \tag{1.11}$$

Finally, let us transform the coordinates to the one which is in the conformally flat form. This can be achieved by introducing new coordinates such that

$$\tilde{p} = v + u, \tilde{q} = v - u \Rightarrow d\tilde{p}\, d\tilde{q} = -du^2 + dv^2, \tag{1.12}$$

Substituting the result back into equation (1.11), the line element becomes

$$ds^2 = 32\mu^3 \frac{e^{-\frac{r}{2\mu}}}{r} (-dv^2 + du^2). \tag{1.13}$$

Now r is a function of u and v. We can see that there are no singularities of the metric at $r = 2\mu$ as well as it satisfies the conformally flat form. The real singularity still explicitly appear in the metric. In order to sketch the spacetime diagram, one has to find the relation between Schwarzschild and Kruskal-Szekeres coordinates. Firstly, let us consider the simple relation

$$v^2 - u^2 = \tilde{p}\,\tilde{q} = \mp e^{\frac{p-q}{4\mu}} = \mp e^{\frac{\tilde{r}}{2\mu}} = \mp e^{\frac{r}{2\mu}} \left(\left| \frac{r}{2\mu} - 1 \right| \right) = -e^{\frac{r}{2\mu}} \left(\frac{r}{2\mu} - 1 \right). \tag{1.14}$$

This relation provide us the lines at the event horizon such that $v = \pm u$. These lines are sketched as crossing sign which is shown in the figure 1.1. These crossing lines separate the spacetime diagram into four regions: left (I'), right (I), upper (II) and lower (II') regions. For any constant r, it is a hyperbolic locus. For $r > 2\mu$, it corresponds to hyperbolic locus in left and right regions of the diagram and , for $r < 2\mu$, it corresponds to hyperbolic locus in upper and lower regions of the diagram which contain the locus of the singularity.

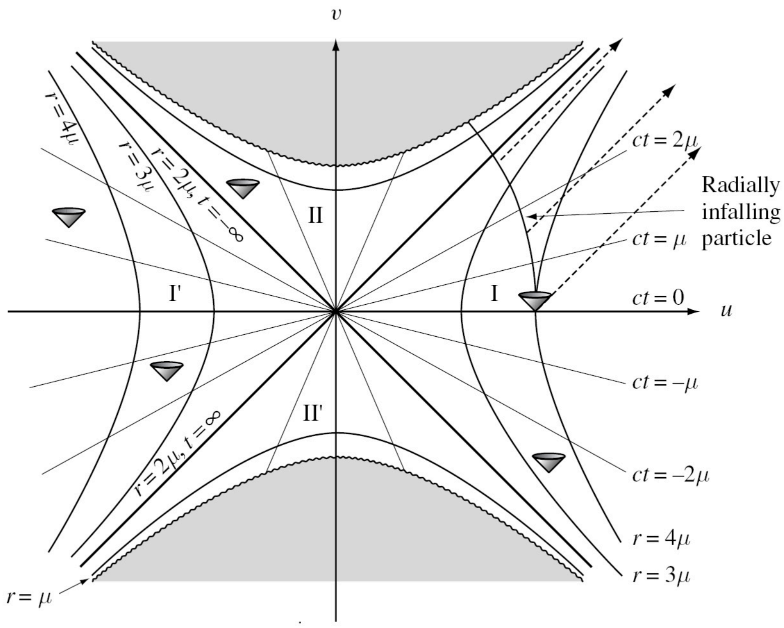

Figure 1.1: Spacetime diagram in Kruskal-Szekeres coordinates

To see more information of the diagram, let us consider the relation for constant time. Firstly, we can find the relation of v in terms of r and t which can be written as

$$\begin{aligned} v &= \frac{1}{2}(\tilde{p}+\tilde{q}) = \frac{1}{2}\left(e^{\frac{p}{4\mu}} \mp e^{\frac{p}{4\mu}}\right) = \frac{1}{2}\left(e^{\frac{ct+\tilde{r}}{4\mu}} \mp e^{-\frac{ct-\tilde{r}}{4\mu}}\right), \\ &= e^{\frac{\tilde{r}}{4\mu}}\left(e^{\frac{ct}{4\mu}} \mp e^{-\frac{ct}{4\mu}}\right) = e^{\frac{r}{4\mu}}\left(\frac{r}{2\mu}-1\right)^{1/2}\frac{1}{2}\left(e^{\frac{ct}{4\mu}} \mp e^{-\frac{ct}{4\mu}}\right), \\ &= e^{\frac{r}{4\mu}}\left(\frac{r}{2\mu}-1\right)^{1/2}\sinh\left(\frac{ct}{4\mu}\right), \quad \text{for } r > 2\mu, \qquad (1.15) \\ &= e^{\frac{r}{4\mu}}\left(1-\frac{r}{2\mu}\right)^{1/2}\cosh\left(\frac{ct}{4\mu}\right), \quad \text{for } r < 2\mu. \qquad (1.16) \end{aligned}$$

We can obtain the relation of u in terms of r and t in the same manner. The results can be expressed as

$$\begin{aligned} u &= e^{\frac{r}{4\mu}}\left(\frac{r}{2\mu}-1\right)^{1/2}\cosh\left(\frac{ct}{4\mu}\right), \quad \text{for } r > 2\mu, \qquad (1.17) \\ &= e^{\frac{r}{4\mu}}\left(1-\frac{r}{2\mu}\right)^{1/2}\sinh\left(\frac{ct}{4\mu}\right), \quad \text{for } r < 2\mu, \qquad (1.18) \end{aligned}$$

