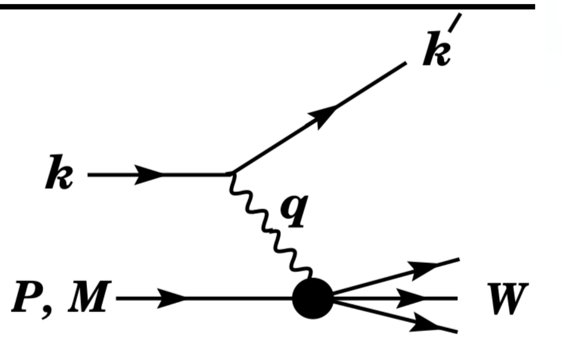
Hadronic tensor on the lattice: idea, status and prospect

Jian Liang, Keh-Fei Liu, Yi-Bo Yang and Terry Draper

χQCD collaboration

09/26/2019 PDFLattice2019 @ MSU Kellogg Biological Station

Hadronic tensor on the lattice

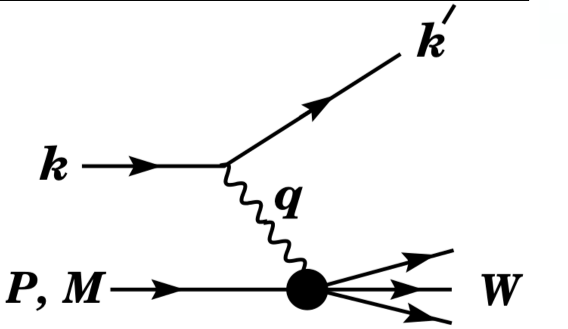


Minkowski $$W_{\mu\nu}\left(p,\vec{q},\nu\right)=\frac{1}{4\pi}\int d^4z e^{iq\cdot z}\left\langle p,s\left|\left[J^\dagger_\mu(z)J_\nu(0)\right]\right|p,s\right\rangle$$

$$=\frac{1}{2}\sum_n\int\prod_i^n\left[\frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}\right]\langle p,s|J^\dagger_\mu(0)|n\rangle\langle n|J_\nu(0)|p,s\rangle(2\pi)^3\boldsymbol{\delta^4(q-p_n+p)}$$

$$\boldsymbol{W_{\mu\nu}}=\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\boldsymbol{F_1(x,Q^2)}+\frac{\hat{P}_\mu\hat{P}_\nu}{P\cdot q}\boldsymbol{F_2(x,Q^2)}$$

Hadronic tensor on the lattice



Minkowski $$W_{\mu\nu}\left(p,\vec{q},\nu\right)=\frac{1}{4\pi}\int d^4z e^{iq\cdot z}\left\langle p,s\left|\left[J_\mu^\dagger(z)J_\nu(0)\right]\right|p,s\right\rangle$$

$$=\frac{1}{2}\sum_n\int\prod_i^n\left[\frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}\right]\langle p,s|J_\mu^\dagger(0)|n\rangle\langle n|J_\nu(0)|p,s\rangle(2\pi)^3\delta^4(q-p_n+p)$$

$$W_{\mu\nu}=\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)F_1(x,Q^2)+\frac{\hat{P}_\mu\hat{P}_\nu}{P\cdot q}F_2(x,Q^2)$$

Lattice correlators are all with **Euclidean time**; time dependent matrix elements are problematic.

Euclidean $$W'_{\mu\nu}\left(p,\vec{q},\nu\right)=\frac{1}{4\pi}\sum_n\int dt e^{\left(\nu-\left(E_n-E_p\right)\right)t}\int d^3\vec{z}e^{i\vec{q}\cdot\vec{z}}\langle p,s|J_\mu^\dagger(\vec{z})|n\rangle\langle n|J_\nu(0)|p,s\rangle$$

$$=\frac{1}{4\pi}\sum_n\frac{e^{\left(\nu-\left(E_n-E_p\right)\right)T}-1}{\nu-(E_n-E_p)}\int d^3\vec{z}e^{i\vec{q}\cdot\vec{z}}\langle p,s|J_\mu^\dagger(\vec{z})|n\rangle\langle n|J_\nu(0)|p,s\rangle$$

A simple change from **Fourier transform** to **Laplace transform** in the time direction **leads to divergences** when $\nu-(E_n-E_p)>0$.

Hadronic tensor on the lattice

four-point function with **3-dimensional Fourier transform**

$$C_4 = \sum_{\vec{x}_f} e^{-i\vec{p}\cdot\vec{x}_f} \sum_{\vec{x}_2\vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)} \left\langle \chi_N(\vec{x}_f, t_f) J^{\dagger}_{\mu}(\vec{x}_2, t_2) J_{\nu}(\vec{x}_1, t_1) \bar{\chi}_N(\vec{0}, t_0) \right\rangle$$

Euclidean hadronic tensor defined as a function of time difference between the currents

$$\tilde{W}_{\mu\nu}(p, \vec{q}, \tau) = \frac{E_p}{m_N} \frac{\mathrm{Tr}[\Gamma_e C_4]}{\mathrm{Tr}[\Gamma_e C_2]} \to \sum_{\vec{x}_2\vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)} \langle p, s \,|\, J_{\mu}(\vec{x}_2, t_2) J_{\nu}(\vec{x}_1, t_1) \,|\, p, s \rangle$$

$$= \sum_{n} A_n e^{-(E_n - E_p)\tau}, \; \tau \equiv t_2 - t_1$$

Solving the **inverse problem** of a Laplace transform to get back to Minkowski space

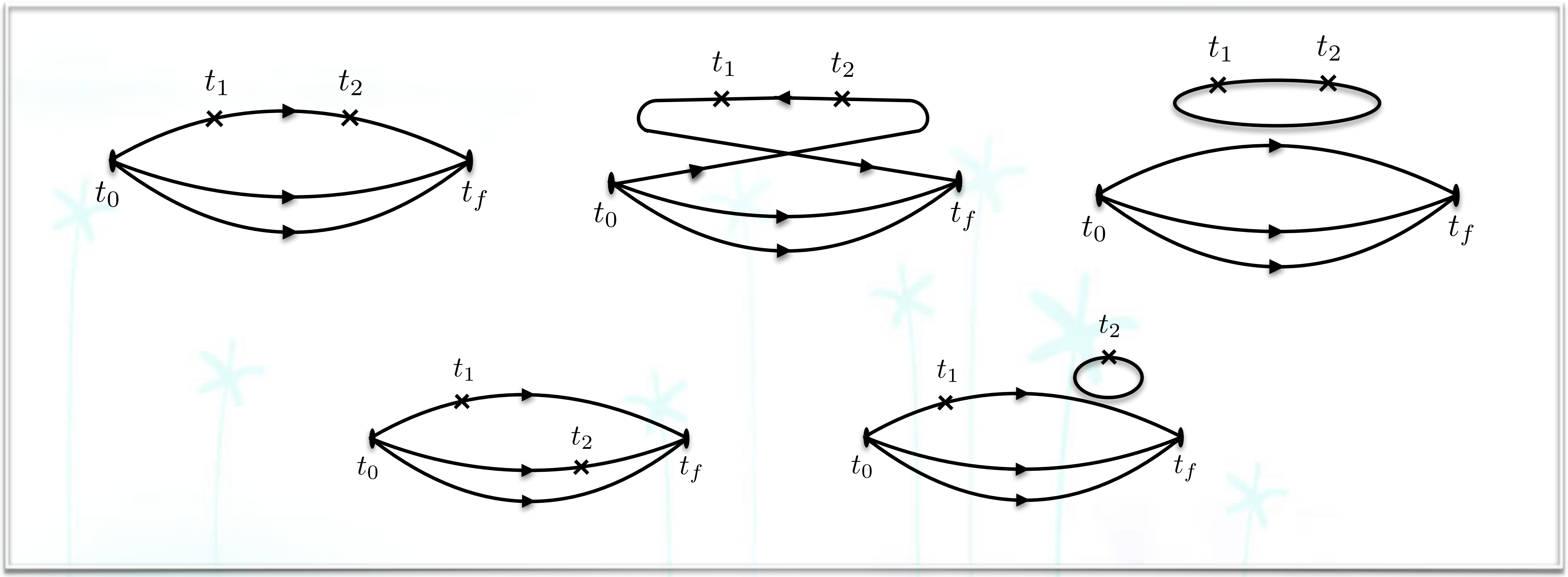
$$\tilde{W}_{\mu\nu}\left(p, \vec{q}, \tau\right) = \int d\nu W_{\mu\nu}\left(p, \vec{q}, \nu\right) e^{-\nu\tau}$$

1. No renormalization
2. Structure functions frame independent
(no need to have highly boosted nucleon)
3. Extract PDFs just as from experimental results
(no new matching)

K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)
K.-F. Liu, PRD62, 074501 (2000)
J. Liang et. al., EPJ Web Conf. 175, 14014 (2018)
J. Liang et. al., arXiv:1906.05312

