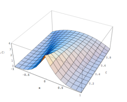





# Pseudo-GPDs

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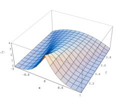
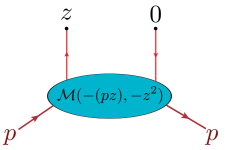
PDFLattice Meeting 2019
September 26, 2019



# Matrix elements and kinematics



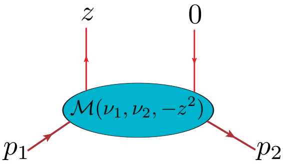





- PDFs: matrix element (ignoring spin)

$$\langle p|\phi(0)\phi(z)|p\rangle = \mathcal{M}(-(pz), -z^2)$$

- Lorentz invariance: $\mathcal{M}$ depends on $z$ through $(pz) \equiv -\nu$ and $z^2$
- Ioffe time $\nu$: $\mathcal{M}(\nu, -z^2) =$ Ioffe-time pseudo-distribution (pseudo-ITD)
- Pseudo $\equiv$ off the light cone

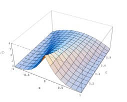



- GPDs: matrix element (ignoring spin)

$$\langle p_2|\phi(0)\phi(z)|p_1\rangle = \mathcal{M}(\nu_1, \nu_2, -z^2)$$

- $\mathcal{M}$ depends on $z$ through 2 Ioffe times $(p_1 z) \equiv -\nu_1$, $(p_2 z) \equiv -\nu_2$ and $z^2$
- Introduce $\mathcal{P} = (p_1 + p_2)/2$, average Ioffe time $-(\mathcal{P}z) = (\nu_1 + \nu_2)/2 \equiv \nu$
- Skewness variable

$$\xi = \frac{(p_1 z) - (p_2 z)}{(p_1 z) + (p_2 z)} = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \Rightarrow \nu_1 - \nu_2 = 2\xi\nu$$



# GPD kinematics on the lattice



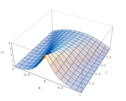


- GPD definition for pion (Ji,1996)

$$\langle p_2|\bar{\psi}(-z/2)\gamma^\alpha \hat{E}\psi(z/2)|p_1\rangle \quad = 2\mathcal{P}^\alpha \int_{-1}^{1} dx\, e^{-ix(\mathcal{P}z)}\, H(x,\xi,t;\mu^2)$$

- On the light cone: $z = z_-$, $\alpha = +$. This choice kills $z^\alpha$ contamination
- For the nucleon: $2\mathcal{P}^+$ substituted by $\bar{u}(p_2)\gamma^+ u(p_1)$
- On the lattice: take $z = z_3$. Then $-z^2 = z_3^2$
- Take $\alpha = 0$ to kill $z^\alpha$ contamination
- Orient $\mathcal{P}$ in $z_3$ direction $\mathcal{P} = \{\mathcal{P}^0, 0_\perp, \mathcal{P}^3\}$
- Then $p_1 = \{E_1, \Delta_\perp/2, P_1\}$ and $p_2 = \{E_2, -\Delta_\perp/2, P_2\}$
- Hence, $-(p_1 z) \equiv \nu_1 = P_1 z_3$ and $-(p_2 z) \equiv \nu_2 = P_2 z_3$
- On the lattice, it is more convenient to take the $\bar{\psi}(0)\ldots\psi(z)$ operator
- Reason: $z_3/2$ on the lattice should be an integer multiple of $a$
- If $z_3/2 = na$, then $z_3 = 2na$
- Total separations $z_3$ given by an odd number of spacings are lost
- Double Ioffe-time pseudo-distribution $M(\nu_1,\nu_2,t;z_3^2)$

$$\langle p_2|\bar{\psi}(0)\gamma^0\ldots\psi(z_3)|p_1\rangle = 2\mathcal{P}^0 M(\nu_1,\nu_2,t;z_3^2)$$

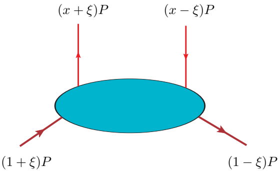






- Skewness variable $\xi$ is now given by

$$\xi = \frac{(p_1 z_3) - (p_2 z_3)}{(p_1 z_3) + (p_2 z_3)} = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} = \frac{P_1 - P_2}{P_1 + P_2}$$




- We may write $P_1 = (1+\xi)P$ and $P_2 = (1-\xi)P$, where $P \equiv \mathcal{P}_3$
- Recalling $\nu_1 + \nu_2 = 2\nu$ and $\nu_1 - \nu_2 = 2\xi\nu$ define generalized Ioffe-time pseudo-distribution (pseudo-GITD)

$$M(\nu_1, \nu_2, t; z_3^2) = \mathcal{M}(\nu, \xi, t; z_3^2)$$

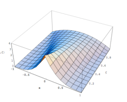
- Using translation invariance

$$\langle p_2 | \bar{\psi}(0) \ldots \psi(z) | p_1 \rangle = e^{-i(p_1 z)/2 + i(p_2 z)/2} \langle p_2 | \bar{\psi}(-z/2) \ldots \psi(z/2) | p_1 \rangle$$