Now, the relations of u and v in terms of t can be written as

$$\tanh\left(\frac{ct}{4\mu}\right) = \frac{v}{u}, \quad \text{for } r > 2\mu, \qquad (1.19)$$

$$\tanh\left(\frac{ct}{4\mu}\right) = \frac{u}{v}, \quad \text{for } r < 2\mu. \qquad (1.20)$$

From these relation, we found that the axis $t = 0$ corresponds to the axis $v = 0$ for $r > 2\mu$ and corresponds to the axis $u = 0$ for $r < 2\mu$. For $r > 2\mu$ in the right region, any value of t corresponds the strength line. The more value of t, the more slope of the strength line until the slope becomes 1 corresponding to $t = \infty$ as shown in figure 1.1. The lightcone structure is still the same everywhere in the diagram and have the future lightcone in the direction of increasing v. This leads to the fact that the upper region correspond to the interior of the black hole while the lower region corresponds to the interior of white hole. We also see from the lightcone at the line $r = 2\mu$ in the upper region that the particle never escaped from black hole while, in the lower region, particle never moved into the white hole. The interesting results of this diagram is that there are two regions which have an asymptotically flat Minkowski spacetime in the left and right regions or in region I and I'. Region I is completely described our spacetime outside black hole. This leads to the fact that there exists another world in region I' which cannot be influenced by our world. We can see that the particle at the origin of the diagram are restricted to go the region II which is the black hole. Thus it is impossible to move from region I to region I' or inversely from region I' to region I. The join between these two region is called "wormhole" which we will discuss in detail in next sections.

Chapter 2

Wormhole and Einstein-Rosen bridge

As we have mentioned in the previous section, there is a join between two worlds which is the origin in Kruskal-Szekeres spacetime diagram. Remembering that each point in spacetime diagram represent 2-sphere of the solid angle. Thus, there are others view point to consider this join. We have learned that it is convenient to consider two-dimensional diagram. Now, we can choose to consider diagram in which $v = 0$ and $\theta = \pi/2$. This corresponds to the line element

$$ds^2 = 32\mu^3 \frac{e^{-\frac{r}{2\mu}}}{r} du^2 + r^2 d\phi^2. \tag{2.1}$$

By using the relation of u^2 and r in equation (1.14) , one obtains

$$ds^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{2.2}$$

It is convenient to consider this surface by embedding it into three-dimensional Euclidian space. Since we have the coordinate ϕ and r which are looks similar to the polar cylindrical coordinates, it is convenient to consider this three-dimensional Euclidian space in polar cylindrical coordinates which can be written as

$$ds^2 = dz^2 + d\rho^2 + \rho^2 d\psi^2. \tag{2.3}$$

In order to embed our two-sphere into this three-dimensional Euclidian space, we have to find a constraint equation to satisfy line element (2.2). This can be achieved by introducing the coordinates such that $\rho = \rho(r)$, $z = z(r)$. Therefore, the line element in equation (2.3) becomes

$$ds^2 = \left(\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2\right) dr^2 + \rho^2 d\psi^2. \tag{2.4}$$

Then, by setting the coordinate such that $\psi = \phi$, $\rho = r$, one obtains

$$ds^2 = \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2. \tag{2.5}$$

Comparing to equation (2.2), the constraint equation in differential form can be written as

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} \Rightarrow \left(\frac{dz}{dr}\right)^2 = \frac{r}{r - 2\mu} - 1 = \frac{2\mu}{r - 2\mu}. \tag{2.6}$$

This leads to the constraint equation,

$$z = (2\mu)^{1/2} \int (r - 2\mu)^{-1/2} dr = (8\mu)^{1/2} (r - 2\mu)^{1/2} + \text{constant}. \tag{2.7}$$

From this constraint equation, the generality of the surface structure is not lost by setting the constant

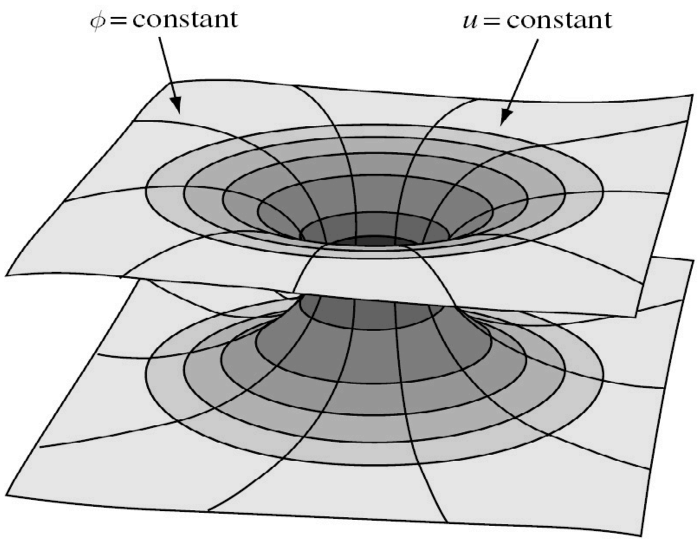

Figure 2.1: Wormhole or Einstein-Rosen bridge [?]. It is another view point of the join between our world and the other which corresponds to the line $v = 0$ in Kruskal-Szekeres diagram.