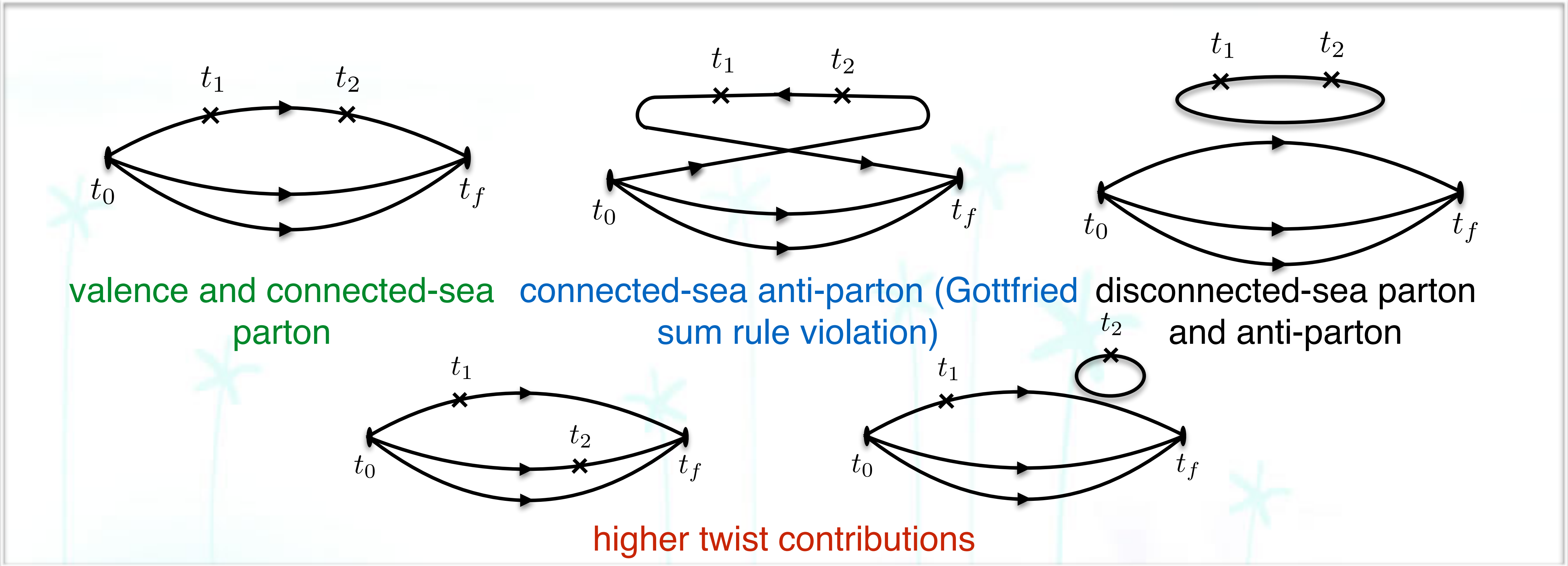
Topologically distinct contributions

$$C_4 = \sum_{\vec{x}_f} e^{-i\vec{p}\cdot\vec{x}_f} \sum_{\vec{x}_2\vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)} \left\langle \chi_N(\vec{x}_f, t_f) J^\dagger_\mu(\vec{x}_2, t_2) J_\nu(\vec{x}_1, t_1) \bar{\chi}_N(\vec{0}, t_0) \right\rangle$$



Topologically distinct contributions

$$C_4 = \sum_{\vec{x}_f} e^{-i\vec{p}\cdot\vec{x}_f} \sum_{\vec{x}_2\vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)} \left\langle \chi_N(\vec{x}_f, t_f) J_\mu^\dagger(\vec{x}_2, t_2) J_\nu(\vec{x}_1, t_1) \bar{\chi}_N(\vec{0}, t_0) \right\rangle$$



K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

1. More d.o.f.'s in global fittings? Better understand sea partons? (arXiv:1901.07526)

2. How higher twist contributions change with increasing the momentum transfer.

Kinematics

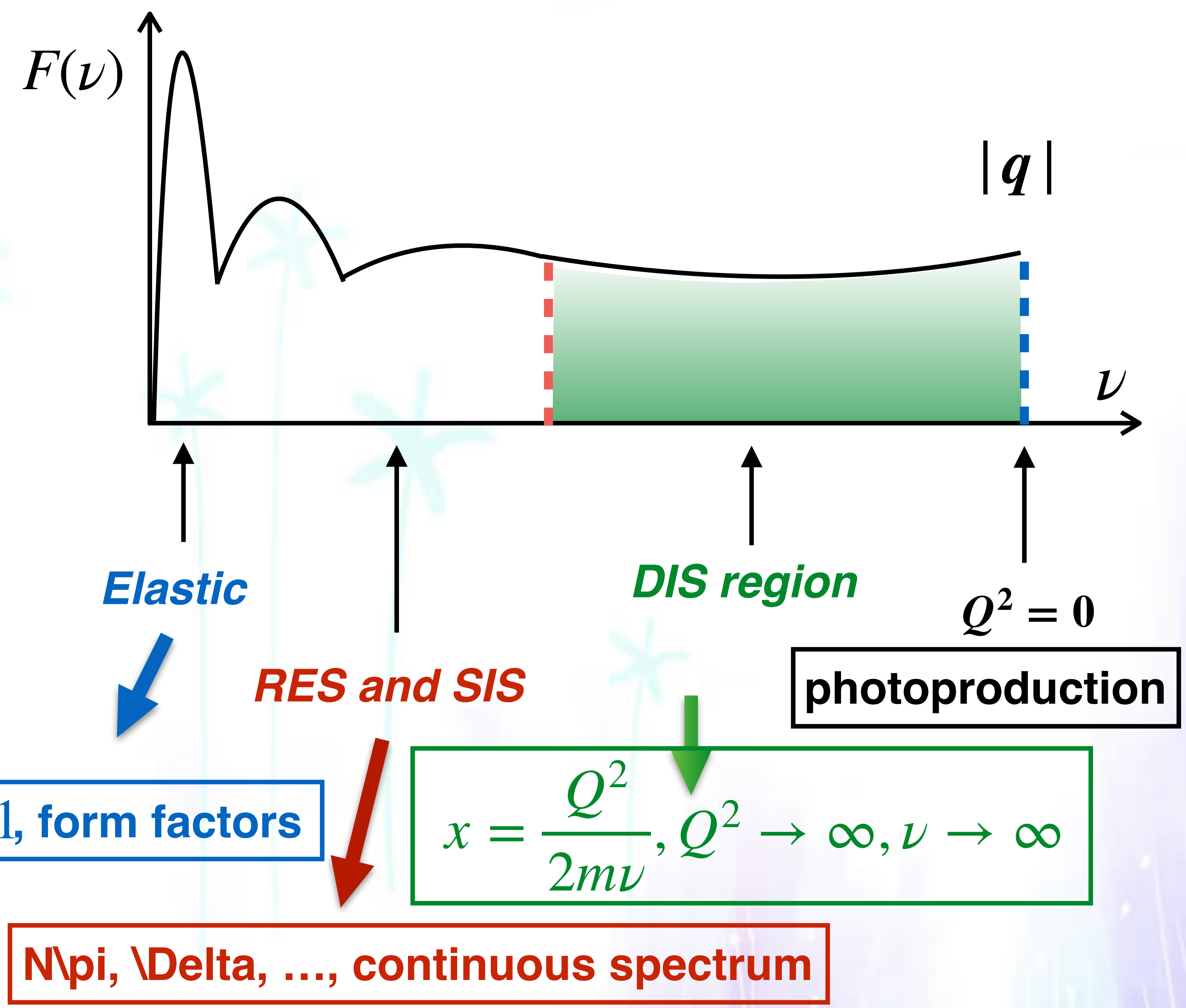
$$\tilde{W}_{\mu\nu}(p,\vec{q},\tau) = \frac{E_p}{m_N}\frac{\mathrm{Tr}[\Gamma_e C_4]}{\mathrm{Tr}[\Gamma_e C_2]} \to \sum_{\vec{x}_2\vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)}\langle p,s|J_\mu(\vec{x}_2,t_2)J_\nu(\vec{x}_1,t_1)|p,s\rangle$$

$$\tilde{\boldsymbol{W}}_{\mu\nu}\left(\boldsymbol{p},\vec{\boldsymbol{q}},\boldsymbol{\tau}\right) = \int d\nu \boldsymbol{W}_{\mu\nu}\left(\boldsymbol{p},\vec{\boldsymbol{q}},\boldsymbol{\nu}\right)e^{-\nu\tau}$$