- Parameterize it by *pseudo-GPD*

$$\mathcal{M}(\nu, \xi, t; z_3^2) = e^{i\xi\nu} \int_{-1}^{1} dx\, e^{ix\nu}\, \mathcal{H}(x, \xi, t; z_3^2)$$

- Check: third momentum component of the quark at $z_3$ is $(x+\xi)P$



# Matching lattice to light cone





- Inverse transformation is given by

$$\mathcal{H}\left(x,\xi,t;z_3^2\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{-ix\nu}\left[e^{-i\xi\nu}\mathcal{M}(\nu,\xi,t;z_3^2)\right]$$

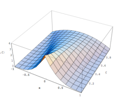
- Measure $e^{-i\xi\nu}\mathcal{M}(\nu,\xi,t;z_3^2) \equiv \widetilde{\mathcal{M}}(\nu,\xi,t;z_3^2)$
- Form reduced pseudo-GITD

$$\widetilde{\mathfrak{M}}(\nu,\xi,t,z_3^2) \equiv \frac{\widetilde{\mathcal{M}}(\nu,\xi,t,z_3^2)}{\widetilde{\mathcal{M}}(0,0,0,z_3^2)}$$

- First scenario to proceed: standard pseudo-PDF approach
- Obtain "data points" for light-cone GITD $\widetilde{\mathcal{I}}(\nu,\xi,t,\mu^2)$ from the data points for $\widetilde{\mathfrak{M}}(\nu,\xi,t,z_3^2)$ using matching relations (AR, last week)

$$\widetilde{\mathcal{I}}(\nu,\xi,t,\mu^2) = \widetilde{\mathfrak{M}}(\nu,\xi,t,z_3^2) + \frac{\alpha_s}{2\pi}C_F\int_0^1 dw\,\widetilde{\mathfrak{M}}(w\nu,\xi,t,z_3^2)$$
$$\left\{\ln\left[z_3^2\mu^2\frac{e^{2\gamma_E+1}}{4}\right]\left(\left[\frac{2w}{1-w}\right]_+\cos(\bar{w}\xi\nu)+\frac{\sin(\bar{w}\xi\nu)}{\xi\nu}-\frac{1}{2}\delta(\bar{w})\right)\right.$$
$$\left.+4\left[\frac{\ln(1-w)}{1-w}\right]_+\cos(\bar{w}\xi\nu)-2\frac{\sin(\bar{w}\xi\nu)}{\xi\nu}+\delta(\bar{w})\right\}$$

- Find/fit the $\alpha_s$ value that produces "flat" $\sim z_3$-independent values for $\widetilde{\mathcal{I}}(\nu,\xi,t,\mu^2)$



# Matching lattice to light cone, cont.



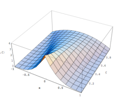
- Build model for $H(x, \xi, t; \mu^2)$ to fit the "data points" for $\widetilde{\mathcal{I}}(\nu, \xi, t, \mu^2)$ using

$$\widetilde{\mathcal{I}}(\nu, \xi, t, \mu^2) = \int_{-1}^{1} dx\, e^{ix\nu} H(x, \xi, t; \mu^2)$$

- Second scenario: "global analysis" strategy
- Write matching relation for $\widetilde{\mathfrak{M}}(\nu, \xi, t, z_3^2)$ in terms of light-cone GITD

$$\widetilde{\mathfrak{M}}(\nu, \xi, t, z_3^2) = \widetilde{\mathcal{I}}(\nu, \xi, t, \mu^2) - \frac{\alpha_s}{2\pi} C_F \int_0^1 dw\, \widetilde{\mathcal{I}}(w\nu, \xi, t, \mu^2)$$
$$\left\{ \ln\left[ z_3^2 \mu^2 \frac{e^{2\gamma_E+1}}{4} \right] \left( \left[ \frac{2w}{1-w} \right]_+ \cos(\bar{w}\xi\nu) + \frac{\sin(\bar{w}\xi\nu)}{\xi\nu} - \frac{1}{2}\delta(\bar{w}) \right) \right.$$
$$\left. + 4\left[ \frac{\ln(1-w)}{1-w} \right]_+ \cos(\bar{w}\xi\nu) - 2\frac{\sin(\bar{w}\xi\nu)}{\xi\nu} + \delta(\bar{w}) \right\}$$

- Again, write light-cone GITD in terms of GPD, then fit $\alpha_s$ and model for $H(x, \xi, t; \mu^2)$ from the data for $\widetilde{\mathfrak{M}}(\nu, \xi, t, z_3^2)$



# Modeling GPDs and polynomiality





- Subtlety: GPD should satisfy the polynomiality condition

$$\int_{-1}^{1} dx\, x^N\, H(x,\xi,t;\mu^2) = \sum_{k=0}^{N} h_k(t,\mu^2)\,\xi^k$$