to be zero and we can see that this is the locus of parabolic line. At $z = 0$ corresponds to $r = 2\mu$ which is the closet point to the z-axis and recognizing that for $v = 0$ corresponds to $r \geq 2\mu$. To obtain the surface one can turn the parabolic locus around z-axis and then the surface can be shown in figure (2.1). From this figure, the two world can be connected together through the throat of surface at $z = 0$ corresponding to origin of the Kruskal-Szekeres diagram. If we return to consider the Kruskal-Szekeres diagram, we cannot stay at the origin point and are forced to move to region II which $v > 0$. Since, in region II, the coordinate r becomes timelike coordinates, the surface which depends on r will become to be dynamics. At $v = v_0 =$ constant where $0 < v_0 < 1$, the metric is still in the same form and then the surface also has the same form. However, from the Kruskal-Szekeres diagram, when we scan $-\infty < u < \infty$, the minimum of r is in the range $0 < r < 2\mu$. Therefore the throat of the surface becomes more and more narrow as v increasing until $v = 1$ corresponding to $r = 0$, the throat is pinched off. The crosse section of the surface with various v are shown in figure 2.2. From point of

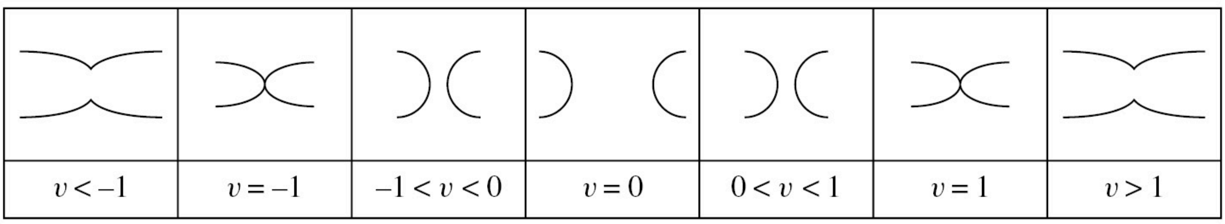

Figure 2.2: Cross section of Wormhole or Einstein-Rosen bridge with various value of v [?]

view of Einstein-Rosen bridge, it may be possible to travel from one world to the other before the throat of the bridge is pinched off. However, the information from the Kruskal-Szekeres still valid and provides us that the throat is pinched off too quickly for any timelike particle can cross it from one to the other world. It is important to note that this is only the solution of Einstein equation in empty spacetime. It may be possible to construct the spacetime geometry by introducing exotic matter in which the wormhole will not pinch off too quickly. This is also the basic study of the time traveling which is commonly mention in pop-science movie. Research area of this subject is still active and it is interesting for presentation and report of the students in the course.

Chapter 3

Reissner-Nordstrom black hole

It is interesting to ask how is the spacetime geometry if the star has a charge? Even though stars are usually neutral, it is instructive to investigate the property of spacetime when the star carries a charge. This investigation will provide the black hole solution called "Reissner-Nordstrom black hole" or "charged black hole". This is the main objective of this talk.

3.1 Electromagnetism

Considering the electromagnetic force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}), \tag{3.1}$$

or we can write in the component form

$$F^i = q(E^i + \epsilon^{jki} v^j B^k), \tag{3.2}$$

promoting to be a 4-force

$$f^{(4)} = q \,\square\, u^{(4)} \tag{3.3}$$

where $\square$ should be a quantity depending on $\vec{E}$ and $\vec{B}$. Moreover, it must be a 2-rank tensor in order to contract with a vector ($u^{(4)}$) then obtain another vector ($f^{(4)}$). Hence, it is possible to assume that

$$\square \equiv F^{\mu\nu}, \tag{3.4}$$

then,

$$f^\mu = qF^{\mu\nu} u_\nu. \tag{3.5}$$

Considering the case of pure force (a force which preserves the rest mass) as

$$f^\mu u_\mu = 0 \propto \frac{\mathrm{d}m}{\mathrm{d}\tau}. \tag{3.6}$$

Substituting with (3.5), we obtain a condition,

$$f^\mu u_\mu = qF^{\mu\nu} u_\nu u_\mu = 0. \tag{3.7}$$

This implies that $F^{\mu\nu}$ must be an anti-symmetric tensor, $F^{\mu\nu} = -F^{\nu\mu}$. In addition, $F^{\mu\nu}$ contains 6 independent components (it is true only in 4 dimensional spacetime).

From the fact that the fields $\vec{E}$ and $\vec{B}$ are written in the potentials

$$\vec{E} = \vec{\nabla}\phi - \frac{\partial A}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}, \tag{3.8}$$

where ϕ and $\vec{A}$ are the scalar and vector potentials respectively. In the same fashion, they are promoted to 4-vector potential

$$A^{\mu} = \left(\frac{\phi}{c}, A^i\right), \qquad \rightarrow \qquad A_{\mu} = \left(-\frac{\phi}{c}, A^i\right). \tag{3.9}$$

According to discussion about the feature of the tensor $F^{\mu\nu}$, it is thus possible to write down

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{3.10}$$

By comparing (3.8) and (3.10), it is found that

$$E_i = cF_{i0} = -cF_{0i}, \qquad B_i = \epsilon_{ijk}F_{jk}. \tag{3.11}$$

The matrix form of $F_{\mu\nu}$ reads

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}, \tag{3.12}$$

or

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}, \tag{3.13}$$

Next, we will consider the source in electromagnetism. The source must depend on the density, ρ and current $\vec{j} = \rho\,\vec{u}$. From $u^{\mu} = \gamma(c, u^i)$, then we define the 4-current density as