$$Q^2 = -q^2 = |\vec{q}|^2 - \nu^2$$

$$x = \frac{Q^2}{2m\nu}$$

$$W^2 = m^2 + 2m\nu - Q^2$$

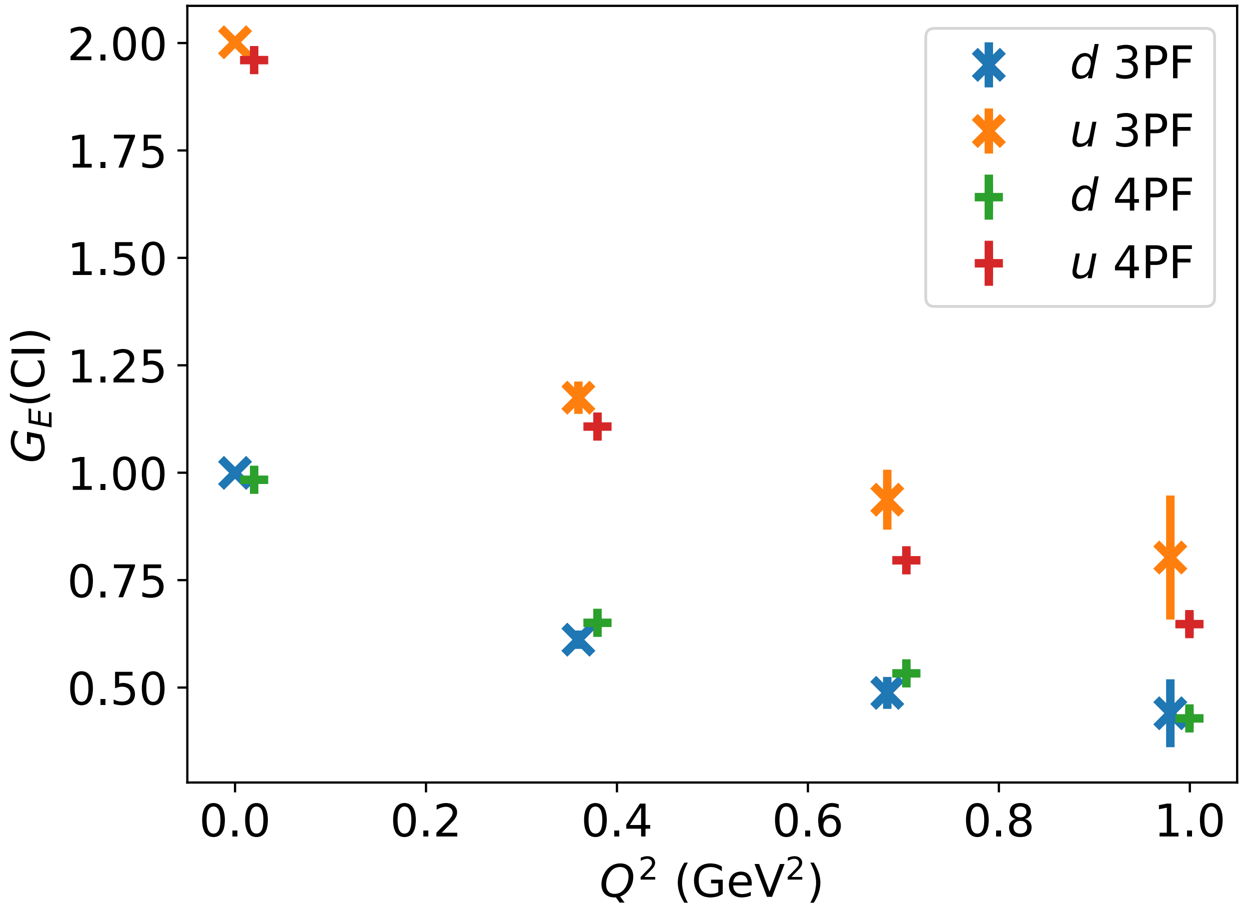


1. Both Q^2 and ν need to be large (difficulty?)
2. Will have a range of ν (feature?)

Elastic FFs

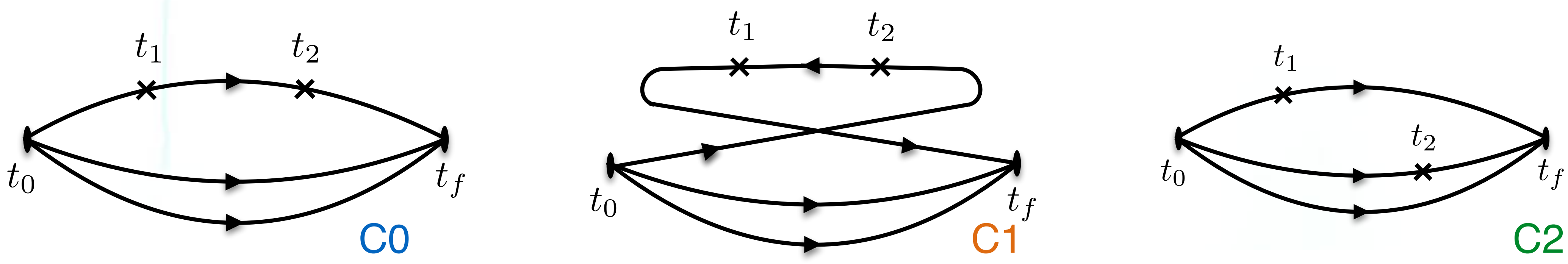
$$\tilde{W}_{44}(\vec{p}, \vec{q}, \tau) = \sum_{\vec{x}_2 \vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2 - \vec{x}_1)} \langle p, s | J_\mu(\vec{x}_2, t_2) J_\nu(\vec{x}_1, t_1) | p, s \rangle = \sum_n A_n e^{-\nu_n \tau}$$

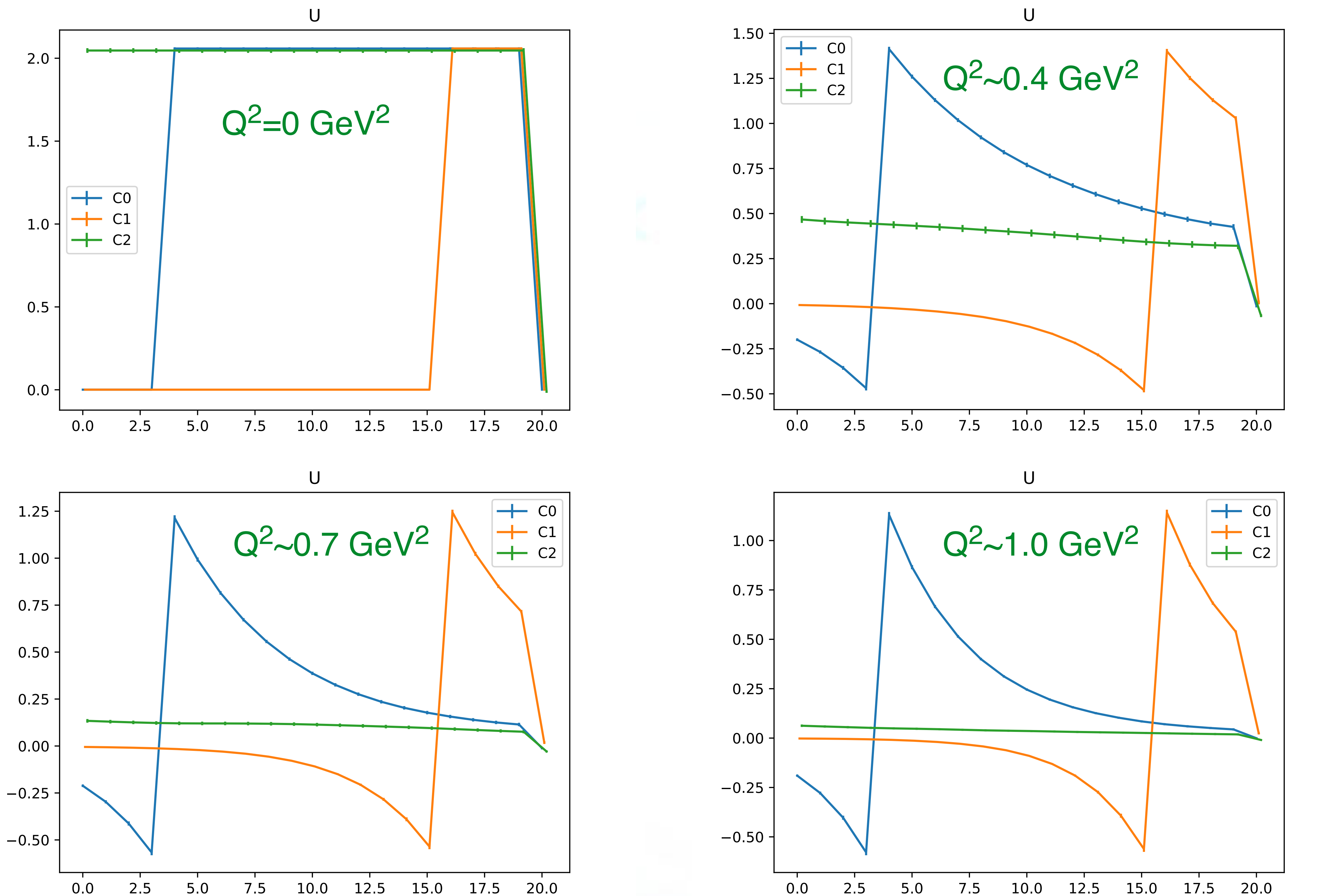
$$A_0 = \langle p, s | J_4(\vec{q}) | n = 0 \rangle \langle n = 0 | J_4(-\vec{q}) | p, s \rangle = G_E^2(Q^2)$$



FFs extracted from 3-point functions and 4-point functions show consistency.
(32IF lattice, a ~0.063 fm, pion mass ~370 MeV)