- May be satisfied using the double distribution (DD) representation

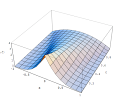
$$H(x,\xi,t;\mu^2) = \int_{|\alpha|+|\beta|\leq 1} \delta(x-\beta-\xi\alpha)\, f(\beta,\alpha;t,\mu^2)$$

- Possible model for DD is given by factorized DD Ansatz

$$f(\beta,\alpha;t,\mu^2) = f(\beta;t,\mu^2) h(\beta,\alpha)$$

- With $f(\beta;t=0,\mu^2)$ being relevant PDF,
  and $h(\beta,\alpha)$ - the profile function, e.g.,

$$h(\beta,\alpha) = \frac{3}{4}\frac{(1-|\beta|)^2-\alpha^2}{(1-|\beta|)^3}$$



# Illustration of DD→GPD conversion

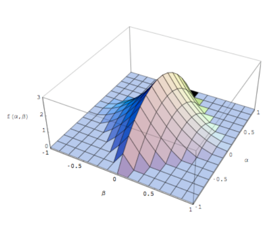


**Factorized model for DDs:**

($\sim$ usual parton density in $\beta$-direction) $\otimes$
($\sim$ distribution amplitude in $\alpha$-direction)

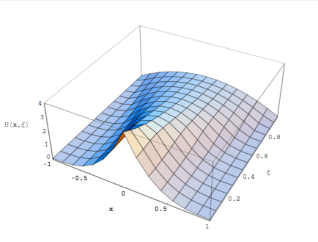
**Toy model for double distribution**

$$f(\beta,\alpha) = 3[(1-|\beta|)^2-\alpha^2]\,\theta(|\alpha|+|\beta|\le 1)$$

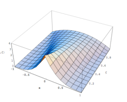


- Corresponds to toy “forward” distribution $f(\beta) = 4(1-|\beta|)^3$

**GPD $H(x,\xi)$ resulting from toy DD**



- For $\xi = 0$ reduces to usual parton density
- For $\xi = 1$ has shape like meson distribution amplitude



# Remarks on lattice implementation





- On the lattice: discrete sets of coordinates $z_3 = n_z a$ and longitudinal momenta $P_1 = 2\pi N_1/L$, $P_2 = 2\pi N_2/L$ (where $L = na$ is the lattice size)
- Possible values of the Ioffe-time parameters are limited to discrete sets $\nu_1 = 2\pi n_z N_1/n$ and $\nu_2 = 2\pi n_z N_2/n$
- Possible values for skewness are given by a set of rational numbers

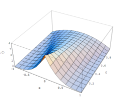
$$\xi = \frac{P_1 - P_2}{P_1 + P_2} = \frac{N_1 - N_2}{N_1 + N_2}$$

- Changing $N_1$ and $N_2$ from 0 to 6, gives 13 possible values $(0, 1/11, 1/9, \ldots 2/3, 5/7, 1)$ for $\xi$
- Varying $\xi$ changes momentum transfer $t$: taking longitudinal momenta gives

$$t = -\frac{2M^2(P_1 - P_2)^2}{M^2 + P_1 P_2 + \sqrt{M^2 + P_1^2}\sqrt{M^2 + P_2^2}} \equiv t_0(P_1, P_2, M)$$

- In the $\{\xi, P\}$ variables,

$$t_0 = -\frac{8\xi^2 M^2}{1 - \xi^2 + \frac{M^2}{P^2} + \sqrt{(1 - \xi^2 + \frac{M^2}{P^2})^2 + 4\xi^2 \frac{M^2}{P^2}}}$$



# Remarks on lattice implementation, cont.





- Value of $t_0$ is $\xi$-dependent, while we need GPDs as functions of $x$ for fixed $\xi$ and $t$
- Add transverse component $\Delta_\perp$, e.g., use $p_1 = \{E_1, \Delta_\perp/2, P_1\}$ and $p_2 = \{E_2, -\Delta_\perp/2, P_2\}$. Then

$$t = 2M^2 + 2P_1P_2 - \Delta_\perp^2/2 - 2\sqrt{M^2 + P_1^2 + \Delta_\perp^2/4}\sqrt{M^2 + P_2^2 + \Delta_\perp^2/4}$$

- Strategy: choose first some particular values of $P_1$ and $P_2$
- This fixes the value of $\xi = (P_1 - P_2)/(P_1 + P_2)$ and $t_0$
- Then take several different values of $\Delta_\perp$ to change $t$
- That will give $t$-dependence for this $\xi$ and some $\nu_0 = (P_1 + P_2)a/2$ corresponding to one lattice spacing
- Changing $z_3 = n_z a$, we vary $\nu = n_z \nu_0$ leaving $\xi$ and $t$ unchanged
- Using matching relations convert the $\nu$-dependence of GITD into the $x$-dependence of GPD
- End up with $H(x, \xi, t; \mu^2)$ for a fixed $\xi$ as a function of $x$ and $t$
- Before starting to do all this, pray to get enough computer time!