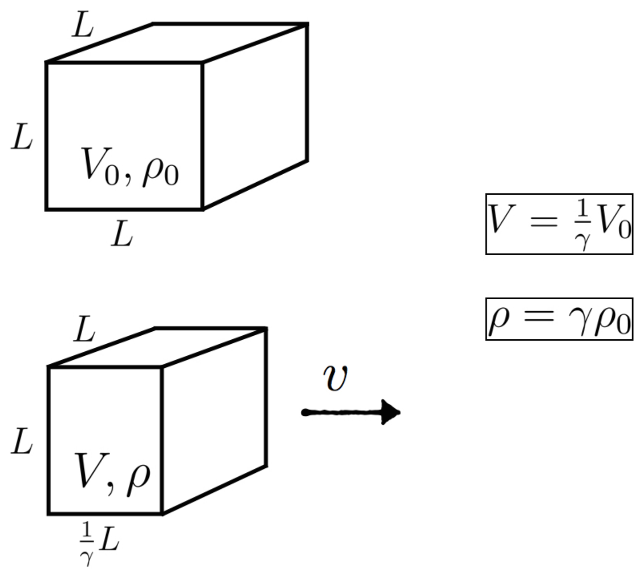

Figure 3.1: aaa

$$j^{\mu} = \rho_0\, u^{\mu} = \rho(c, u^i). \tag{3.14}$$

From the invariant quantity $u^{\mu}u_{\mu} = -c^2$, we also have another one

$$j^{\mu}j_{\mu} = \rho_0^2\, u^{\mu}u_{\mu} = -\rho_0^2\, c^2. \tag{3.15}$$

As the ordinary form of Maxwell's equations, they read

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \tag{3.16}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{3.17}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \tag{3.18}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}. \tag{3.19}$$

Notice that these equations contain the first derivative of $\vec{E}$ and $\vec{B}$. We first consider the equations with source, (3.16) and (3.19). To obtain the left side of them, the 2-rank anti-symmetric tensor should be taken the first derivative. Thus, the equation of motion should be written as

$$\partial_\mu F^{\nu\mu} = kj^\nu. \tag{3.20}$$

where k is a constant and the contraction of indices in $\partial_\mu F^{\nu\mu}$ is just a convention.

Considering each component of (3.20),
For $\nu = 0$,

$$\begin{aligned}
\partial_\mu F^{0\mu} &= kj^0, \\
\partial_i F^{0i} &= kj^0, \qquad (F^{00} = 0.) \\
\partial_i\left(\frac{E_i}{c}\right) &= k\rho c, \\
\vec{\nabla}\cdot\vec{E} &= kc^2\rho.
\end{aligned} \tag{3.21}$$

Comparing to (3.16), the constant k is determined as

$$k = \frac{1}{c^2\epsilon_0}. \tag{3.22}$$

For $\nu = i$,

$$\begin{aligned}
\partial_\mu F^{i\mu} &= kj^i, \\
\partial_0 F^{i0} + \partial_k F^{ik} &= \frac{1}{c^2\epsilon_0}j^i, \\
\frac{1}{c}\partial_t\left(-\frac{E_i}{c}\right) + \partial_k(\epsilon_{ikl}B_l) &= \frac{1}{c^2\epsilon_0}\vec{j}, \\
-\frac{1}{c^2}\partial_t E_i + (\vec{\nabla}\times\vec{B})_i &= \mu_0 j_i. \qquad (c^2 = \frac{1}{\epsilon_0\mu_0}.)
\end{aligned} \tag{3.23}$$

We have already obtained 2 of 4 Maxwell's equations. It is found that (3.17) and (3.19) are obtained from the identity of the tensor $F_{\mu\nu}$ as follows

$$,\partial_{[\rho}F_{\mu\nu]} = 0, \qquad \rightarrow \qquad \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0. \tag{3.24}$$

Considering each component of (3.24),
For $\rho = 0, \mu = i, \nu = j$,

$$\begin{aligned}
\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} &= 0, \\
\partial_0(\epsilon_{ijk}B_k) + \frac{1}{c}\partial_i(E_j) + \frac{1}{c}\partial_j(-E_i) &= 0, \\
\partial_0(\epsilon_{ijk}B_k) + \frac{1}{c}(\partial_i E_j - \partial_j E_i) &= 0, \\
\frac{1}{c}\epsilon_{ijk}\partial_t B_k + \frac{1}{c}\epsilon_{ijk}(\vec{\nabla}\times\vec{E})_k &= 0, \\
\partial_t B_k + (\vec{\nabla}\times\vec{E})_k &= 0.
\end{aligned} \tag{3.25}$$

For $\rho = i, \mu = j, \nu = k$,

$$\begin{aligned} \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} &= 0, \\ \partial_i(\epsilon_{jkl} B_l) + \partial_j(\epsilon_{kil} B_l) + \partial_k(\epsilon_{ijl} B_l) &= 0, \\ \epsilon_{jkl}\partial_i B_l + \epsilon_{kil}\partial_j B_l + \epsilon_{ijl}\partial_k B_l &= 0. \end{aligned}$$

Contracting with ϵ^{ijk} and using the identity $\epsilon^{ijk}\epsilon^{ijl} = 2!\delta^k_l$, then

$$\begin{aligned} \epsilon^{ijk}\epsilon_{jkl}\partial_i B_l + \epsilon^{ijk}\epsilon_{kil}\partial_j B_l + \epsilon^{ijk}\epsilon_{ijl}\partial_k B_l &= 0, \\ (\epsilon^{jki}\epsilon_{jkl}\partial_i B_l + \epsilon^{kij}\epsilon_{kil}\partial_j B_l + \epsilon^{ijk}\epsilon_{ijl}\partial_k B_l) &= 0, \\ (2\delta^i_l\partial_i B_l + 2\delta^j_l\partial_j B_l + 2\delta^k_l\partial_k B_l) &= 0, \\ 6\partial_i B_i &= 0, \\ \partial_i B_i &= 0. \end{aligned} \tag{3.26}$$