Higher twist contribution

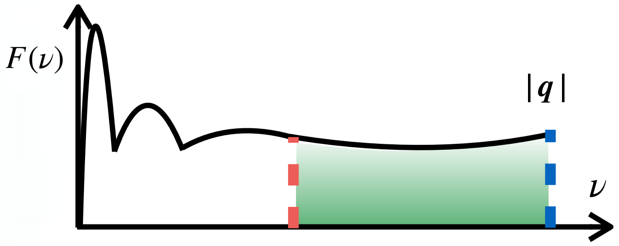


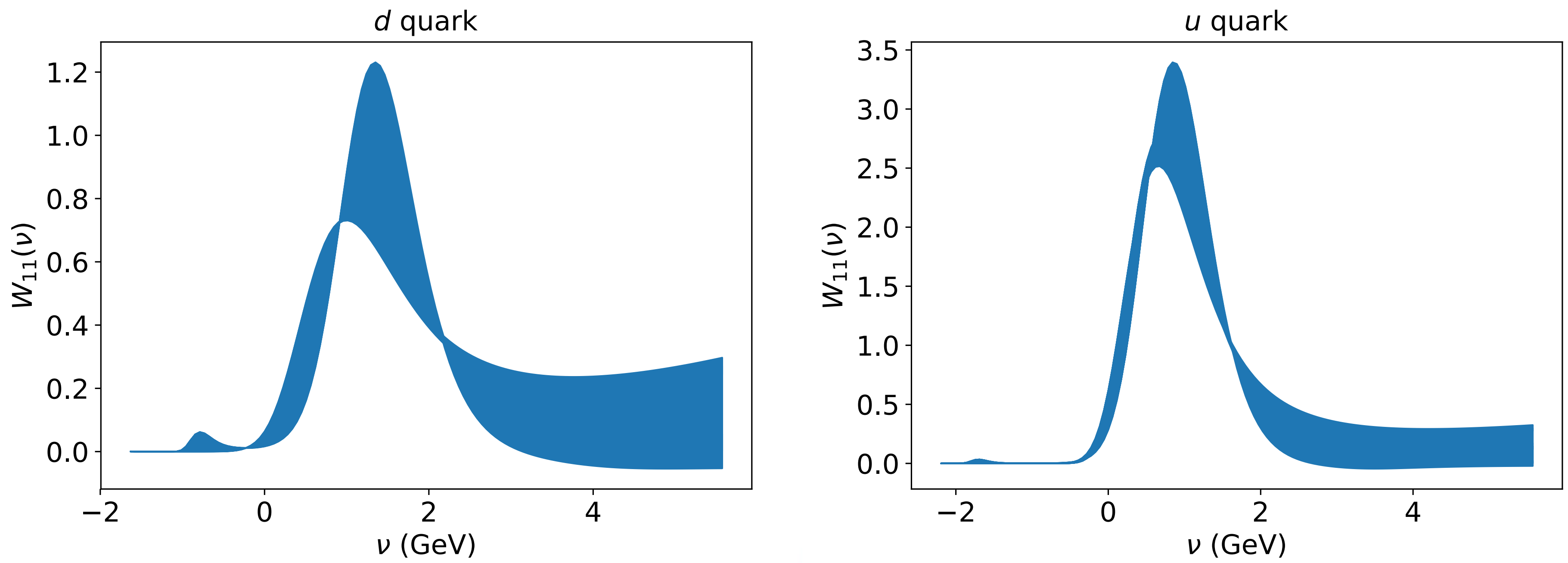


Hadronic tensor with large momentum transfer

clover anisotropic lattice, $24^3 \times 128,\ a_t \sim 0.035\ \mathrm{fm},\ \xi = 3.7,\ m_\pi \sim 380\ \mathrm{MeV},\ \frac{2\pi}{L} \sim 0.42\ \mathrm{GeV}$

$|\vec{p}| \sim 1.78\ \mathrm{GeV} \quad |\vec{q}| \sim 3.57\ \mathrm{GeV} \quad \nu \sim 3\ \mathrm{GeV} \quad Q^2 \sim 3\ \mathrm{GeV}^2 \quad x \sim 0.1$





1. Elastic contribution is suppressed by the large momentum transfer.
2. RES contribution at ~ 1 GeV is large and relatively stable.
3. Large error in the SIS and DIS region (> 2 GeV), no enough constraint from the data.

Smaller lattice spacings (< 0.04 fm) are essential to have larger energy transfers.

J. Liang et. al., arXiv:1906.05312

Summary and prospect

1. Calculating the hadronic tensor on the lattice is one way (potentially a good one) to study parton physics from first principle.

2. The inverse problem is a problem, but a common problem.

3. We are working on lattices with small lattice spacing (HISQ ~0.04 fm, quenched anisotropic lattice ~0.02 fm) to check more systematically the "requirement" for our method to work well and to get some results can be compared with experiments.

4. Pion/kaon case might be easier since we don't need to boost the pion/kaon too much.

5. Can be generalized to GPDs: $W_{\mu\nu}\left(p, p', \vec{q}, \nu\right) = \frac{1}{4\pi}\int d^4 z e^{iq\cdot z}\left\langle p, s \left| \left[J_\mu^\dagger(z) J_\nu(0)\right] \right| p', s\right\rangle$

Summary and prospect

1. Calculating the hadronic tensor on the lattice is one way (potentially a good one) to study parton physics from first principle.

2. The inverse problem is a problem, but a common problem.

3. We are working on lattices with small lattice spacing (HISQ ~0.04 fm, quenched anisotropic lattice ~0.02 fm) to check more systematically the “requirement” for our method to work well and to get some results can be compared with experiments.

4. Pion/kaon case might be easier since we don’t need to boost the pion/kaon too much.

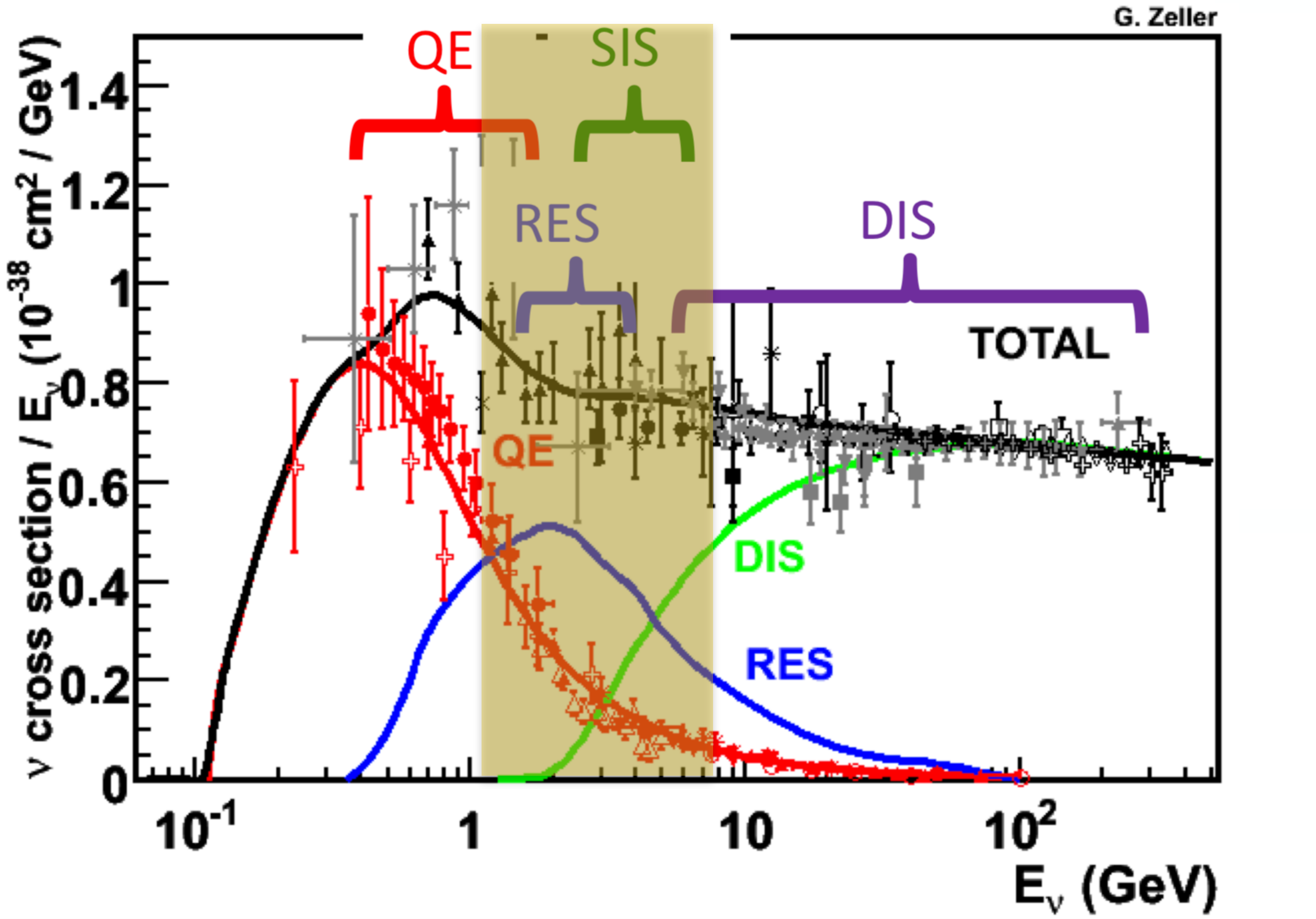
5. Can be generalized to GPDs: $W_{\mu\nu}\left(p, p', \vec{q}, \nu\right) = \frac{1}{4\pi}\int d^4 z e^{iq\cdot z}\left\langle p, s \left| \left[J_\mu^\dagger(z) J_\nu(0)\right] \right| p', s\right\rangle$

Thank you

Backups

Motivation 2: neutrino-nucleus scattering

- one of the most important tasks in High Energy Physics is to understand the properties of neutrinos.
- DUNE@LBNF FERMILAB with neutrino energy ~1-~7 GeV



J.A. Formaggio and G.P. Zeller, RMP84, 1307 (2012); Teppei Katori's talk

- $\nu A \to \nu N$, theoretical input about nucleon structure is needed to help map out the original neutrino beam energy and flux.
- For elastic contribution, nucleon FFs can be calculated by lattice or models.
- But soon enough, one will not be able to tell one state from another and will need INCLUSIVE hadron tensor (the resonance and shallow inelastic scattering (SIS) region).
- The only way that lattice QCD can help as far as we know.

More about the inverse problem

A general form

$$c(\tau) = \int k(\tau, \nu)\omega(\nu)d\nu$$

where it is hard to have the inverse function. In our case, the Laplace kernel

$$k(\tau, \nu) = e^{-\nu\tau}$$

$\omega(\nu)$ is a continuous function, but numerically we can discretize it.

$$c(\tau) = \sum_{\nu} k(\tau, \nu)\omega(\nu)\Delta\nu$$

If the number of nu is less than the number of tau, chi square fitting

If the number of nu is equal to the number of tau, linear equations

If the number of nu is larger than the number of tau, no unique solution

plug in Bayesian prior information?

Inverse problems are everywhere

✦ Extracting spectral functions from lattice data

✦ Global fittings of PDFs

✦ Lattice calculation of Quasi-PDFs

$$\tilde{q}_\Gamma(x, P_z, \tilde{\mu}) = \int_{-\infty}^{\infty} \frac{dz}{2\pi}\, e^{ixP_z z} \langle P | O_\Gamma(z) | P \rangle$$

$$\tilde{q}(x, P_z, p_z^R, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|}\, C\left(\frac{x}{y}, r, \frac{yP_z}{\mu}, \frac{yP_z}{p_z^R}\right) q(y, \mu)$$

H.-W. Lin et.al., PRL121, 242003(2018)

✦ Lattice calculation of Pseudo-PDFs

$$\mathfrak{M}_R(\nu, \mu^2) \equiv \int_0^1 dx\, \cos(\nu x)\, q_v(x, \mu^2)$$

K. Orginos et al., PRD96, 094503 (2017)

✦ Lattice cross sections

$$\sigma_n(\omega, \xi^2, P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x, \mu^2) \times K_n^a(x\omega, \xi^2, x^2P^2, \mu^2) + O(\xi^2 \Lambda_{\rm QCD}^2)$$

Y.-Q. Ma and J.-W. Qiu, PRL120, 022003 (2018)

Solving the inverse problem

$$c(\tau) = \sum_{\nu} k(\tau, \nu)\omega(\nu)\Delta\nu$$

✦ **Backus-Gilbert (BG)**

G. Backus and F. Gilbert, Geophysical Journal International 16, 169 (1968)

If the kernels can span a complete function basis

$$\sum_{\tau} c(\tau, \nu_0)k(\tau, \nu) \sim \delta(\nu - \nu_0)$$

$$\sum_{\tau} c(\tau, \nu_0)c(\tau) \sim \int \delta(\nu - \nu_0)\omega(\nu)d\nu = \omega(\nu_0)$$

The actual incompleteness of the kernels leads to bad resolution.

✦ **Maximum Entropy (ME)**

E Rietsch et. al., JOURNAL OF GEOPHYSICS, 42:489 (1977)

M. Asakawa et. al., Prog. Part. Nucl. Phys. 46, 459 (2001)

$$P[\omega | D, \alpha, m] \propto \frac{1}{Z_S Z_L} e^{Q(\omega)} \qquad Q = \alpha S - L$$

$$S = \sum_{\nu} \left[\omega(\nu) - m(\nu) - \omega(\nu)\log\left(\frac{\omega(\nu)}{m(\nu)}\right)\right] \Delta\nu$$

Maximum search is using SVD in a reduced parameter space ($O(10^1)$).

Hyper parameter alpha is averaged over based on assumptions.

Solving the inverse problem

$$c(\tau) = \sum_{\nu} k(\tau, \nu)\omega(\nu)\Delta\nu$$

✦ **Bayesian Reconstruction (BR)**

Y. Burnier and A. Rothkopf, PRL 111, 182003 (2013)

$$P[\omega | D, \alpha, m] \propto e^{Q'(\omega)} \quad Q' = \alpha S - L - \gamma(L - N_\tau)^2$$

No over fitting

$$S = \sum_{\nu} \left[1 - \frac{\omega(\nu)}{m(\nu)} + \log\left(\frac{\omega(\nu)}{m(\nu)} \right) \right] \Delta\nu$$

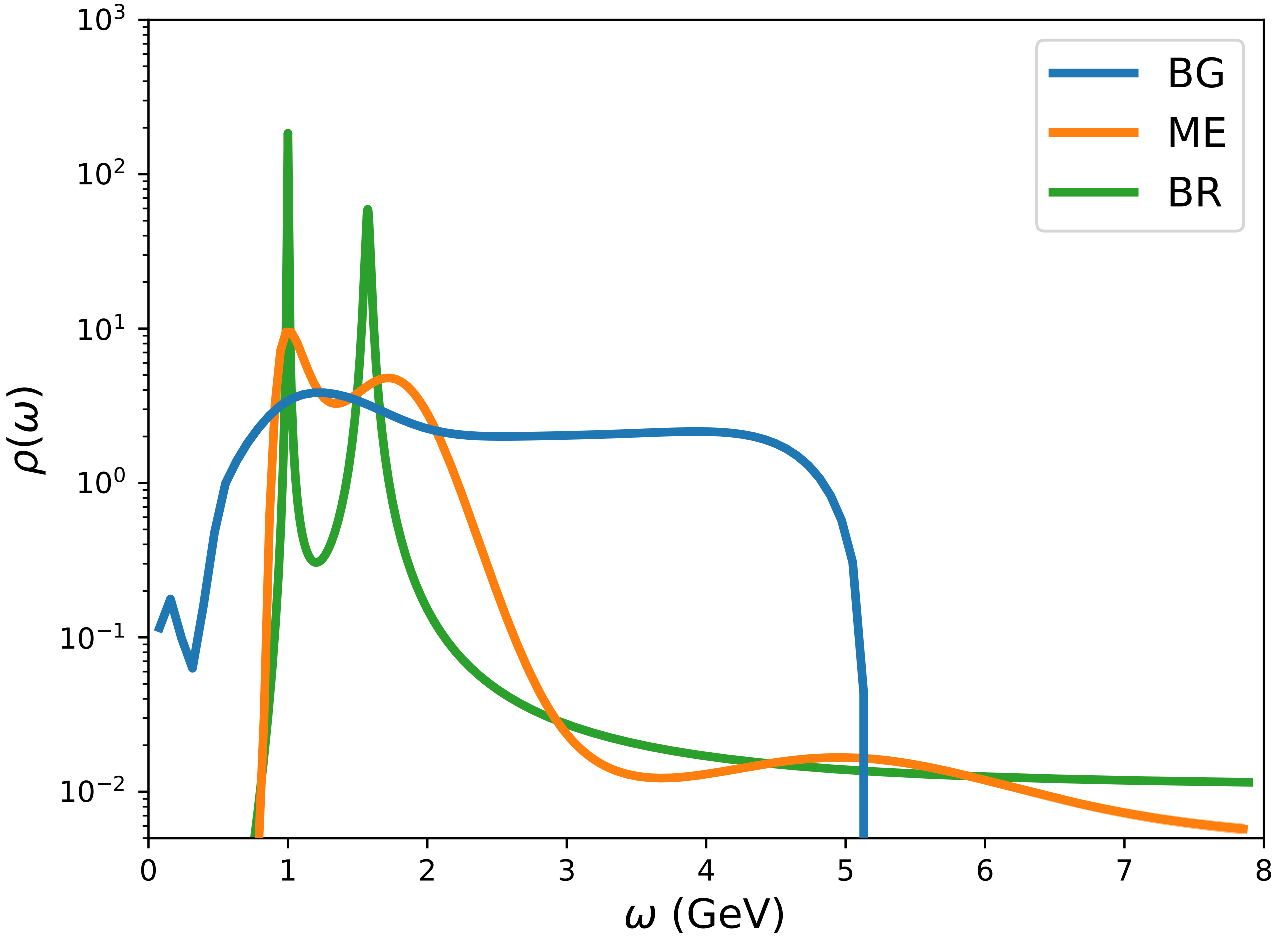
$$P[\omega | D, m] = \frac{P[D | \omega, I]}{P[D | m]} \int d\alpha P[\alpha | D, m]$$