We have seen that another form of Maxwell's equations are

$$\partial_\mu F^{\nu\mu} = \mu_0 j^\nu, \tag{3.27}$$

$$\partial_{[\rho} F_{\mu\nu]} = 0. \tag{3.28}$$

The 2-rank antisymmetric tensor $F^{\mu\nu}$ is called the electromagnetic stress-energy tensor or Maxwell stress tensor. Moreover, the dual tensor of $F_{\mu\nu}$ which is defined as

$$\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{3.29}$$

This dual tensor can be construct the same Maxwell's equations, $\partial_\mu \tilde{F}^{\nu\mu} = \mu_0 j^\nu$ and $\partial_{[\rho}\tilde{F}_{\mu\nu]} = 0$.

In other frame, the Maxwell stress tensor is transformed under the Lorentz transformation as

$$F^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}. \tag{3.30}$$

Notice that $F_{\mu\nu}F^{\mu\nu}$ is invariant under the Lorentz transformation.

In field theory, the Lagrangian density can be construct as follows

$$\mathcal{L} = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{3.31}$$

3.2 Black hole solution with charges

In order solve the Einstein equation analytically, we will restrict our attention in spherically static spacetime. Therefore, the left hand side of Einstein field equation is still the same. We can adopt the results from two-three previous talk. Our task now is to find the form of energy momentum tensor for charged object. Considering electric and magnetic charges, their energy momentum tensor can be written in terms of field strength tensor as

$$T_{\mu\nu} = \mu_0^{-1}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{3.32}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and A_μ are components of four-vector potential. This form of the energy momentum tensor can be obtained from description of quantum field theory. You will see explicitly how to obtain this energy momentum tensor later (properly in the chapter of filed theory for general relativity). Note that the overall factor μ_0^{-1} is obtained by comparing T_{00} with the energy density of the electromagnetic wave.

In order to obtain the energy momentum tensor satisfying spherical symmetry, the electric components E_θ, E_ϕ and magnetic components B_θ, B_ϕ must vanish. With using the spherical coordinates, the field strength tensor can be written as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_r/c & 0 & 0 \\ E_r/c & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^2\sin\theta B_r \\ 0 & 0 & r^2\sin\theta B_r & 0 \end{pmatrix}. \tag{3.33}$$

Factor $r^2 \sin\theta$ in front of B_r comes from the covariant components of field strength tensor in spherical coordinates,

$$F_{ij} = \epsilon_{ijk} B^k = \sqrt{g}\tilde{\epsilon}_{ijk} B^k \quad \Rightarrow \quad F_{\theta\phi} = r^2 \sin\theta \tilde{\epsilon}_{\theta\phi r} B_r = r^2 \sin\theta B_r. \tag{3.34}$$

Therefore, general form of the field strength tensor satisfying spherical symmetry can be written as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E(r)/c & 0 & 0 \\ E(r)/c & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^2 \sin\theta B(r) \\ 0 & 0 & r^2 \sin\theta B(r) & 0 \end{pmatrix}, \tag{3.35}$$

where $E(r)$ and $B(r)$ are now arbitrary well-defined function. Even though $E(r)$ and $B(r)$ are arbitrary functions, we expect that $E(r)$ will play the role of electric field and $B(r)$ will play the role of magnetic field. These functions are supposed to be solved by using the the equations of motion for EM-gauge field. It is important to note that, in physics we have learned, the usual magnetic field cannot be written in such form since we have not observed magnetic monopole yet. However, this is an instructive study of the geometry. It is just the toy model for theoretical study. We emphasis here also that our consideration does not seem to be realistic situation since usual stars are neutral which is not charged object. Thus we have considered the toy model from the beginning of our study. It is worthwhile to study charged body with magnetic monopoles.

For curved spacetime, the equation of motion for this field can be written as

$$\nabla_\mu F^{\mu\nu} = 0, \tag{3.36}$$
$$\nabla_\rho F_{\mu\nu} + \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} = 0. \tag{3.37}$$

Considering equation (3.36), we have

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho}, \\ &= \partial_\mu F^{\mu\nu} + F^{\rho\nu} \frac{1}{\sqrt{-g}} \partial_\rho \sqrt{-g} + \frac{1}{2} \Gamma^\nu_{\mu\rho} \left(F^{\mu\rho} - F^{\rho\mu} \right), \\ &= \frac{1}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g} F^{\rho\nu} \right) + \frac{1}{2} \left(\Gamma^\nu_{\mu\rho} F^{\mu\rho} - \Gamma^\nu_{\rho\mu} F^{\rho\mu} \right), \\ &= \frac{1}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g} F^{\rho\nu} \right). \end{aligned} \tag{3.38}$$

For $\nu = 0$, one obtains

$$\begin{aligned} \nabla_\mu F^{\mu 0} &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} F^{\mu 0} \right), \\ &= \frac{1}{e^{\alpha+\beta} r^2 \sin\theta} \partial_1 \left(e^{\alpha+\beta} r^2 \sin\theta F^{10} \right), \\ &= \frac{1}{e^{\alpha+\beta} r^2 \sin\theta} \partial_1 \left(e^{\alpha+\beta} r^2 g^{11} g^{00} \sin\theta F_{10} \right), \\ &= \frac{1}{c e^{\alpha+\beta} r^2} \partial_r \left(e^{-(\alpha+\beta)} r^2 E \right) = 0, \\ \therefore e^{-(\alpha+\beta)} r^2 E &= C, \quad \Rightarrow \quad E = \frac{C e^{(\alpha+\beta)}}{r^2}, \end{aligned} \tag{3.39}$$

where C is a constant. This constant can be obtained by considering the metric with large r. For large r, the metric becomes to be Minkowski and the function must reduce to the electric field of a point charge as follow

$$E = \frac{Q}{4\pi\epsilon_0 c^2 r^2} = \frac{C}{r^2}, \quad \Rightarrow \quad C = \frac{Q}{4\pi\epsilon_0}. \tag{3.40}$$