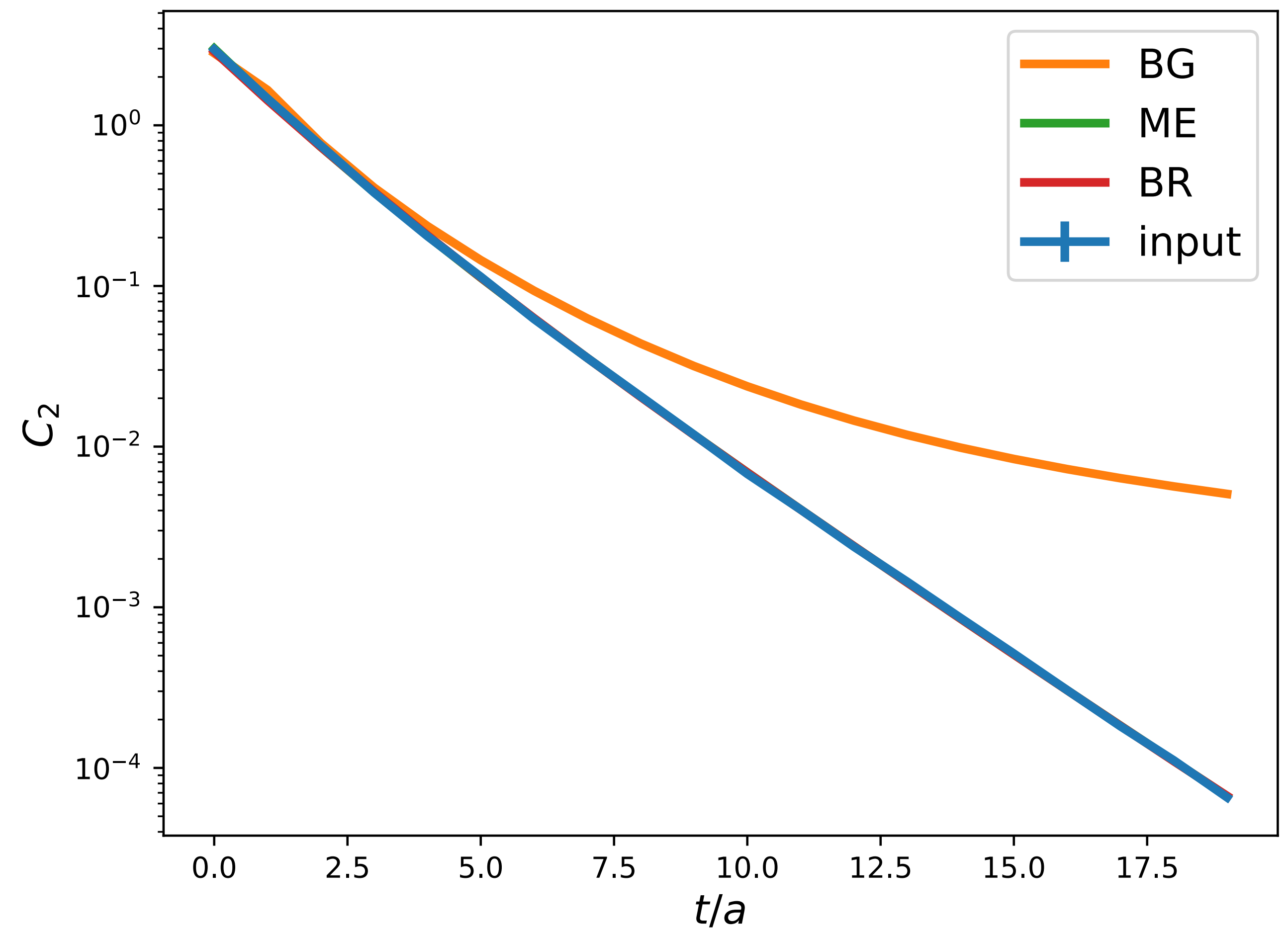
Hyper parameter alpha is integrated over.

Maximum search is in the entire parameter space ($O(10^3)$).

High precision architecture is needed (e.g., 512-bit floating point number).

Tests on nucleon two-point functions





- mock two-point function data: three single exponentials with mass 1.0, 1.5 and 1.8 GeV respectively, a~0.1 fm, Nt=20, S/N=100
- expecting peaks at ~1 GeV and ~1.6 GeV
- bad resolution of BG
- BR is shaper and more stable than ME

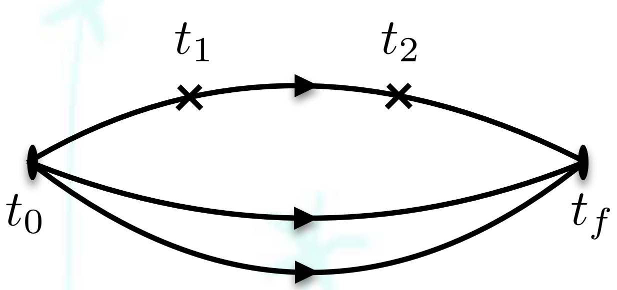
Lattice setups

clover anisotropic lattice, $24^3 \times 128,\ a_t \sim 0.035\ \text{fm},\ m_\pi \sim 380\ \text{MeV},\ \frac{2\pi}{L} \sim 0.42\ \text{GeV}$

H.-W. Lin et al., PRD 79, 034502 (2009)

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P \cdot q} F_2(x, Q^2)$$

$\mu = \nu = 1$ and $p_1 = q_1 = 0 \quad W_{11}(\nu) = F_1(x, Q^2)$



two sequential-sources for each 4-point function
554 configurations, 16 source positions

The *x*-range can be reached on this lattice is roughy [0.05, 0.3] by combining different kinematic setups.

This calculation:

$\boldsymbol{p}$	$\boldsymbol{q}$	E_p	$E_{n=0}$	$\lvert\boldsymbol{q}\rvert$	ν	Q^2	x
(0,3,3)	(0,-6,-6)	2.15	2.15	3.57	[2.96, 3.28]	[4, 2]	[0.16, 0.07]

More on the setups

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \sum_n A_n e^{-(E_n - E_p)\tau}$$

energy of the intermediate state n | **external nucleon energy**

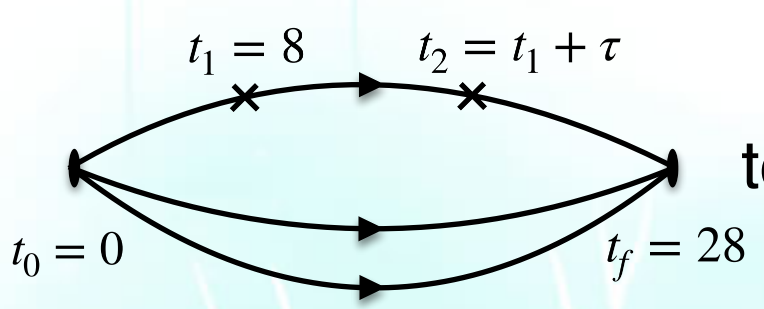
p=(033), ***q***=(0-6-6) $\boldsymbol{p} + \boldsymbol{q} = -\boldsymbol{p}$

$$E_0 = (m_N^2 + |\boldsymbol{p} + \boldsymbol{q}|^2) = E_p$$

the lowest energy of intermediate states

for small τ, higher intermediate states contribute, exponentially decay

for large τ, lowest intermediate state (elastic contribution) dominates, constant

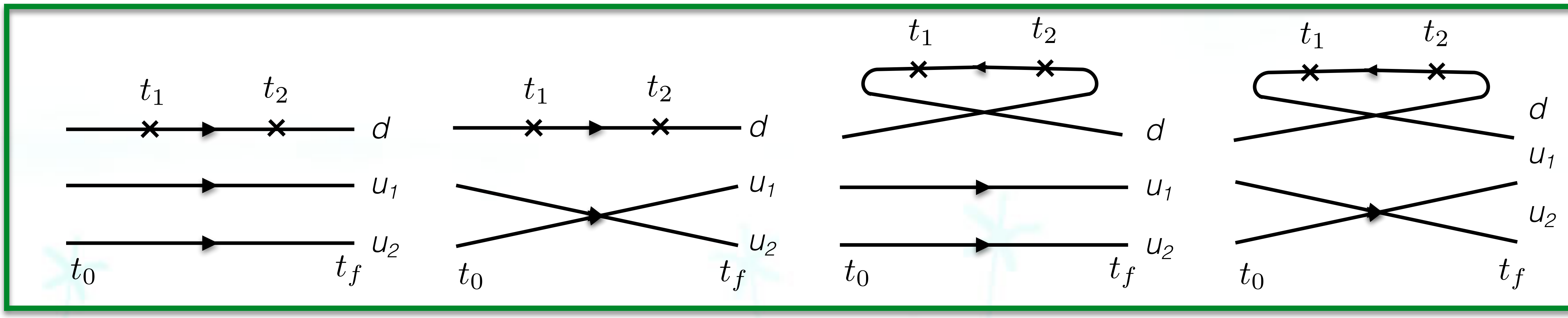


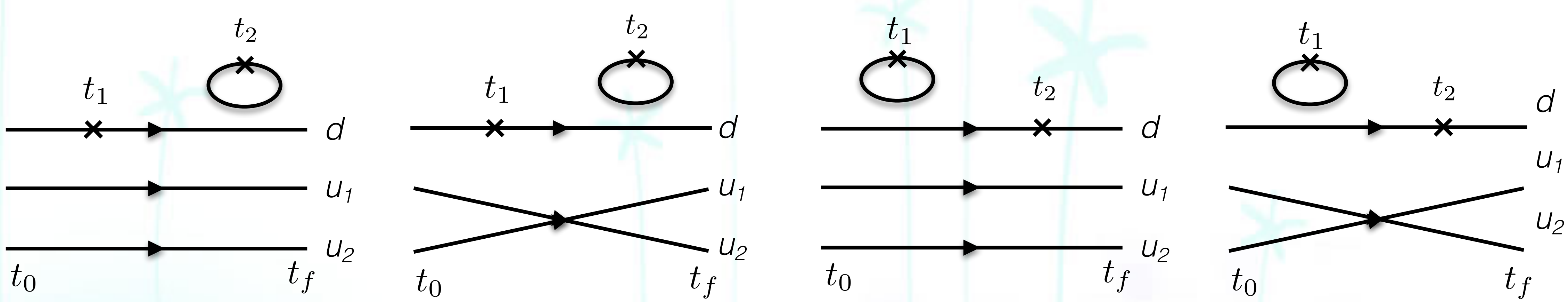
to avoid the contact point and sink excited stats $\tau \in [1,\ 12]$

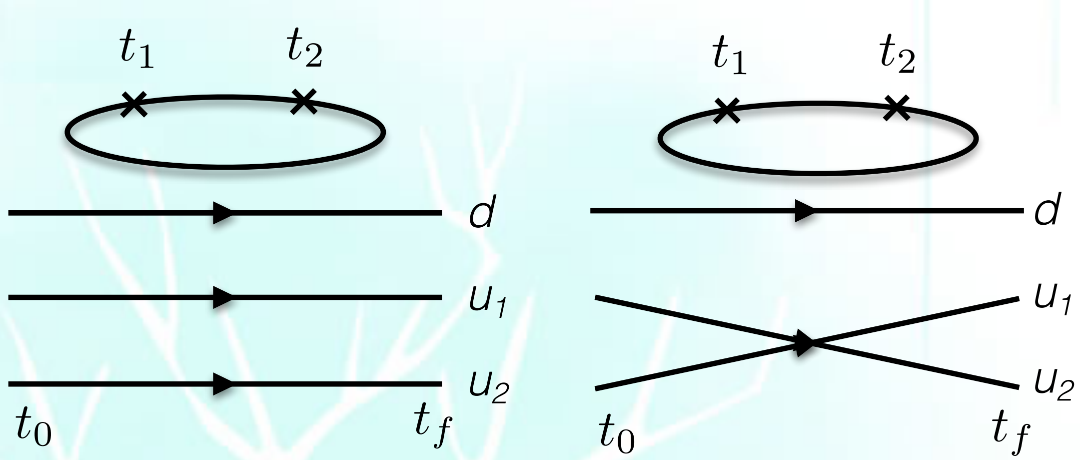
Contractions (*d* quark)

$$C_4 = \sum_{\boldsymbol{x_f}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x_f}} \sum_{\boldsymbol{x_2}\boldsymbol{x_1}} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x_2}-\boldsymbol{x_1})} \left\langle \chi_N(\boldsymbol{x_f}, t_f) J_\mu(\boldsymbol{x_2}, t_2) J_\nu(\boldsymbol{x_1}, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

$$\chi_N = \left[u_1^T C\gamma_5 d\right] u_2$$





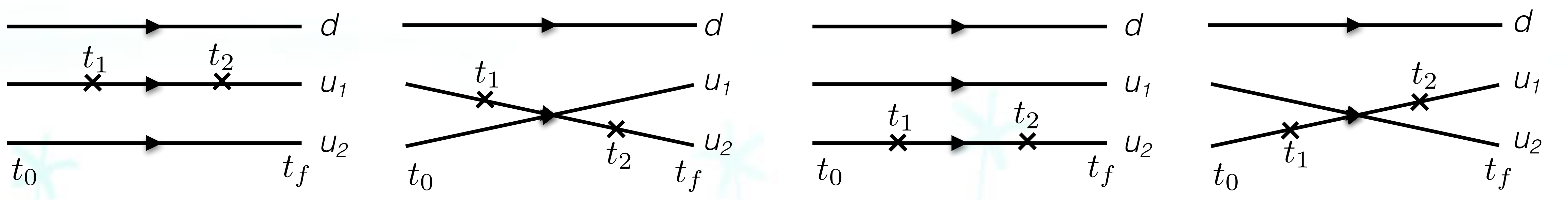


connected insertions only for now

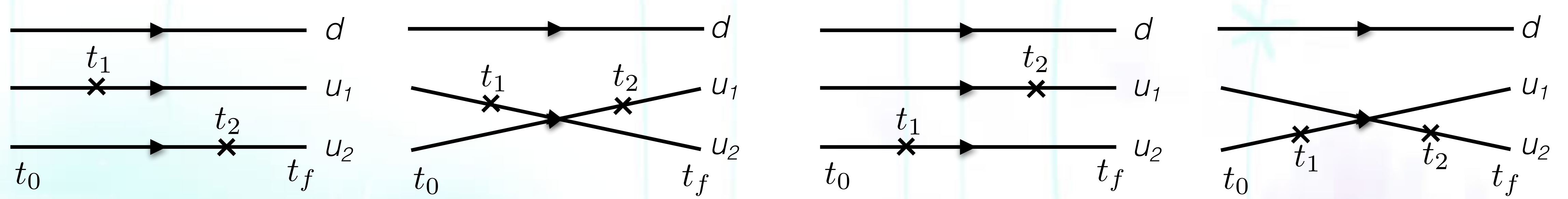
Contractions (*u* quark)

$$C_4 = \sum_{\boldsymbol{x}_f} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}_f} \sum_{\boldsymbol{x}_2\boldsymbol{x}_1} e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} \left\langle \chi_N(\boldsymbol{x}_f, t_f) J_\mu(\boldsymbol{x}_2, t_2) J_\nu(\boldsymbol{x}_1, t_1) \bar{\chi}_N(\boldsymbol{0}, t_0) \right\rangle$$

$$\chi_N = \left[u_1^T C\gamma_5 d\right] u_2$$

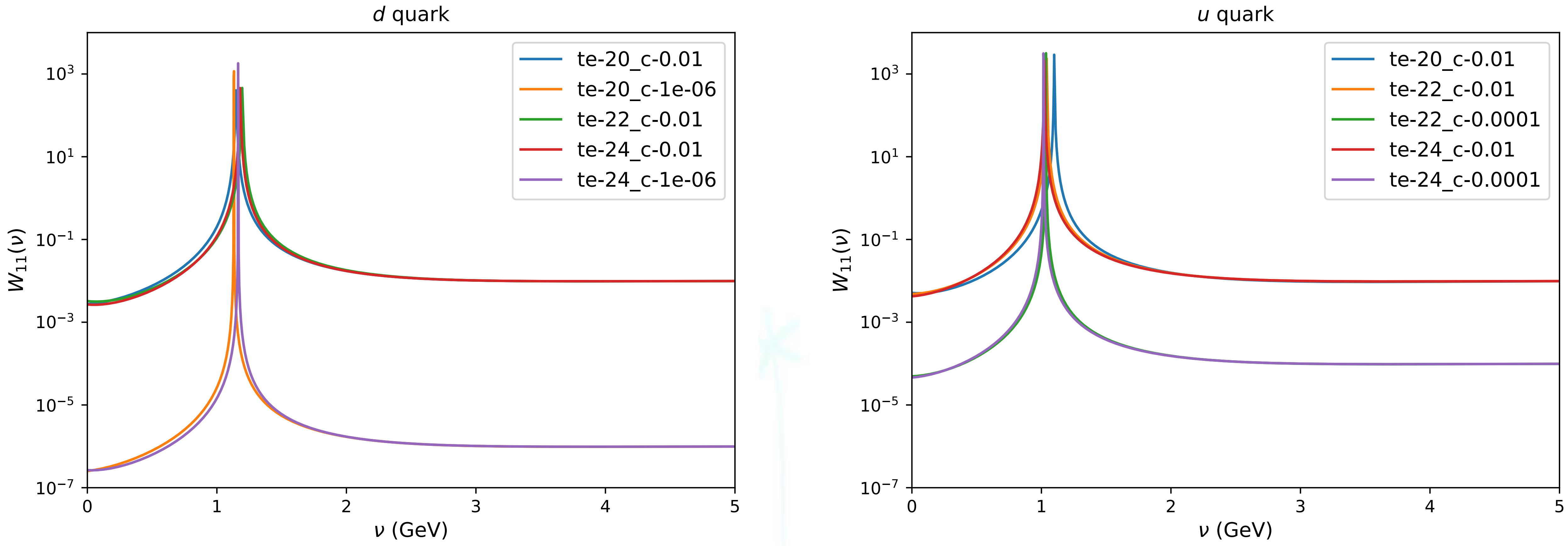


plus all possible backward propagating ones



currents can be on two different quark lines respectively (cat ear diagrams)