Therefore, the function E representing the electric field of a charged object can be written as

$$E = \frac{Q}{4\pi\epsilon_0} \frac{e^{(\alpha+\beta)}}{r^2}. \tag{3.41}$$

In order to find the form of B, let us consider the second equation of motion with component $(\rho,\mu,\nu)=(r,\theta,\phi)$. Therefore, equation (3.37) becomes

$$\begin{aligned}\nabla_\rho F_{\mu\nu}+\nabla_\mu F_{\nu\rho}+\nabla_\nu F_{\rho\mu} &= 0,\\ \partial_\rho F_{\mu\nu}+\partial_\mu F_{\nu\rho}+\partial_\nu F_{\rho\mu} &= 0,\\ \partial_r F_{\theta\phi} &= 0,\\ \partial_r(r^2\sin\theta B) &= 0, \quad\Rightarrow\quad B=\frac{C}{r^2},\end{aligned}\tag{3.42}$$

where C is a constant. In the same fashion with finding the constant of the electric field, the constant C can be written as

$$C=\frac{\mu_0 P}{4\pi},\tag{3.43}$$

where P is a source representing the magnetic monopole. Therefore, the function B representing the magnetic field of a point magnetic monopole can be written as

$$B=\frac{\mu_0 P}{4\pi}\frac{1}{r^2}.\tag{3.44}$$

Now we will return to consider the energy momentum tensor of the charged object in equation (3.32). Considering the second term in equation (3.32), one has

$$\begin{aligned}F_{\rho\sigma}F^{\rho\sigma} &= g^{\rho\rho'}g^{\sigma\sigma'}F_{\rho\sigma}F_{\rho'\sigma'},\\ &= 2g^{00}g^{11}F_{01}F_{01}+2g^{22}g^{33}F_{23}F_{23},\\ &= 2\left(-e^{-2(\alpha+\beta)}E^2/c^2+B^2\right).\end{aligned}\tag{3.45}$$

By using this relation, component $(0,0)$ reads

$$\begin{aligned}T_{00} &= \mu_0^{-1}\left(F_{0\rho}F_0{}^\rho-\frac{1}{4}g_{00}F_{\rho\sigma}F^{\rho\sigma}\right),\\ &= \mu_0^{-1}\left(e^{-2\beta}E^2/c^2+\frac{1}{2}e^{2\alpha}\left(-e^{-2(\alpha+\beta)}E^2/c^2+B^2\right)\right),\\ &= \frac{\mu_0^{-1}}{2}e^{2\alpha}\left(e^{-2(\alpha+\beta)}E^2/c^2+B^2\right),\\ \therefore T^0_0 &= g^{00}T_{00}=-\frac{\mu_0^{-1}}{2}\left(e^{-2(\alpha+\beta)}E^2/c^2+B^2\right).\end{aligned}\tag{3.46}$$

Using the same procedure calculation, other non-zero components of the energy momentum tensor can be written as

$$T^1_1 = T^0_0=-\frac{\mu_0^{-1}}{2}\left(e^{-2(\alpha+\beta)}E^2/c^2+B^2\right),\tag{3.47}$$

$$T^2_2 = T^3_3=\frac{\mu_0^{-1}}{2}\left(e^{-2(\alpha+\beta)}E^2/c^2+B^2\right).\tag{3.48}$$

Substituting these components into Einstein field equation and using the results of Einstein tensor in equations one obtains

$$2r\beta'e^{-2\beta}-e^{-2\beta}+1 = \frac{4\pi G}{c^4\mu_0}r^2\left(e^{-2(\alpha+\beta)}E^2/c^2+B^2\right)=\frac{q^2+p^2}{r^2},\tag{3.49}$$

$$2r\alpha'e^{-2\beta}-e^{-2\beta}+1 = \frac{4\pi G}{c^4\mu_0}r^2\left(e^{-2(\alpha+\beta)}E^2/c^2+B^2\right)=\frac{q^2+p^2}{r^2},\tag{3.50}$$

$$r\alpha'^2-r\beta'\alpha'+r\alpha''+(\alpha'-\beta') = \frac{4\pi G}{c^4\mu_0}e^{2\beta}r\left(e^{-2(\alpha+\beta)}E^2/c^2+B^2\right)=\frac{q^2+p^2}{r^3}e^{2\beta},\tag{3.51}$$

where

$$q^2 = \frac{GQ^2}{4\pi\epsilon_0 c^4}, \quad p^2 = \frac{GP^2}{4\pi\epsilon_0 c^6}. \tag{3.52}$$