Minkowski hadronic tensor (after BR)



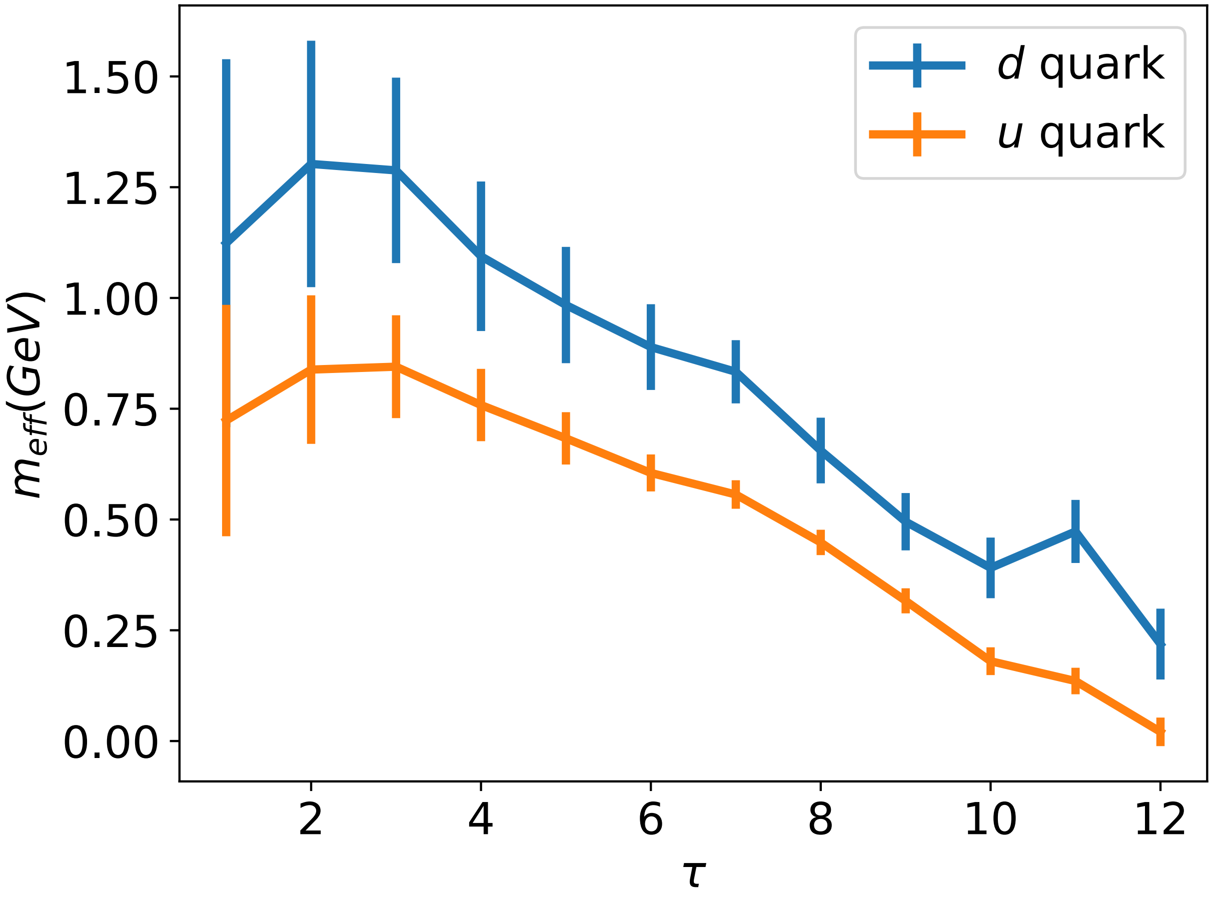
- similar structures show around 1 GeV (but shaper)
- quickly approach to the default model after 2 GeV

Check the effective mass

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau) = \sum_n A_n e^{-(E_n - E_p)\tau} \qquad E_p \sim 2.15 \text{ GeV}$$

one can check the effective mass of the Euclidean hadronic tensor

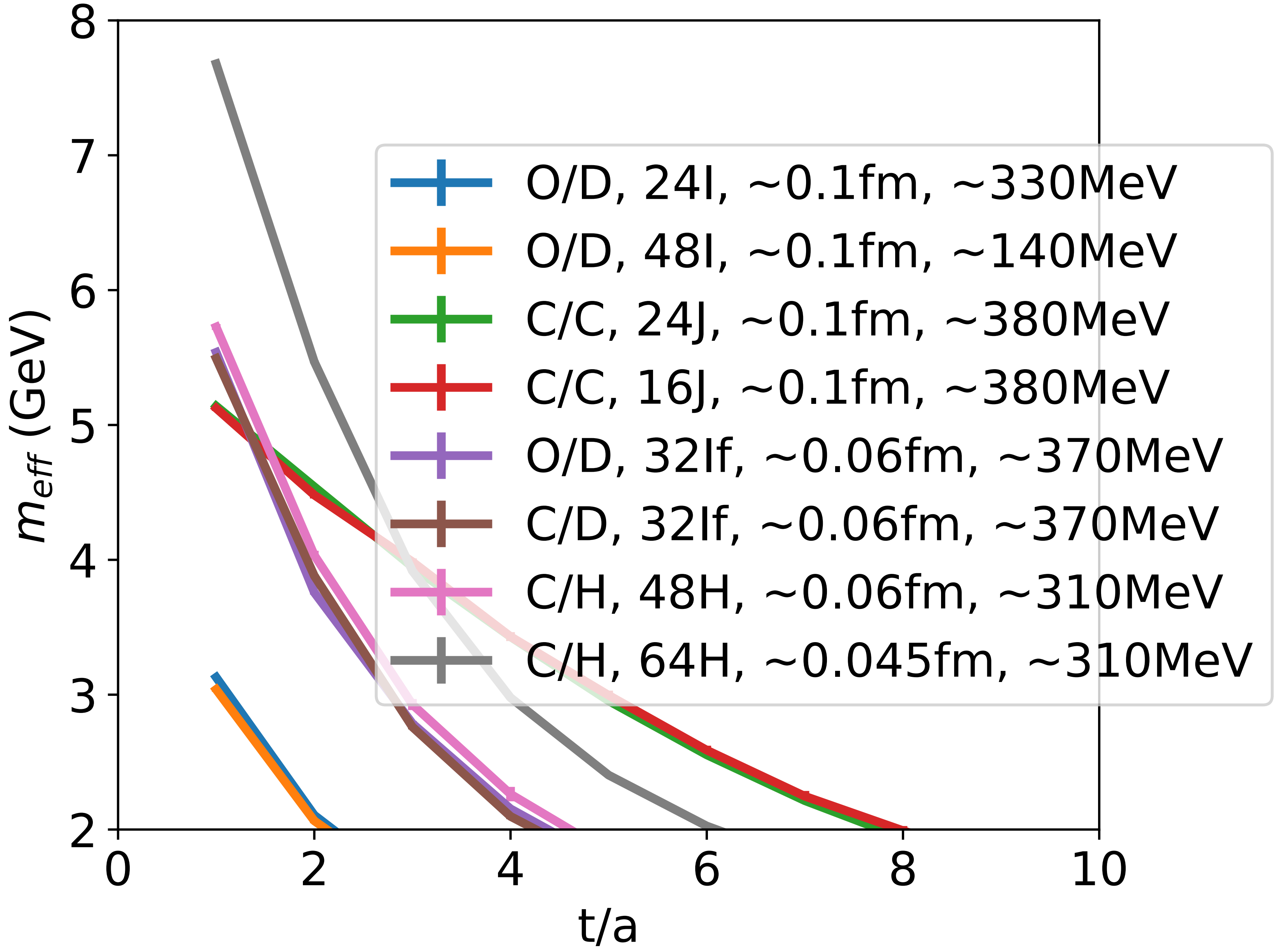
$$m_{\text{eff}} = \log[\tilde{W}(\tau)/\tilde{W}(\tau+1)]$$



$\nu \sim E_n - E_p \sim 1$ GeV $\;E_n \sim 3.2$ GeV NOT large enough energy transfer!

lattice artifacts: finite volume (resulting in discrete momenta and discrete spectrum)? finite lattice spacing (an UV cutoff)? and/or unphysical pion mass (unphysical multi-particle states)?

Learn more from two-point functions



- ✦ **It seems how high we can reach is mainly connected to the lattice spacings.**
- ✦ **Other factors are not significant.**
- ✦ **The a~0.045 fm lattice can be a much better choice.**