Combining equations (3.49) and (3.50), one obtains

$$\alpha' + \beta' = 0, \;\Rightarrow\; \alpha + \beta = \text{constant} = 0, \;\Rightarrow\; \alpha = -\beta \tag{3.53}$$

where we have used the asymptotic Minkowski metric as $r \to \infty$ to obtain constant $= 0$. From equation (3.49), one can solve this equation for B as follow

$$\begin{aligned}
2r\beta' e^{-2\beta} - e^{-2\beta} + 1 &= \frac{q^2 + p^2}{r^2}, \\
-\frac{d}{dr}\left(re^{-2\beta}\right) + 1 &= \frac{q^2 + p^2}{r^2}, \\
\frac{d}{dr}\left(re^{-2\beta}\right) &= 1 - \frac{q^2 + p^2}{r^2}, \\
re^{-2\beta} &= r + \frac{q^2 + p^2}{r} + C, \\
e^{-2\beta} &= 1 + \frac{q^2 + p^2}{r^2} + \frac{C}{r}, \\
e^{-2\beta} &= 1 + \frac{q^2 + p^2}{r^2} - \frac{2\mu}{r},
\end{aligned} \tag{3.54}$$

where we have used the condition in which the metric must recover the Schwarzschild metric as q and p vanish to obtain $C = -2\mu$. Therefore, the solution can be expressed as

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2\mu}{r} + \frac{q^2 + p^2}{r^2}\right) c^2 dt^2 + \left(1 - \frac{2\mu}{r} + \frac{q^2 + p^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \\
&= -\left(1 - \frac{2\mu}{r} + \frac{q_{em}^2}{r^2}\right) c^2 dt^2 + \left(1 - \frac{2\mu}{r} + \frac{q_{em}^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \\
&= -\Delta(r)\, c^2 dt^2 + \Delta(r)^{-1} dr^2 + r^2 d\Omega^2.
\end{aligned} \tag{3.55}$$

For simplicity we have set $q_{em}^2 = q^2 + p^2$. Without electric and magnetic charges corresponding to $q_{em} = 0$, this metric is reduced to Schwarzschild metric. The intrinsic singularity still take place at $r = 0$. This singularity can be inferred from the curvature square $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ as the same argument discussed in the Schwarzschild case. The coordinate singularity will modify by changing

$$1 - \frac{2\mu}{r} = 0 \quad \Rightarrow \quad 1 - \frac{2\mu}{r} + \frac{q_{em}^2}{r^2} = 0. \tag{3.56}$$

Therefore, the maximum number of the singularity points is extended from one to two. This possible two points can be found by solving the above equation as follow

$$r^2 - 2\mu r + q_{em}^2 = 0 \quad \Rightarrow \quad r = r_\pm = \mu \pm \left(\mu^2 - q_{em}^2\right)^{1/2}. \tag{3.57}$$

It is convenient to separate our analysis into three cases; $\mu^2 < q_{em}^2$, $\mu^2 > q_{em}^2$ and $\mu^2 = q_{em}^2$.

3.3 Case 1: $\mu^2 < q_{em}^2$

From the equation (3.57), the singularity points, $r_\pm$, will be imaginary. This implies that there are no coordinate singularities in this case. Since there are no event horizons, the coordinate t is still the timelike coordinate in all points of the spacetime inferring also from the fact that $\Delta(r)$ does not change its sign. The importantly consequent result is that one can travel from asymptotically flat region to the intrinsic singularity point, $r = 0$, and then return back to asymptotically flat region again.

An intrinsic singularity without covering of the coordinate singularity like this case is called "naked singularity". In physics, the information at an initial event will provide influence of the information other future timelike events. This means that the physical information occurring at the singularity point will affect the physics in the asymptotically flat region. However, at singularity point, physics is not well-defined. Therefore, all results at asymptotically flat region receiving information from singularity point are not well-defined. This situation will contradict with our traveling since we can come back from the singularity point as mentioned above. We expect that quantum gravity theory will solve this problem of classical general relativity. One of possible arguments to avoid this problem is suggested by Penrose which is known as "cosmic censorship conjecture (CCC)"; "naked singularities cannot form in gravitational collapse from generic, initially nonsingular state in an asymptotically flat spacetime obeying the dominant energy condition". Note that the CCC does not imply that the naked singularity cannot exist, it just provide that the naked singularity cannot be formed. Even though CCC is just a conjecture which has not been proof yet, the great effort to find the convincing counterexamples is not successive. Until now, the precise proof of this conjecture is one of the outstanding problems of general relativity.

One of simple example of the star in this case is that the star forming by collecting only huge amount of electrons. It is clear that $q_{em} > \mu$ since electron mass is very tiny. However, it is well-known that coulomb force is repulsive and much strength than gravitational force. In this sense, it is impossible to form the star belonging to this case. This simple thinking is one of the examples sporting the argument of the cosmic censorship conjecture. Note that this situation is not really concise argument since collecting electrons to form the star may use the effect of quantum theory.

To understand more about the formation of this kind of the star, let us compare the result with the Schwarzschild one as follow

$$\Delta(r) = \left(1 - \frac{2\mu}{r} + \frac{q_{em}^2}{r^2}\right) = 1 - \frac{2G}{c^2 r}\left(m - \frac{c^2 q_{em}^2}{2Gr}\right) = 1 - \frac{2G}{c^2 r} m_{eff}(r). \tag{3.58}$$

Now we have Schwarzschild-like metric with effective mass m_{eff} depending on r. We can see that, in the case of large q_{em}^2, the effective mass will becomes negative. This corresponds to repulsive gravitational force which break the mechanism to form the star. Moreover, in quantum theory, the negative mass corresponds to the unstable state and may lead to an unphysical state.

3.4 Case 2: $\mu^2 > q_{em}^2$

From the equation (3.57), there exist two real solutions for coordinate singularity. At large r, the metric tends to recover Minkowski metric. A particle radially fall inward from this region will encounter the coordinate singularity point r_+. Since the sign $\Delta(r)$ is changed, crossing this horizon, the coordinates t and r will change their role similar to Schwarzschild case. The coordinate r becomes timelike coordinate and the particle is forced to move in the direction with decreasing radial. This implies that the observer at rest far away from back hole will experience the same situation with the observer from Schwarzschild case. However, the particle motion will change when the particle reaches the second horizon, r_-. It is found that $\Delta(r)$ will change its sign when $r = r_-$. This means that the coordinate t becomes to be the timelike coordinate and r will be the spacelike coordinate again. Therefore, particle inside the radius r_- will be forced to move in the direction with increasing time t. Therefore, particle now can decide to move inward to the singularity point or move outward to asymptotic region. In the case of outward moving, the particle will move back to r_- and the coordinate t and r will change their role again. The coordinate r becomes timelike coordinate. The particle is unavoidably move in the direction with increasing radial. Then particle will reach the point with radius r_+ and the coordinates t and r will change their roles again. Outside the outer horizon, the coordinate t becomes timelike coordinate again. At this point, particle can decide to go back into the black hole or go to the asymptotic region. However, this asymptotic region is other regions at which it came from in the previous one. In the case of going into the black hole, particle may repeat its traveling as many times. However, the black holes and the asymptotic regions are different.

3.5 Case 3: $\mu^2 = q_{em}^2$

From the equation (3.57), there exists only one real solution for coordinate singularity. It can imagine that it is a special case of the case $\mu^2 > q_{em}^2$ in the sense that the outer and inner horizon becomes the same value $r = r_+ = r_- = \mu$. Comparing to case 2, the region between the inner and outer horizon will disappear. Therefore, there is no region for r to be a timelike coordinate. In other words, t is always timelike coordinate except at $r = \mu$, both t and r becomes null coordinates. This case is known as "extreme Reissner-Nordstrom black hole". It is found that the timelike particle motion is similar to case 2. The difference is that there are no regions between r_- and r_+. The black hole tunnels in which the particle can move from our world to other worlds also exist in this case. It is important to note that this kind of black hole is very famous among theoretical toy models. The reason may come from the simplicity of calculation while the phenomenon is still similar to the complicated case. Moreover, it is also compatible with supersymmetric theories where it may exist when supersymmetries are unbroken. However, it not easy to exist in the realistic situation since adding a little bit mass will lead to another case. This means that this configuration of black hole is not stable.

Note: for RN geometry, there exist the black hole tunnels which one can move from our world to other worlds as found in conformal diagram. However, it is found that these tunnels are very sensitive to the static assumption and spherical symmetry we impose. If we move into the black hole, it means that we perturb the assumption and symmetry. This will destroy the structure of the tunnel and make it unstable. Therefore, the tunnel cannot exist in the realistic situation.

As we mentioned before, the conformal diagram provides us that the particle can move inward to the real singularity point at $r = 0$. One may ask that how much energy of the particle to take itself to this point. In order to obtain the answer, let us consider the effective potential of a radial motion of particle. By playing in the same way with we have done in Schwarzschild case, we have

$$\frac{1}{2}\dot{r}^2 + \left(\frac{q_{em}^2}{2r^2} - \frac{\mu}{r}\right) = \frac{c^2(k^2-1)}{2}, \quad \Rightarrow \quad V_{eff} = \frac{q_{em}^2}{2r^2} - \frac{\mu}{r}. \tag{3.59}$$

Note that this effective potential can be used for all cases of our consideration. The potential can be sketched in three cases with parameters $q_{em}^2 = \mu^2/2, q_{em}^2 = \mu^2, q_{em}^2 = 2\mu^2$ as shown in figure 3.2.

From this effective potential, it is found that, in the region $r < r_- < r_{min}$ where $r_{min} = \mu/2, \mu, 2\mu$ respectively, the slope of the potential is negative. This means that the effective force acting on the particle will be repulsive. This behavior can be also inferred from the negative effective mass as r small in equation (3.58). The value of effective potential becomes infinity as $r \to 0$. Therefore, the energy of the particle must be infinity in order to move to the real singularity. In other words, it is impossible to move to the real singularity.

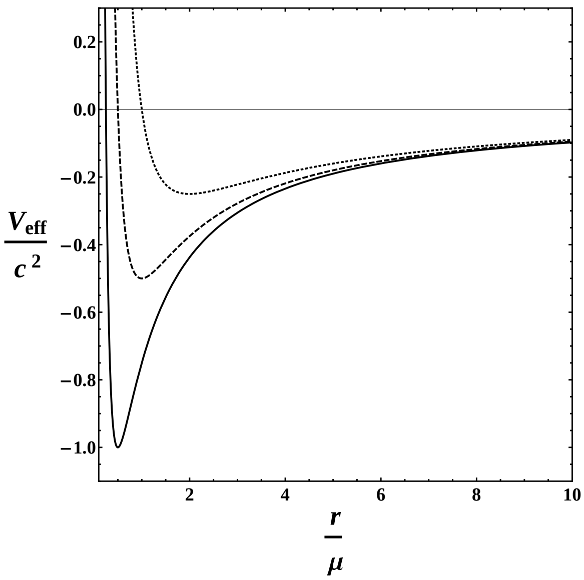

Figure 3.2: Effective potential for radial motion of massive particle in three cases. The solid, dashed and dotted lines represent the plot with parameters $q_{em}^2 = \mu^2/2$, $q_{em}^2 = \mu^2$, and $q_{em}^2 = 2\mu^2$ corresponding to case1, case2 and case3 respectively.